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"Quasi Quantum Groups and Monoidal Categories"

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The specific research activities in the 2022 stage of the project were materialized through a number of 6 scientific articles; 3 of them are submitted for publication and 3 are in an advanced development stage.

[1] D. Bulacu, B. Torrecillas, 1-Homology for coalgebras in Yetter-Drinfeld categories, submitted for publication.

[2] D. Bulacu, D. Popescu, B. Torrecillas, Double wreath quasi-Hopf algebras, advanced development stage.

[3] S. Dăscălescu, C. Năstăsescu, L. Năstăsescu, Graded Frobenius rings, submitted for publication.

[4] M. Joiţa, Finsler locally C^*-modules, advanced development stage.

[5] L. Liu, A. Makhlouf, C. Menini, F. Panaite, BiHom-NS-algebras, twisted Rota-Baxter operators and generalized Nijenhuis operators, submitted for publication.

[6] A. Makhlouf, D. Ştefan, Deformations of algebraic structures in monoidal categories, advanced development stage.

The scientific description of the results in the 2022 stage and the degree of achievement of the specific scientific objectives

Each • refers to results connected to one of the three scientific activities considered in the stage 2022 of the project.

• The paper [1] relates to the objectives assumed in **activity 1.1** of the realization plan of the grant. Namely, the defining and the study of the 1-cycles associated to a coalgebra in a category of Yetter-Drinfeld modules over a quasi quantum group (qQG for short) and the description of them in the case of Yetter-Drinfeld coalgebras obtained from the so called symplectic-fermionic qQGs, as well as their connections with the structure of the qQGs with a coalgebra projection. The objectives assumed at 1.1 were realized 100%. For short, the results obtained in [1] are the following.

The 1-cycles are the formal dual version of the 1-cocycles. The latter are defined for module algebras, so the former are defined for comodule coalgebras over Hopf algebras. Although the comodule coalgebra notion does not make sense in the quasi-Hopf algebra setting, we were able in [1] to introduce the 1-homology but for coalgebras in Yetter-Drinfeld categories (instead of coalgebras in categories of corepresentations) with coefficients in a given quasi-Hopf algebra; actually this is the framework that suits to the quasi-Hopf case and appeared naturally in [2]. We have obtained in this way the notion of alternative 1-cycle for a Yetter-Drinfeld coalgebra.

We added the term "alternative" because the H-linearity of ζ (with respect to the adjoint action $\triangleright$) is broken when we pass from Hopf to quasi-Hopf algebras. But we can deform an alternative 1-cycle ζ by a special element of $H\otimes H$ in order to produce an H-linear map, and the new map obtained in this way from ζ is denoted by $\overline{\zeta}$. The map $\overline{\zeta}$, called in what follows a 1-cycle for C, satisfies new relations deduced form the correspondence $\zeta\mapsto\overline{\zeta}$. In the Hopf case $\mathrm{AZ}^1_H(C,H)$ (the space of alternative 1-cycles) and $\mathrm{Z}^1_H(C,H)$ (the set of 1-cycles of H) are equal; in the quasi-Hopf case the correspondence $\zeta\mapsto\overline{\zeta}$ is one to one. We should stress the fact that the H-linearity is essential for the definition of an invertible (alternative) 1-cycle: H with $\triangleright$ gives an algebra (denoted by H_0) in the monoidal category ${}_H\mathcal{M}$ of left H-modules, and since C is a coalgebra in ${}_H\mathcal{M}$ one can consider the convolution monoid $\mathrm{Hom}_H(C,H_0)$; then $\overline{\zeta}$ is (convolution) invertible if it is an invertible element of this monoid. As $\zeta\mapsto\overline{\zeta}$ is bijective, one can consider invertible alternative 1-cycles, too. Furthermore, we show that any 1-cycle is invertible, and therefore so is any alternative 1-cycle.

The algebra H_0 in ${}_H\mathcal{M}$ is, moreover, an algebra in ${}_H^H\mathcal{Y}D$, so we expect a 1-cycle $\overline{\zeta}$ to be not only left H-linear, but to be a morphism in ${}_H^H\mathcal{Y}D$ between C and H_0. This is not always the case; in turn, to any left H-linear morphism $\overline{\zeta}:C\to H_0$ one can associate a new left H-coaction $\lambda_C^{\overline{\zeta}}$ on C and if we denote by $C^{\overline{\zeta}}$ the H-module coalgebra C equipped with this new left H-coaction then $\overline{\zeta}$ is a 1-cycle if and only if $C^{\overline{\zeta}}$ is an object in ${}_H^H\mathcal{Y}D$, in which case $\overline{\zeta}$ becomes a morphism between $C^{\overline{\zeta}}$ and H_0 in ${}_H^H\mathcal{Y}D$. Furthermore, $C^{\overline{\zeta}}$ is a right C-comodule within ${}_H^H\mathcal{Y}D$ and the correspondence $\overline{\zeta}\mapsto\lambda_C^{\overline{\zeta}}$ produces a bijective map between $\mathrm{Z}^1_H(C,H_0)$ and $\mathrm{Com}^r_{{}_H^H\mathcal{Y}D}(C,\underline{\Delta}_C)$; $\mathrm{Z}^1_H(C,H_0)$ is a new notation for $\mathrm{Z}^1_H(C,H)$, justified by the above considerations, while $\mathrm{Com}^r_{{}_H^H\mathcal{Y}D}(C,\underline{\Delta}_C)$ is the set of left H-coactions on the left H-module coalgebra C that turns C into a right C-comodule in ${}_H^H\mathcal{Y}D$ via the comodule structure morphism equals $\underline{\Delta}_C$. Consequently, one can define homologous 1-cycles: $\overline{\zeta_1}\sim\overline{\zeta_2}$ (our notation for being homologous) if and only if $\zeta_1\sim\zeta_2$, if and only if $C^{\overline{\zeta_1}}$, $C^{\overline{\zeta_2}}$ are isomorphic as right C-comodules in ${}_H^H\mathcal{Y}D$, if and only if ζ_1, ζ_2 are related somehow through a convolution invertible morphism $\varsigma\in\mathrm{Hom}_H(C,k)$. As a byproduct, we obtain a characterization for $\mathcal{AH}^1_H(C,H)\simeq\mathcal{H}^1_H(C,H_0)$, the set of equivalence classes of alternative 1-cycles (resp. 1-cycles) modulo $\sim$.

The coalgebras in ${}_H^H\mathcal{Y}D$ define a category $\mathrm{Coalg}({}_H^H\mathcal{Y}D)$ that is isomorphic to the category $H-\mathrm{BimCoalg}(\pi)$; the latter has as objects couples (D,π) consisting of a coalgebra D in ${}_H\mathcal{M}_H$ and an H-bimodule coalgebra morphism $\pi:D\to H$. To a coalgebra C in ${}_H^H\mathcal{Y}D$ one associates the coalgebra $D=C\rtimes H$, the so called smash product coalgebra of C and H, and the morphism $\pi=\underline{\varepsilon}_C\otimes\mathrm{Id}_H$. A natural problem that arises is the following: for $(D,\pi)\in H-\mathrm{BimCoalg}(\pi)$ determine all the coalgebras $\mathfrak{C}$ in ${}_H^H\mathcal{Y}D$ that realizes D; that is, for $\varrho:\mathfrak{C}\rtimes H\to H$ a morphism of coalgebras in ${}_H\mathcal{M}_H$, $(\mathfrak{C}\rtimes H,\varrho)$ and $(C\rtimes H,\underline{\varepsilon}_C\otimes\mathrm{Id}_H)$ are isomorphic objects in $H-\mathrm{BimCoalg}(\pi)$. To find the couples $(\mathfrak{C},\varrho)$ that realize a given D (or C) we must first describe the morphisms between two smash product coalgebras over a given quasi-Hopf algebra H. Inspired by the work of Schauenburg we showed that a morphism $F : C \bowtie H \to C' \bowtie H$ of coalgebras in ${}_H\mathcal{M}_H$ identifies to a pair (Ω, ζ) consisting of a morphism $\Omega : C \to C'$ of coalgebras in ${}_H\mathcal{M}$ and an alternative 1-cycle ζ of C that are compatible in a certain way. Then a brute characterization for $\mathrm{Aut}_{\overline{H}}(C \bowtie H)$, the group of coalgebra automorphisms of $C \bowtie H$ in ${}_H\mathcal{M}_H$, is provided; the group of coinner automorphisms of the H-bimodule coalgebra $C \bowtie H$, $\mathrm{Coinn}_{\overline{H}}(C \bowtie H)$, is characterized, too. A nicer description for the elements of $\mathrm{Aut}_{\overline{H}}(C \bowtie H)$ is given latter one, after the study of invertible (alternative) 1-cycles of a Yetter-Drinfeld coalgebra. But the true meaning of an element of $\mathrm{Aut}_{\overline{H}}(C \bowtie H)$ is uncovered after the solving of the problem mantioned above. More precisely, giving a coalgebra morphism $\varrho : \mathfrak{C} \bowtie H \to H$ in ${}_H\mathcal{M}_H$ is equivalent to giving an alternative 1-cycle for $\mathfrak{C}$; then, in terms of ζ, there exists a Yetter-Drinfeld coalgebra $\mathfrak{C}^{(\zeta)}$ derived from $\mathfrak{C}$ and ζ such that $(\mathfrak{C} \bowtie H, \varrho)$ and $(\mathfrak{C}^{(\zeta)} \bowtie H, \underline{\varepsilon}_{\mathfrak{C}^{(\zeta)}} \otimes \mathrm{Id}_H)$ are isomorphic objects in $H - \mathrm{BimCoalg}(\pi)$. We deduce from here that $(\mathfrak{C}, \varrho)$ realizes D if and only if $\mathfrak{C}^{(\zeta)}$ and C are isomorphic as coalgebras in ${}_H^H\mathcal{Y}D$, where ζ is the alternative 1-cycle of $\mathfrak{C}$ that defines ϱ, if and only if $\mathfrak{C} \times H$ and $C \times H$ are isomorphic coalgebras in ${}_H\mathcal{M}_H$. Therefore, $\mathrm{Aut}_{\overline{H}}(C \bowtie H)$ is described by pairs (Ω, ζ) consisting of $\Omega \in \mathrm{Aut}_H(C)$, an automorphism of the coalgebra C in ${}_H\mathcal{M}$, and an alternative 1-cycle ζ for C such that Ω is, moreover, a Yetter-Drinfeld coalgebra isomorphism between $C^{(\zeta)}$ and C.

An element ζ of $\mathrm{AZ}^1_H(C, H)$ is called good if $C^{(\zeta)}$ and C are isomorphic as coalgebras in ${}_H^H\mathcal{Y}D$. If $\mathrm{AZ}^{1,g}_H(C, H)$ stands for the set of good alternative 1-cycles of C, it follows that $\mathrm{AZ}^{1,g}_H(C, H)$ identifies to the set of left cosets of $\mathrm{AutCoalg}_{{}_H^H\mathcal{Y}D}(C)$ (the group of coalgebra automorphisms of C in ${}_H^H\mathcal{Y}D$) in $\mathrm{Aut}_{\overline{H}}(C \bowtie H)$. More generally, we proved that $\mathrm{Aut}_{\overline{H}}(C \bowtie H)$ acts on the set $\mathrm{AZ}^1_H(C, H)$ from the left and that the orbits of this group action gives the types of the Yetter-Drinfeld coalgebras $C^{(\zeta)}$: $C^{(\zeta_1)}$ and $C^{(\zeta_2)}$ are isomorphic as coalgebras in ${}_H^H\mathcal{Y}D$ if and only if ζ_1, ζ_2 belong to the same orbit. Note that, the alternative 1-cycles that realize the same coalgebra as C in $H - \mathrm{BimCoalg}(\pi)$ are precisely the good ones, and they define the orbit of the trivial alternative 1-cycle of C, namely $C \ni c \mapsto \underline{\varepsilon}_C(c)1_H \in H$; 1_H is the unit of H. Furthermore, this group action induces a group action of $\mathrm{Out}_{\overline{H}}(C \bowtie H)$ (the group of outer coalgebra automorphisms of C in ${}_H\mathcal{M}$, defined as the quotient of $\mathrm{Aut}_{\overline{H}}(C \bowtie H)$ by the normal subgroup $\mathrm{Coinn}_{\overline{H}}(C \bowtie H)$) on $\mathcal{AH}^1_H(C, H)$ and the orbits corresponding to this new action are in a one to one correspondence to those produced by the group action of $\mathrm{Aut}_{\overline{H}}(C \bowtie H)$ on $\mathrm{AZ}^1_H(C, H)$. This new description of the orbits is used in the case when $\mathrm{Aut}_{\overline{H}}(C \bowtie H)$ and $\mathcal{AH}^1_H(C, H)$ are finite sets, when we determined the number N of the orbits, that is the number of types of Yetter-Drinfeld coalgebras of the from $C^{(\zeta)}$.

In [1], we have also computed the alternative 1-cycles of the Yetter-Drinfeld coalgebras that describe the symplectic fermion quasi-Hopf algebras as what we called a double wreath quasi-quantum group. The symplectic fermion quasi-Hopf algebra A contains a 4-dimensional quasi-Hopf subalgebra H and C has a coalgebra structure in ${}_H^H\mathcal{Y}D$ given by a certain morphism of

H-bimodule coalgebras $\pi : A \to H$; thus A identifies to $C \rtimes H$ as a coalgebra in ${}_H\mathcal{M}_H$. We showed that an alternative 1-cycle ζ of C with coefficients in H is determined by a certain value of ζ on the unit of A and a family of complex scalars (a part of them arbitrary and the other part defined inductively). Although $\mathrm{AZ}^1_H(C, H)$ is an infinite set, $\mathcal{AH}^1_H(C, H)$ has 4 elements and this fact allowed to show that any alternative 1-cycle of C is good. Consequently, for our coalgebra C in ${}_H\mathcal{M}$ its comultiplication $\underline{\Delta}_C$ determines 4 non-isomorphic C-comodule structures for C in ${}^H_H\mathcal{Y}H$ and $C^{(\zeta)} \simeq C$ as coalgebras in ${}^H_H\mathcal{Y}D$, for any $\zeta \in \mathrm{AZ}^1_H(C, H)$. Hence $N = 1$ and the group action is transitive.

The main results of [1] were presented at the international conference "New trends in Hopf algebras and Monoidal categories", 6-9 september, Turin, Italy (see https://www.hopf-turin-22.it/speakers/bulacu).

• For continuity, we will explain now how the results obtained in [2, 5] realize the objectives of the **activity 1.3** of the stage 2022 of the project in the percent of 90% (and shortly in percent of 100%, by submitting [2] for publication).

By [2] all the objectivies of the **activity 1.3** are realized: the description of a qQG with a weak projection, the defining of a 2-cocycle for a qQG, a deformation theory by 2-cocycles for qQGs, examples. Below we describe all these results of [2].

As far as we know, there is no natural way to define, in general, 2-cocycles for a quasi-Hopf algebra. Nevertheless, there are situations when we can do this for Hopf like objects lying in suitable categories. Such a situation occurs for instance when we deal with Hopf algebras in braided categories. Another one, which fits perfectly to our case, is when we deal with the so-called bimonoids in duoidal categories. To be more concise, if A is a bimonoid in a duoidal category $\mathcal{C}$, the Hopf case adapted to a certain categorical monoidal setting leads us naturally to the concept of a normalized (invertible) 2-cocycle σ for A; and also to the deformation of A by σ (denoted in what follows by A^σ), which is as well a bimonoid in $\mathcal{C}$. Our problem is then solved by taking $\mathcal{C}$ equals ${}_H\mathcal{M}_H$, the category of bimodules over a quasi-bialgebra H. Consequently, for a biproduct quasi-Hopf algebra $R \times H$ we have a categorical way to introduce normalized (invertible) 2-cocycles (over H), as well to perform a deformation theory for them. The latter gives rise to quasi-Hopf algebras with a weak projection which are not, in general, biproduct quasi-Hopf algebras. Their structure is described in [2] in terms of what we called a pre-Hopf algebra with 1-cycle in a category of Yetter-Drinfeld modules. Alternatively, up to isomorphism, a quasi-Hopf algebra with a weak projection is a wreath algebra in ${}_k\mathcal{M}$, the category of vector spaces, that is at the same time an op-wreath coalgebra within the category of bimodules defined by a quasi-Hopf algebra H. Again, there is a natural way that leads to this result. Namely, conceptually, a quasi-bialgebra is an associative algebra H such that the category of H-bimodules is monoidal and H has a natural coalgebra structure within this category. Now, for a given quasi-bialgebra H, one can consider wreath algebras $R\#_\gamma H$ determined by wreath algebras R in ${}^{\#}_{H}\mathcal{T}({}_k\mathcal{M})$ and, dually, op-wreath coalgebras $R \bowtie^{\delta} H$ if R is a op-wreath coalgebra in $\mathcal{E}({}_H\mathcal{M}_H)^{H}_{\#}$, too. Both $R\#_\gamma H$ and $R \bowtie^{\delta} H$ are build on the vector space $R \otimes H$, the former is an associative k-algebra and the latter is a coalgebra in ${}_H\mathcal{M}_H$. Thus, it is natural to ask when the two structures afford a quasi-bialgebra (resp. quasi-Hopf algebra) structure on $R \otimes H$ such that the natural embedding $\iota : H \to R \otimes H$ becomes a quasi-bialgebra (resp. quasi-Hopf algebra) morphism. We proved that this happens if and only if R is a pre-Hopf algebra with 1-cycle in ${}^H_H\mathcal{Y}D$.

In [2], we also completed the Yetter-Drinfeld coalgebra structure of the symplectic fermion quasi-Hopf algebras $\mathcal{O}_q(N)$ up to a double wreath quasi-Hopf algebra structure, by computing explicitly the multiplication and the 1-cycle corresponding to the diagram R of $\mathcal{O}_q(N)$. Secondly, for some 8-dimensional quasi-Hopf algebras we found coalgebra projections onto the unique 2-dimensional quasi-Hopf algebra $H(2)$ which is not twist equaivalent to a Hopf algebra, determined their double wreath quasi-Hopf algebra structure and then the 2-cocycles on them and the deformed quasi-Hopf algebras corresponding to them.

The results obtained in [2] will be presented at the international conference UMA-RMA, Ronda (Spain), 12-16/12/2022 (http://www.rsmeuma2022.uma.es/index.php/programa-general/).

Another objective of the **activity 1.3** is the study of some Rota-Baxter operators and dendriform algebras and their generalizations, in connection to shuffle algebras. By submitting [5] for publication, this objective was achieved in percent of 100%. For short, the content of [5] is the following.

NS-algebras (corresponding to associative algebras) have been introduced by Leroux and independently by Uchino as algebras with three operations $\prec$, $\succ$ and $\vee$ satisfying certain axioms that imply that the new operation $* = \prec + \succ + \vee$ is associative. NS-algebras generalize both Loday's dendriform and Loday-Ronco's tridendriform algebras. Examples are obtained via so-called twisted Rota-Baxter operators, which are a generalization of $\mathcal{O}$-operators involving a Hochschild 2-cocycle, and via Nijenhuis operators. We recall that a Nijenhuis operator $N : A \to A$ on an associative algebra (A, μ) with multiplication denoted by $\mu(x \otimes y) = xy$, for $x, y \in A$, is a linear map satisfying

$$N(x)N(y) = N(N(x)y + xN(y) - N(xy)), \quad \forall\, x, y \in A. \tag{0.1}$$

As discovered by Leroux, if one defines $x \prec y = xN(y)$, $x \succ y = N(x)y$ and $x \vee y = -N(xy)$, then $(A, \prec, \succ, \vee)$ is an NS-algebra, and in particular the new multiplication defined on A by $x * y = xN(y) + N(x)y - N(xy)$ is associative. Basic examples are obtained by taking a fixed element $a \in A$ and defining $N_1, N_2 : A \to A$ by $N_1(x) = ax$ and $N_2(x) = xa$, for all $x \in A$; it turns out that N_1, N_2 are Nijenhuis operators and in each case the new multiplication $*$ as above boils down to $x * y = xay$, for all $x, y \in A$. This property can be regarded also in the following (converse) way: the fact that the new multiplication on A given by $x * y = xay$ is associative (usually, this new operation $*$ is said to be a "perturbation" of the old multiplication of A via the element a) can be obtained as a consequence of a property of Nijenhuis operators (or, alternatively, that it can be given a Nijenhuis operator interpretation).

Hom-type and BiHom-type algebras are certain algebraic structures (of growing interest in recent years) whose study began in some early papers by Makhlouf and Silvestrov and more recently in the paper G. Graziani, A. Makhlouf, C. Menini, F. Panaite, *BiHom-associative algebras, BiHom-Lie algebras and BiHom-bialgebras*, Symmetry Integrability Geom. Methods Appl. (SIGMA) **11** (2015), 086, and can be roughly described as being defined by some identities obtained by twisting a classical algebraic identity (such as associativity) by one or two maps. For instance, a BiHom-associative algebra (A, μ, α, β) is an algebra (A, μ), with notation $\mu : A \otimes A \to A$, $\mu(x \otimes y) = xy$, together with two (multiplicative with respect to μ) commuting linear maps (called structure maps) $\alpha, \beta : A \to A$ such that $\alpha(x)(yz) = (xy)\beta(z)$ for all $x, y, z \in A$. There exist BiHom analogues of many types of algebras, in particular of (tri)dendriform algebras, infinitesimal bialgebras etc. Examples of (Bi)Hom-type algebras can be obtained from classical types of algebras by a procedure called "Yau twisting".

The BiHom analogue of the "perturbations" mentioned above has been introduced in the paper L. Liu, A. Makhlouf, C. Menini, F. Panaite, *Tensor products and perturbations of BiHom-Novikov-Poisson algebras*, J. Geom. Phys. **161** (2021), 104026 as follows. Let (A, μ, α, β) be a BiHom-associative algebra and let $a \in A$ be such that $\alpha^2(a) = \beta^2(a) = a$. Define a new operation on A by $x * y = \alpha(x)(\alpha(a)y)$; then $(A, *, \alpha^2, \beta^2)$ is a BiHom-associative algebra.

The starting point of the paper [5] was to look for a "Nijenhuis operator interpretation" of this fact. One can notice that Nijenhuis operators defined by the relation (0.1) can be considered on any type of algebra (not necessarily associative), so we were trying to find a Nijenhuis operator on (A, μ, α, β) depending on the given element a and that would lead to the operation $*$, but this did not work. It turns out that the solution to this problem was to consider a generalized version of Nijenhuis operators on BiHom-associative algebras. And indeed, the operators $N_1, N_2 : A \to A$, $N_1(x) = \alpha(a)x$ and $N_2(x) = x\alpha(a)$, for $x \in A$, are such generalized Nijenhuis operators from which one can obtain the multiplication $*$ in a certain way.

We were then led to introduce the concept of BiHom-NS-algebra, the BiHom analogue of Leroux's and Uchino's NS-algebras. They generalize BiHom-(tri)dendriform algebras, and it turns out that the generalized Nijenhuis operators that we introduced lead to BiHom-NS-algebras. We defined as well the BiHom analogue of twisted Rota-Baxter operators and proved that they also lead to BiHom-NS-algebras and that, moreover, just as in the classical case, there is an adjunction between BiHom-NS-algebras and twisted Rota-Baxter operators.

The last objectives of **activity 1.3** refer to the **initiation** of a study regarding the relationship between almost dual pairs in Yetter-Drinfeld module categories and 2-cocycles in certain braided categories. This study was indeed initiated, in the sense that we have defined almost dual pairs in Yetter-Drinfeld module categories and described, in a particular case, the structure of a 2-cocycle on a quasi-quantum group of biproduct type (a particular case of a quasi-quantum group with weak projection). Thus, we can say that this objective has been achieved in percent of 100%.

• We mention now the results that are related to the **activity 1.2** of the stage 2022 of the project. In this direction, the first objective, the description of the braided tensor Hopf algebra associated to a vector space V (considered as Yetter-Drinfeld module over a Hopf group algebra deformed by a 3-cocycle) was realized in percent of 100%. Furthermore, we have described 2-dimensional braided Hopf algebras in a category of Yetter-Drinfeld modules over a qQG and we have obtained concrete examples of this type. Also, we have initiated the description of their tensor algebras. Another objective of **activity 1.2** was the defining of a class of braided Hopf algebras that are not just vector spaces. More precisely, they are Frobenius algebras in categories of corepresentations over a (q)QG; and that in this direction a structure theorem for them is essential for our study. This problem is solved in [3], whose content we will detail in what follows. We would like to mention that, by submitting [3] for publication, this objective of **activity 1.2** was achieved in percent of 100%.

Originated in the work of Frobenius on group representations, Frobenius algebras and their relatives, quasi-Frobenius algebras, have been objects of intense study after the influential work of Brauer, Nesbitt and Nakayama around 1940. The initial interest was algebraic, but Frobenius algebras occurred, sometimes unexpectedly, in topology, differential geometry, knot theory, homological algebra, topological quantum field theory, Hopf algebra theory, etc. A step towards a deeper understanding of Frobenius algebras from a ring theoretical perspective was the study of (quasi-)Frobenius rings.

There are certain Frobenius algebras equipped with more structure that occur in a natural way, for example Frobenius algebras endowed with a grading. Inspired by an equivalent characterization of Frobenius algebras given by Abrams, one can consider Frobenius algebras in an arbitrary monoidal category as algebras A, endowed with a coalgebra structure whose comultiplication is a morphism of A-bimodules. In particular, one can look at Frobenius algebras in the monoidal category of G-graded vector spaces, where G is a group; these are called graded Frobenius algebras, and they were investigated in previous work of the authors, as well as a version modified by a shift, called σ-graded Frobenius algebras. Such objects occur in noncommutative geometry, where certain connected graded algebras are n-graded Frobenius for a positive integer n. For example, if A is a connected Noetherian graded algebra which is Artin-Schelter regular and Koszul, of global dimension n, then the Koszul dual algebra $A^!$ of A is n-graded Frobenius.

The structure and representation theory of graded Frobenius algebras have been used to classification results for certain algebras playing a role in non-commutative geometry, and for proving a non-commutative Bernstein-Gelfand-Gelfand correspondence.

In order to understand the structure of graded Frobenius algebras, our initial aim was to fill in a missing piece of the Frobenius puzzle, by defining and investigating graded quasi-Frobenius algebras. In developing the theory, we realized that it is interesting to consider ring theoretical versions of the concepts. The aim of the paper is to introduce graded quasi-Frobenius rings and (σ-)graded Frobenius rings, and to investigate them and their representations. As expected, a finite dimensional graded algebra turns out to be (σ-)graded Frobenius if and only if it is (σ-)graded Frobenius as a ring.

Some results about graded rings and graded modules may give the impression that graded theory is a simple extension of the un-graded one. This is true up to a point, and a reason is that the category of modules over a ring and the category of graded R-modules over a graded ring R are both Grothendieck categories. However, the category of graded R-modules is equipped with a family of category isomorphisms, the shifts by group elements, and this adds an extra level of complexity to the structure of this category and its objects. As an example in support of this idea, we mention the theory of the graded Grothendieck group of an algebra graded by an abelian group. On the other hand, even in the case where the category of graded R-modules is equivalent to the category of modules over a ring A, this ring has usually a much more complicated structure than R.

In Section Section 2 of [3] we discussed the structure of a graded ring $A = M_n(\Delta)(g_1, \dots, g_n)$ associated with a graded division ring Δ, and some group elements $g_1, \dots, g_n$. A is graded simple and graded Artinian, so any two graded simple left A-modules are isomorphic up to a shift. We count the isomorphism types of graded simple left A-modules, and how many of them embed into A. In Section 3 of [3] we considered the graded versions of the Jacobson radical and the singular radical, and we prove some of their properties related to finiteness conditions and to injectivity. We also give an alternative proof of the structure theorem for graded simple graded left Artinian rings, which says that any such ring is isomorphic to $M_n(\Delta)(g_1, \dots, g_n)$ for some n, Δ and $g_1, \dots, g_n$. In Section 4 of [3] we considered the decomposition of a graded left Artinian ring into a sum of graded indecomposable left modules, and obtain some consequences on the graded simple modules when we factor by the graded Jacobson radical. A structure result for projective objects in the category of graded modules is derived. In Section 5 of [3] we defined graded quasi-Frobenius rings by proving several equivalent characterizations. In the case of a graded ring R of finite support, we show that R is graded quasi-Frobenius if and only if it is quasi-Frobenius. More properties of graded quasi-Frobenius rings were investigated in Section 6 of [3], where we also associated a certain set of data with a graded quasi-Frobenius ring, including a version of the Nakayama permutation. This set of data is used in Section

7 of [3] to introduce graded Frobenius rings and to give equivalent characterizations. In fact, we define the more general version of a σ-graded Frobenius ring, which matches with the shift-modified version of graded Frobenius algebra mentioned above. At this point it will be clear that there is a higher degree of complexity of the concept, compared to the un-graded one. In the un-graded case, the Nakayama permutation and the multiplicities of the isomorphism types of principal indecomposable modules is all that we need for deciding whether a quasi-Frobenius ring is Frobenius, while in the graded case it turns out that one needs more information, related to the inertia groups of the graded simple modules and certain shifts. We note that for developing the theory of graded (quasi-)Frobenius rings, we need many times to work not with isomorphism types of graded modules, but with isoshift types.

The last objective of **activity 1.2** refers to balanced structures and twists on categories of Yetter-Drinfeld modules and their connections to pivotal, sovereign and/or spherical ones. Here, we have described the balanced and ribbon structures of a category of Yetter-Drinfeld modules over a quasi-quantum group. At first sight, they are extremely complicated and, therefore, difficult to work with them; in what follows we will try to refine them making their connections with the pivotal, sovereign and/or spherical structures natural. For this reason, this last objective was achieved in percent of 90%. We expect the problem to be completely solved in the near future (these results are viewed as support for certain assumed objectives in the two coming years).

We conclude this part of the report with the description of the scientific content of the papers [4, 6]. The results from [4, 6] are related to assumed objectives for the following years, but we will include them here because they were obtained in this stage of the project.

An objective of the project (from the following years) is to get concrete examples of shuffle quasi-quantum groups by using the study of certain functors between categories of Hilbert modules over C^*-local algebras; the paper [4] fits under this topic.

Finsler C^*-modules are right modules over a C^*-algebra A equipped with a map with values in A which has the properties of a norm. In the commutative case, Finsler C^*-modules are a natural generalization of Hilbert C^*-modules. Indeed, let X be a Hausdorff compact space. If $B = \bigcup_{t\in X} H_t$ is a bundle of Hilbert spaces over X satisfying appropriate continuity properties, then the set E of continuous sections (that is, continuous maps $f : X \to B$ such that $f(t) \in H_t$ for $t \in X$) has a natural structure of Hilbert C^*-module over $C(X)$, the C^*-algebra of all complex continuous functions on X, with the inner product given by $(f, g) \in E \times E \to \langle f, g\rangle \in C(X)$ with $\langle f, g\rangle (t) = \langle f (t), g (t)\rangle_{H_t}$. Moreover, any Hilbert C^*-module over a unital commutative C^*-algebra is of this form. If $B = \bigcup_{t\in X} B_t$ is a bundle of Banach spaces over X, then E is a Finsler C^*-module with the map $\rho_{C(X)} : E \to C(X)$ given by $\rho_{C(X)}(f)(t) = \|f(t)\|_{B_t}$, and any Finsler C^*-module over a commutative unital C^*-algebra is isomorphic to one of this form.

Therefore, if the Hilbert C^*-modules serve as basis for "noncommutative Hilbert bundles", the Finsler C^*-modules serves as basis for "noncommutative Banach bundles".

The notion of locally C^*-algebra is a generalization of the notion of C^*-algebra. A locally C^*-algebra is a complete Hausdorff complex topological $*$-algebra A whose topology is determined by an upward filtered family $\{p_\lambda\}_{\lambda\in\Lambda}$ of C^*-seminorms. If X is a direct limit of a countable family of Hausdorff compact spaces $\{X_n\}_n$, then $C(X)$ is a unital locally C^*-algebra with respect to the family of C^*-seminorms $\{p_n\}_n$, where $p_n(f) = \sup\{|f(x)|; x \in X_n\}$. Moreover, any unital Frechét locally C^*-algebra is of this form.

An element $a \in A$ is bounded if $\sup\{p_\lambda(a); \lambda \in \Lambda\} < \infty$. Then $b(A) := \{a \in A; \sup\{p_\lambda(a); \lambda \in \Lambda\} < \infty\}$ is a C^*-algebra with respect to the C^*-norm $\|\cdot\|_\infty = \sup\{p_\lambda(\cdot); \lambda \in \Lambda\}$. Moreover, $b(A)$ is dense in A.

Let $(\Upsilon, \leq)$ be a directed poset. A quantized domain in a Hilbert space $\mathcal{H}$ is a triple $\{\mathcal{H}; \mathcal{E}; \mathcal{D}_\mathcal{E}\}$, where $\mathcal{E} = \{\mathcal{H}_\iota; \iota \in \Upsilon\}$ is an upward filtered family of closed subspaces with dense union $\mathcal{D}_\mathcal{E} = \bigcup_{\iota\in\Upsilon} \mathcal{H}_\iota$ in $\mathcal{H}$. Put

$C^*(\mathcal{D}_\mathcal{E}) := \{T \in L(\mathcal{D}_\mathcal{E}); T(\mathcal{H}_\iota) \subseteq \mathcal{H}_\iota, T(\mathcal{H}_\iota^\perp \cap \mathcal{D}_\mathcal{E}) \subseteq \mathcal{H}_\iota^\perp \cap \mathcal{D}_\mathcal{E}, T|_{\mathcal{H}_\iota} \in B(\mathcal{H}_\iota), \forall \iota \in \Upsilon\}$.

If $T \in C^*(\mathcal{D}_\mathcal{E})$, then there exists $T^\bigstar$:Dom$(T^\bigstar) \to \mathcal{H}$, Dom$(T^\bigstar) \supset \mathcal{D}_\mathcal{E}$. Let $T^* := T^\bigstar|_{\mathcal{D}_\mathcal{F}} \in C^*(\mathcal{D}_\mathcal{E})$. Then, $C^*(\mathcal{D}_\mathcal{E})$ has a natural structure of a unital $*$-algebra. For each $\iota \in \Upsilon$, the map $p_\iota^{C^*(\mathcal{D}_\mathcal{E})} : C^*(\mathcal{D}_\mathcal{E}) \to [0, \infty)$ defined by $p_\iota^{C^*(\mathcal{D}_\mathcal{E})}(T) = \left\| T|_{\mathcal{H}_\iota} \right\|$ is a C^*-seminorm, and $C^*(\mathcal{D}_\mathcal{E})$ is a locally C^*-algebra with respect to the family of C^*-seminorms $\{p_\iota^{C^*(\mathcal{D}_\mathcal{E})}\}_{\iota\in\Upsilon}$.

A local $*$-representation of A on the the quantized domain $\{\mathcal{H}, \mathcal{E} = \{\mathcal{H}_\iota\}_{\iota\in\Upsilon}, \mathcal{D}_\mathcal{E}\}$ is a $*$-morphism $\varphi : A \to C^*(\mathcal{D}_\mathcal{E})$ with the property that for each $\iota \in \Upsilon$, there exists $\lambda \in \Lambda$ such that $p_\iota^{C^*(\mathcal{D}_\mathcal{E})}(\varphi(a)) \leq p_\lambda(a)$ for all $a \in A$. For every locally C^*-algebra A there is a quantized domain $\{\mathcal{H}, \mathcal{E}, \mathcal{D}_\mathcal{E}\}$ and a local isometric $*$-homomorphism $\pi : A \to C^*(\mathcal{D}_\mathcal{E})$. This result can be regarded as an unbounded analog of the Gelfand-Naimark theorem. So, the concrete models for locally C^*-algebras are $*$-subalgebras of unbounded linear operators on a Hilbert space, which satisfy some properties.

A Finsler locally C^*-module over A is a right A-module equipped with a map $\rho_A : E \to A_+$ satisfying the following conditions:

(1) For each $\lambda \in \Lambda$, the map $p_\lambda^E : E \to \mathbb{R}_+$ given by $p_\lambda^E(x) = p_\lambda(\rho_A(x))$ is a seminorm on E;
(2) E is complete with respect to the topology defined by the family of seminorms $\{p_\lambda^E\}_{\lambda\in\Lambda}$;
(3) $\rho_A(xa)^2 = a^* \rho_A(x)^2 a$ for all $x \in E$ and for all $a \in A$.

Let $\{\mathcal{H}, \mathcal{E} = \{\mathcal{H}_\iota\}_{\iota\in\Upsilon}, \mathcal{D}_\mathcal{E}\}$ and $\{\mathcal{K}, \mathcal{F} = \{\mathcal{K}_\iota\}_{\iota\in\Upsilon}, \mathcal{D}_\mathcal{F}\}$ be two quantized domains. Put

$C^*(\mathcal{D}_\mathcal{E}, \mathcal{D}_\mathcal{F}) := \{T \in L(\mathcal{D}_\mathcal{E}, \mathcal{D}_\mathcal{F}); T(\mathcal{H}_\iota) \subseteq \mathcal{K}_\iota, T(\mathcal{H}_\iota^\perp \cap \mathcal{D}_\mathcal{E}) \subseteq \mathcal{K}_\iota^\perp \cap \mathcal{D}_\mathcal{F}, T|_{\mathcal{H}_\iota} \in B(\mathcal{H}_\iota, \mathcal{K}_\iota), \forall \iota \in \Upsilon\}$.

If $T \in C^*(\mathcal{D}_\mathcal{E}, \mathcal{D}_\mathcal{F})$, then there exists $T^\bigstar$:Dom$(T^\bigstar) \to \mathcal{H}$, Dom$(T^\bigstar) \supset \mathcal{D}_\mathcal{F}$ and $T^* := T^\bigstar\big|_{\mathcal{D}_\mathcal{F}} \in C^*(\mathcal{D}_\mathcal{F}, \mathcal{D}_\mathcal{E})$. Moreover, $C^*(\mathcal{D}_\mathcal{E}, \mathcal{D}_\mathcal{F})$ has a natural structure of Hilbert locally C^*-module over $C^*(\mathcal{D}_\mathcal{E})$ with the action of $C^*(\mathcal{D}_\mathcal{E})$ on $C^*(\mathcal{D}_\mathcal{E}, \mathcal{D}_\mathcal{F})$ given by $(T, S) \in C^*(\mathcal{D}_\mathcal{E}, \mathcal{D}_\mathcal{F}) \times C^*(\mathcal{D}_\mathcal{E}) \to TS \in C^*(\mathcal{D}_\mathcal{E}, \mathcal{D}_\mathcal{F})$ and the inner product given by $(T_1, T_2) \in C^*(\mathcal{D}_\mathcal{E}, \mathcal{D}_\mathcal{F}) \times C^*(\mathcal{D}_\mathcal{E}, \mathcal{D}_\mathcal{F}) \to \langle T_1, T_2 \rangle = T_1^* T_2 \in C^*(\mathcal{D}_\mathcal{E})$. Therefore, $C^*(\mathcal{D}_\mathcal{E}, \mathcal{D}_\mathcal{F})$ is a Finsler locally C^*-module over $C^*(\mathcal{D}_\mathcal{E})$ with the map $\rho_{C^*(\mathcal{D}_\mathcal{E})} : C^*(\mathcal{D}_\mathcal{E}, \mathcal{D}_\mathcal{F}) \to C^*(\mathcal{D}_\mathcal{E})$ defined by $\rho_{C^*(\mathcal{D}_\mathcal{E})}(T) = (T^*T)^{\frac{1}{2}}$.

The notion of Finsler locally C^*-modules was introduced by A. Khosravi and M. Azhini (2004). In a series of papers, they and others investigated some properties of Finsler locally C^*-modules. In this paper, we present other properties of Finsler locally C^*-modules. We show that the bounded part $b(E)$ of a Finsler locally C^*-module E over A has a canonical structure of Finsler C^*-module over $b(A)$. Moreover, $b(E)$ is dense in E.

We introduce the notion of local quasi-representations of a Finsler locally C^*-module on Hilbert spaces. A local quasi-representation of a Finsler locally C^*-module over A, (E, ρ_A), on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ with the quantized domains $\mathcal{D}_\mathcal{E} = \bigcup_{\iota \in \Upsilon} \mathcal{H}_\iota$, respectively $\mathcal{D}_\mathcal{F} = \bigcup_{\iota \in \Upsilon} \mathcal{K}_\iota$, is a map $\Phi : E \to C^*(\mathcal{D}_\mathcal{E}, \mathcal{D}_\mathcal{F})$, with the property that there exists a local $*$-representation φ of A on $\mathcal{H}$ such that $(\Phi(x)^* \Phi(x))^{\frac{1}{2}} = \varphi(\rho_A(x))$ for all $x \in E$. We show that every Finsler locally C^*-module has a local quasi-representation.

Also, we show that a Finsler locally C^*-module can be realized as an inverse limit of Finsler C^*-modules. As an application of this result, we show that, under some conditions, a derivation of a locally C^*-algebra in a Finsler locally C^*-module is approximately inner. In addition, we show that the derivations of the Frechét locally C^*-algebra $C^*(\mathcal{D}_\mathbb{E})$, where $\mathcal{D}_\mathbb{E} = \bigcup_n \mathbb{C}^{k(n)}$ and $(k(n))_n$ is an increasing sequence of positive integers, in Finsler locally C^*-modules are approximately inner.

Another objective of the project (from the coming years) aims to develop a theory of deformations for (co)algebras in monoidal categories and to construct such objects that verify, moreover, a certain universal property. The final aim is to obtain new classes of braided Hopf tensorial algebras and to describe the quotients of them. In [6], we initiated the investigation of deformation of algebras in a monoidal category $(\mathcal{M}, \otimes, 1)$, extending a part of the work from the article *On the Deformation of Rings and Algebras* (Annals of Mathematics 79(1064), 59-103), by M. Gerstenhaber. In our approach, for a K-coalgebra C we construct a new monoidal category $(\mathcal{M}_C, \otimes, 1)$ in such a way that $\mathcal{M}$ and $\mathcal{M}_C$ have the same objects, but the hom-sets in $\mathcal{M}_C$ depend on C as well. Moreover, every morphism of coalgebras $\sigma : C \to D$ induces a monoidal functor $\sigma^* : \mathcal{M}_D \to \mathcal{M}_C$. Since the categories that we work with are monoidal, we can talk about algebras (aka monoids) in $\mathcal{M}_C$ and $\mathcal{M}_D$. The functor σ^* maps an algebra in $\mathcal{M}_D$ to an algebra in $\mathcal{M}_C$, inducing another functor $\sigma^* : \mathrm{Alg}(\mathcal{M}_D) \to \mathrm{Alg}(\mathcal{M}_C)$ between the categories of algebras in $\mathcal{M}_D$ and $\mathcal{M}_C$, respectively. Let us remark that the setting from Gerstenhaber's work can be recovered taking σ to be the inclusion of $C = K$ into the coalgebra $D = K[q]$ of polynomials in one variable q. Of course, in this particular case, $\mathcal{M}$ is the category of K-linear spaces.

Given σ as above, a σ-deformation of $A \in \mathrm{Alg}(\mathcal{M}_C)$ is, by definition, an algebra in the fiber of σ^* over A. Two deformations can be identified via a canonical equivalence relation. In one of our main results, one shows that there is a one-to-one correspondence between the set of equivalence classes of σ-deformations of A and the set of coalgebra maps from D to a certain coalgebra $\mathrm{Def}^\sigma(A)$, which satisfies a suitable universal property. As application, one proves that certain types of deformations (e.g. infinitesimal ones) corresponds to some distinguished elements in $\mathrm{Def}^\sigma(A)$, such as group-like or skew-primitive elements. We also related σ-deformations with a Hochschild-type cohomology, and discussed several examples.

II. Summary of progress

- The objectives assumed within the 3 activities of the stage 2022 of the project were achieved in percentage of 100%, 95% and 95%, respectively. Until the end of this stage (31.12.2022), the second and the third activity will be fulfilled in percentage of 100%, so all the objectives of the stage 2022 will be realized in percentage of 100% . Moreover, two other objectives corresponding to the forthcoming stages of the grant were initiated.

- 3 articles were submitted for publication, all to journals from Q2.

- 3 other articles are in an advanced stage of preparation, and will be submitted for publication in the near future.

- The results we have obtained were presented at 5 international conferences, all of them being well appreciated.

- We developed the local network and improved the equipment by purchasing routers and high-performance laptops.

- We have ensured a good environment for the realization and the dissemination of our scientific papers.

- The budget allocated to this stage was spent in full.

Date, | Project leader,

November 2022 | Prof. dr. Daniel Bulacu