Project: CNCSIS PN-III-P4-PCE-2021-0282, contract number 47/2022
"Quasi Quantum Groups and Monoidal Categories"
Project leader: D. Bulacu
SCIENTIFIC PROJECT, NOVEMBER 2023

Among the articles reported in 2022 as being sent for publication, during 2023 the following were accepted (the first two have appeared in journals from the Q2 area, the third was published in a journal from the Q3 area):

[1] S. Dăscălescu, C. Năstăsescu, L. Năstăsescu: Graded (quasi-) Frobenius rings, J. Algebra 620 (2023), 392-424.

[2] L. Liu, A. Makhlouf, C. Menini, F. Panaite, BiHom-NS-algebras, twisted Rota-Baxter operators and generalized Nijenhuis operators, Results in Mathematics 78 (2023), article number: 251.

[3] M. Joiţa, Finsler locally C^*-modules, Bull. Malays. Math. Sci. Soc. 46 (2023), 86.

Also, we are still waiting for the report for the article

[4] D. Bulacu, B. Torrecillas, 1-Homology for coalgebras in Yetter-Drinfeld categories,
elaborated and sent for publication in 2022 (at the last exchange of messages we had with the editor on September 25 this year, he assured us that "the paper is with a referee; a report is expected in two/three months").

The article

[5] D. Bulacu, D. Popescu, B. Torrecillas, Double wreath quasi-Hopf algebras,
declared in 2022 as being in an advanced stage of development, was sent for publication in 2023. Compared to the version reported in 2022, we added new examples, found characterizations for the antipode of the pre-biallgebra with 1-cycle and studied its "deformed version" defined by the initial 2-cocycle.

The research on the topics proposed in the project for the year 2023 has materialized in the following 7 papers, of which 3 are sent for publication in journals with a high impact factor (from the Q2 area) and 4 are in an advanced stage of completion.

[6] A. Makhlouf, D. Ştefan, Deformations of algebraic structures in monoidal categories, advanced development stage.

[7] D. Bulacu, D. Popescu, B. Torrecillas, Double biproduct quasi-quantum groups, submitted for publication.

[8] D. Bulacu, B. Torrecillas, The quasi-Hopf analog of the Drinfeld-Jimbo quantum groups, advanced development stage.

[9] S. Dăscălescu, C. Năstăsescu, L. Năstšescu, Picard groups of quasi-Frobenius algebras and a question on Frobenius strongly graded algebras, submitted for publication.

[10] D. Bulacu, C. Menini, M. Misuratti, Biproduct quasi-quantum groups of rank 2, advanced development stage.

[11] D. Bulacu, C. Menini, M. Misuratti, Quasi-quantum groups obtained from Nichols algebras of diagonal type, work in progress.

[12] M. Joiţa, I. Simon, Injective envelopes for locally C^*-algebras, advanced development stage.

The scientific description of the results in the 2023 stage and the degree of achievement of the specific scientific objectives

Each • refers to results connected to one of the three scientific activities considered in the stage 2023 of the project.

• Regarding the article [6], it was declared in 2022 as being in the finalization stage; I mentioned that it aims an objective of the project from the current year **(see the objectives of activity 2.2 from the year 2023)**. We realized at the end of the last last year that many of the results obtained in [6] at that time can be considerably improved. This fact caused his submission for publication to be postponed for the end of this year. The changes we made are major, so we will include here the results obtained in its new version.

In this article a new interpretation is given to the concept of deformation of an associative $\mathbb{K}$-algebra A with multiplication m, using coalgebra theory and category theory as the main tools of study. By definition, a deformation of A is a $\mathbb{K}[[t]]$-associative algebra defined on the vector space $A[[t]]$, with multiplication $m_t : A[[t]] \times A[[t]] \to A[[t]]$, which is a bilinear map of the form $m_t := \sum_{n=0}^{\infty} m_i t^i$, where $m_i : A \times A \to A$ is a K-bilinear map, and $m_0 = m$.

Our approach starts with the simple observation that formal series m_t as above can be identified with linear functions $f : \mathbb{K}[t] \to \mathrm{Hom}_{\mathbb{K}}(A \otimes A, A)$, associating to f the formal series $\sum_{n=0}^{\infty} f(t^n)t^n$. Another fundamental observation is that the theory of deformations in the classical case makes sense for algebras in an arbitrary monoidal $\mathbb{K}$-linear category $\mathcal{M}$. The above-mentioned identification suggests constructing a new $\mathbb{K}$-linear monoidal category $\mathcal{M}_t$, which has the same objects as $\mathcal{M}$, but morphisms from X to Y in are linear maps $f : \mathbb{K}[t] \to \mathrm{Hom}_{\mathcal{M}}(X, Y)$. Using the category $\mathcal{M}_t$, a deformation $(A[[t]], m_t)$ can be viewed as an associative algebra in $\mathcal{M}_t$ (and vice versa). The main objective is to show that a theory of deformations can be developed where the coalgebra $K[t]$ is replaced with a cocommutative coalgebra.

For greater flexibility, a class $\mathcal{T}$ of coalgebras is introduced. The coalgebras in $\mathcal{T}$ will be considered as being of a certain type (for example, $\mathcal{T}$ could be the class of cocommutative or pointed coalgebras). It is shown that in $\mathcal{T}$ there always exists a cofree coalgebra $F_{\mathcal{T}}(V)$ over any vector space V. In turn, cofree coalgebras in $\mathcal{T}$ will be the main tool for studying deformations of an associative algebra, as well as other related concepts, such as ι-factorizations of a linear transformation f from a coalgebra D of type $\mathcal{T}$ to a given vector space V. By definition, a ι-factorization of f is a linear transformation $g : C \to V$ such that $g \circ \iota = f$. In a certain sense, these can be viewed as the underlying structure of a deformation, obtained by forgetting the multiplication map. One of the main results in this part of the article shows that ι-factorizations of f are in bijective correspondence with the set of coalgebra morphisms $\widehat{g}$ from C to $F_{\mathcal{T}}(V)$ with a certain property (see Theorem 2.3). As connected coalgebras of type $\mathcal{T}$ will be important for applications, a similar result is demonstrated by constructing a kind of connected cofree coalgebra of type $\mathcal{T}$ (see Theorem 2.8).

Next, a new monoidal category $\mathcal{M}_C$ is defined, which generalizes $\mathcal{M}_t$, and some basic properties of it are presented (including a method for producing invertible morphisms, inspired by a well-known result of Takeuchi). Furthermore, any coalgebra morphism $\iota : D \to C$ induces a functor ι^* from the category of algebras in $\mathcal{M}_C$ to the category of algebras in $\mathcal{M}_D$. The notion of ι-deformation of an algebra (A, m) in the category $\mathcal{M}_D$ is now defined as an algebra in the fiber of the functor ι^* over (A, m). For example, ordinary deformations are recovered by taking the morphism ι to be the inclusion of the coalgebra K into $K[t]$. Based on results regarding factorizations, it is shown that deformations of an algebra $m_D : D \to \mathrm{Hom}_K(A^{\otimes 2}, A)$ are governed by a coalgebra $\mathrm{Def}_{\mathcal{T}}(m_D, \iota)$ of type $\mathcal{T}$ satisfying a certain universality property (see Theorem 4.8). A similar result is demonstrated for connected coalgebras (Theorem 4.9).

The central part of the article is the fifth section, where a connection is made with a certain cohomology theory $H^*(m_D, \iota)$. Assuming that C is an extension of the coalgebra D, a 3-cocycle ξ (obstruction to deformation) is first associated with an associative multiplication m_D. It is then shown that there exists a deformation of m_D if and only if ξ is a 3-cocoboundary. The fundamental result is Theorem 5.3, which shows that the set of equivalence classes of deformations is in bijective correspondence with $H^2(m_D, \iota)$.

The mentioned results are already present in a preprint. In addition to these, several specific cases have been studied, some of which are already known in the specialized literature, while others are new. Here, we mention infinitesimal deformations, corresponding to the case when $\iota : C_0 \to C_1$ is the inclusion given by the filtration of the coradical $\{C_n\}_{n \in \mathbb{N}}$ of C. These are classified by the degree 1 component of the coradical filtration on $\mathrm{Def}_{\mathcal{T}}(m_{C_0}, \iota)$. In the connected case, Taft-Wilson's Theorem allows us to identify infinitesimal deformations with primitive elements in $\mathrm{Def}_{\mathcal{T}}(m_{C_0}, \iota)$. More specifically, if $C = K[t]$, several results from the theory of deformations of associative algebras, initiated by M. Gerstenhaber, are recovered. Moreover, by choosing the category $\mathcal{M}$ appropriately, various well-known types of deformations are rediscovered (e.g., for (co)actions of Hopf algebras or diagrams of (co)algebras). The case $C = K[t_1, \ldots, t_n]$ is also analyzed.

These results are currently being written up and will be included in the mentioned preprint. We estimate that the final version will available and submitted for publication in two or three weeks.

One of the initial motivations for this research activity was to find new examples of quantum groups (Hopf algebras or related structures) using deformations. What we have achieved so far can be viewed as an intermediate step.

• The results obtained in [7, 8] are in connection with the objectives of **activity 2.1** of the plan of realization of the project in 2023. For short, these articles make the objectives assumed to activity 2.1 should be completed in percentage of 100%. Below we present the scientific content of the papers [7, 8].

For an algebra C (resp. B) in the category of left (resp. right) representations over H one can consider the k-algebra $C\#H$ (resp. $H\#B$), the smash product of C, H (resp. B, H). In the same spirit, with the help of C, B we can form another k-algebra, denoted by $C\#H\#B$ and named the two-sided smash product of C, H, B since it contains both $C\#H$ and $B\#H$ as associative k-algebras. We have that $C \times H \times B = C\#H\#B$ as k-algebras, provided that C, B are braided Hopf algebras as in the above. A parallel result we have in the coalgebra case: to a coalgebra $C \in {}_H^H\mathcal{Y}D$ (resp. $B \in \mathcal{Y}D_H^H$) we can associate a monoidal coalgebra within the monoidal category of H-bimodules, ${}_H\mathcal{M}_H$; it is denoted by $C \rtimes H$ (resp. $H \ltimes B$) and called the smash product coalgebra of C, H (resp. B, H). In the same context, one can build a new coalgebra in ${}_H\mathcal{M}_H$, $C \rtimes H \ltimes B$, the so-called two-sided smash product coalgebra of C, H, B, a monoidal coalgebra that contains $C \rtimes H$ and $H \ltimes B$ as H-bimodule coalgebras. When C, B are braided Hopf algebras, $C \times H \times B = C \rtimes H \ltimes B$ as coalgebras in ${}_H\mathcal{M}_H$. Note that the (quasi-co)algebra structure of $C \times H \times B$ looks complicate and impossible to find by brute computation; our categorical way provides the natural way to get it for free, as no computation is required. By using again some categorical results, we show that, moreover, $C \times H \times B$ identifies to a left of right biproduct quasi-Hopf algebra. More explicitly, we construct a braided functor $\mathcal{K} : \mathcal{Y}D_H^H \to {}_H^H\mathcal{Y}D$ who allows to see B as a (co)algebra, bialgebra etc. in ${}_H^H\mathcal{Y}D$, too; $\overline{B}$ stands for B endowed with this new braided structure(s) in ${}_H^H\mathcal{Y}D$. It is remarkable that the braided monoidal equivalences ${}_H^H\mathcal{Y}D \sim {}_H^H\mathcal{M}_H^H \sim \mathcal{Y}D_H^H$ provided by the Structure Theorems in ${}_H^H\mathcal{M}_H^H$ lead to natural identifications between left and right smash product (co)algebras, which in the end entail to a natural identification between double biproduct quasi-quantum groups and (left and/or right) biproduct quasi-quantum groups, as desired. We should stress the fact that the "natural condition" mentioned above resides from the necessary and sufficient condition for the tensor product algebra and coalgebra structures associated to two braided bialgebras (in our case C and $\overline{B}$) to afford a (tensor) braided bialgebra structure. When this "natural condition" holds, we have a new braided Hopf algebra $C\widetilde{\otimes}\overline{B}$ in ${}_H^H\mathcal{Y}D$ and $C \times H \times B \equiv (C\widetilde{\otimes}\overline{B}) \times H$ as quasi-Hopf algebras, through a non-trivial isomorphism χ produced, again, by the Structure Theorems in ${}_H^H\mathcal{M}_H^H$.

Owing to the identification $C \times H \times B \equiv (C\widetilde{\otimes}\overline{B}) \times H$, one can characterize the 2-cocycles on $C \times H \times B$. We shown in [5] that the 2-cocycles ϑ on a biproduct quasi-quantum group $A \times H$ are determined by the almost 2-cocycles $\overline{\vartheta}$ on the braided Hopf algebra A in ${}_H^H\mathcal{Y}D$; we used the term almost since $\overline{\vartheta}$ obeys all the required conditions for a braided 2-cocycle with one exception: $\overline{\vartheta}$ is not H-colinear, and so is not a morphism in ${}_H^H\mathcal{Y}D$. The lack of H-colinearity for ϑ is not a disadvantage for us, although we loose somehow the categorical path. On the contrary, if we assume that ϑ is H-colinear, Theorem 5.2 and its Corollary 5.3 say that the deformation of $A \times H$ by ϑ is, up to identification, the biproduct between the braided deformation of A by $\overline{\vartheta}$ and H, an inconvenience for us since the Drinfeld-Jimbo quantum groups are Hopf algebras with weak projections and not biproducts. Nevertheless, particular examples of almost 2-cocycles on $A = C\widetilde{\otimes}\overline{B}$ are provided by the almost dual skew pairings between C and $\overline{B}$ in ${}_H^H\mathcal{Y}D$ (almost since they are not H-colinear, as expected). By using the isomorphism χ, the latter are given by some H-balanced morphisms from $B \otimes A$ to k, leading thus to the Majid's definition of the double bosonization process but now in the quasi-Hopf setting.

The paper [7] ends with concrete examples of double biproduct quasi-quantum groups which produce, owing to the bosonization process, new examples of quasi-quantum groups. More exactly, in [10] are described the braided Hopf algebras of rank 2 within a category of Yetter-Drinfeld modules over a given quasi-Hopf algebra H; we use them to construct examples of double biproduct quasi-quantum groups. Next, we compute the almost dual skew pairings for two such braided Hopf algebras of rank 2 and the resulting 2-cocycles. Last but not least, we apply the bosonization process presented above. By specializing H, we get a wealth of examples of new quasi-Hopf algebras.

More details about the content of the paper [7] can be found in the pdf of it that we have attached. We also want to mention that [7] was sent for publication recently.

• For continuity, we will now highlight the results that make the objectives of the **activity 2.1** from the implementation plan be made in percentage of 100% . These are the subject of the work [8], which is in an advanced stage of development. Not being finished and because it is neither sent for publication nor presented at a conference, we only want to briefly present the results obtained; a pdf file of it will be made public when the work is completed (we hope that this will happen somewhere towards the end of the current year).

In the classical case, the construction of Drinfeld-Jimbo quantum groups uses, on the one hand, a kind of free braided Hopf algebra in a category of Yetter-Drinfeld modules and, on the other hand, their factorizations through so-called Serre relations associated with a Cartan data.

In the quasi-Hopf case, both problems are difficult to address because in this case the Yetter-Drinfeld module structure is much more complicated: it is a module and has a coaction that is not coassociative, but compatible with the action. However, we noticed in a previous article that in the Hopf case coactions are defined by bilinear forms which offers co-quasi-triangular symmetric structures (CQT for short) on H. For a qQG H the notion of CQT does not make sense because it is equivalent to the fact that the corepresentations over H form a braided category and in the quasi-Hopf case this category does not exist. For this reason we start with an R-matrix on H (the dual case) which gives the coaction of the free Hopf algebra in the category of Yetter-Drinfeld modules and we considered its action defined by a family of characters indexed by the set that gives the alphabet. The set of characters must satisfy a set of conditions (denoted by $\mathcal{A}$), imposed by the fact that the multiplication is associative modulo the reassociator of H. The classic case, that of Hopf algebras, is recovered by considering the left Yetter-Drinfeld module category over the dual of H (which thus becomes CQT). Further, we completed the free algebra structure in ${}^H_H\mathcal{YD}$ up to a Hopf algebra. In this sense, we have defined the comultiplication and the antipode on each letter and shown that these definitions behave well with the conditions in the set $\mathcal{A}$.

The second step, an extremely difficult one, was to determine the Serre conditions for our context. The difficulty consisted in the fact that these conditions must be compatible with those in $\mathcal{A}$, a fact that does not appear in the classic case. After several attempts we found them, first for the left handed version and after for the right handed version. We can say that these involve not only the letters of the alphabet and the fixed character family, but also the components of the reassociator of H. In addition, the two constructions (on the left and on the right) verify the "natural condition" mentioned above; also, the ideal generated by our Serre-type relations is a Hopf braided ideal and, thus, we have two braided Hopf algebras in categories of Yetter-Drinfeld modules (left and right respectively), so we applied the double bosonization construction to them.

It should be noted that all the above results were obtained for an arbitrary qQG QT (H, R). In the classical case this is algebra of functions of the free group $\mathbb{Z}[I]$, where $(I, \cdot)$ is a given Cartan data, viewed as the dual of a group Hopf algebra CQT with the bilinear symmetric form $\mathcal{R}$ defined by a non-zero scalar q in the base field: $\mathcal{R}(K_\mu \otimes K_\nu) = q^{\mu\cdot\nu}$, for all $\mu, \nu \in \mathbb{Z}[I]$. Continuing this game of duality, in the quasi-Hopf case, in place of (H, R) we considered the tensor product between $k^{\mathbb{Z}[I]}$ and an arbitrary qQG QT, obtaining an object more general than in the case of Drinfeld-Jimbo quantum groups. The tensor product is required to leave the Hopf zone: there is no guarantee that, for any I, we can find a non-trivial abelian 3-cocycle for $k[\mathbb{Z}[I]]$ that would produce a qQG CQT structure on it.

This generalization allowed us to demonstrate that many of the existing constructions in the literature are particular cases of our construction. We will continue to work in this direction next year as well, **activity 3.1** from 2024 being strictly related to this. At the time of this report, we are working on drafting the results obtained and improving them, where possible.

• The paper [9] from the list above contains results that complement those obtained within **activities 1.2** from the year 2022 but at the same time, it provides important information and examples regarding this year's objectives of **activities 2.2**, since graded algebras produce examples of Yetter-Drinfeld modules over the group algebra and, thus, are related to the "diagonal case" in the theory of Nichols algebras.

Our initial aim was to answer the question: does the Frobenius (symmetric) property transfers from a strongly graded algebra to its homogeneous component of trivial degree? Related to it, we investigate invertible bimodules and the Picard group of a finite dimensional quasi-Frobenius algebra R. In the special case of Hopf algebras, we prove the following.

Theorem A. *Let H be a finite dimensional Hopf algebra with antipode S. Then the order of $[H^*]$ in $\mathrm{Pic}(H)$ is the least common multiple of the order of the class of S^2 in $\mathrm{Out}(H)$ and the order of the modular element of H^* in the group of grouplike elements of H^*.*

We compute the Picard group, the automorphism group and the group of outer automorphisms of a 9-dimensional quasi-Frobenius algebra $\mathcal{R}$ which is not Frobenius, constructed by Nakayama.

Theorem B. *There is an isomorphism of $\mathcal{R}$-bimodules $\varphi : \mathcal{R}^* \otimes_{\mathcal{R}} \mathcal{R}^* \to \mathcal{R}$, thus $[\mathcal{R}^*]$ has order 2 in $\mathrm{Pic}(\mathcal{R})$. An invertible $\mathcal{R}$-bimodule is isomorphic either to ${}_1\mathcal{R}_\alpha$ or to ${}_1{\mathcal{R}^*}_\alpha$ for some $\alpha \in \mathrm{Aut}(\mathcal{R})$, and $\mathrm{Pic}(\mathcal{R}) \simeq \mathrm{Out}(\mathcal{R}) \times C_2$.*

Theorem C. *$\mathrm{Aut}(\mathcal{R})$ is isomorphic to a semidirect product $(K^2 \times M_{2,1}(K)) \rtimes (K^* \times GL_2(K))$, and $\mathrm{Out}(\mathcal{R}) \simeq K^*$.*

We consider an arbitrary finite dimensional algebra R and a morphism of R-bimodules $\psi : R^* \otimes_R R^* \to R$ which is associative, i.e., $\psi(r^* \otimes_R s^*) \leftharpoonup t^* = r^* \rightharpoonup \psi(s^* \otimes_R t^*)$ for any $r^*, s^*, t^* \in R^*$; here $\rightharpoonup$ and $\leftharpoonup$ denote the usual left and right actions of R on R^*. Then we can form the semitrivial extension $R \rtimes_\psi R^*$, which is the cartesian product $R \times R^*$ with the usual addition, and multiplication defined by

$$(r, r^*)(s, s^*) = (rs + \psi(r^* \otimes_R s^*), (r \rightharpoonup s^*) + (r^* \leftharpoonup s))$$

for any $r, s \in R, r^*, s^* \in R^*$. It has a structure of a C_2-graded algebra with R as the homogeneous component of trivial degree. We prove:

Proposition A. *$R \rtimes_\psi R^*$ is a symmetric algebra.*

If ψ is an isomorphism, which implies that R^* is invertible (therefore R must be quasi-Frobenius), then $R \rtimes_\psi R^*$ is a strongly C_2-graded algebra. Next we show that the isomorphism $\varphi : \mathcal{R}^* \otimes_\mathcal{R} \mathcal{R}^* \to \mathcal{R}$ constructed in Theorem B is associative, and we conclude that the strongly C_2-graded algebra $\mathcal{R} \rtimes_\varphi \mathcal{R}^*$ is symmetric, thus also Frobenius, while its component of trivial degree is not Frobenius. This answers in the negative our initial question, for both Frobenius and symmetric properties.

We investigate associativity of isomorphisms $R^* \otimes_R R^* \simeq R$ for quasi-Frobenius algebras R.

This has been sent for publication this year to a journal from the Q2 area.

• The papers [10] and [11], together with paper [6], make the objectives of **activity 2.2** from the year 2023 to be achieved in percent of 90% (in a short time they will be realized in percent of 100%, by completing the work in [10] and sending it for publication).

The paper [10] is a good source of examples of braided Hopf algebras in categories of Yetter-Dinfeld modules, many classes of examples obtained here have already been used to produce examples for the work done in [7, 8, 11]. It was initiated as part of last year's 1.2 activity This year it became a work that can be sent for publication. Furthermore, the results obtained here were presented by the third author at the "Hopf Days in Brussels" conference, September 4-6, 2023, https://sites.google.com/view/hopfdaysbrussels2023/home

For short, the content of [10] is as follows. We classified the 2-dimensional braided Hopf algebras B within a category of Yetter-Drinfeld modules ${}_H^H\mathcal{Y}D$, H a fixed qQG. We have obtained that B can have one of the following forms:

1. $k[S]$, where $S = \{1, s\}$ and $s^2 = s$ is group-like, with H-action and H-coaction trivial (not a Hopf algebra Hopf but a braided Hopf algebra);
2. $k[S]$, where $S = \{1, s\}$ and $s^2 = 1$, s a group-like, with H-action and H-coaction trivial (is both a Hopf algebra Hopf and a braided Hopf algebra);
3. of type $B_{\alpha,y}$, defined by a pair (α, y) consisting of a character α of H and $y \in H$ satisfying certain conditions (is both a Hopf algebra Hopf and a braided Hopf algebra).

Through the bosonization process, each type above corresponds to a quasi-Hopf algebra that we have completely determined in terms of generators and relations. Moreover, in order to find more such examples of qQGs, we considered H to be the group algebra of a finite cyclic group, viewed as a qQG via ω, a nontrivial 3-cocycle of the group. When ω is defined by a primitive root of unit, surprisingly perhaps, all braided Hopf algebras are trivial (the group algebra of a cyclic group with two elements). The non-trivial case is when ω is not defined by a primitive root of unit of k: there are two types in this case and, therefore, two classes of qQGs of rank 2.

The obtained results helped us to show that, in the quasi-Hopf case, we do not have an analogue of Sweedler's 4-dimensional algebra: the only 4-dimensional qQG of rank 2 is the tensor product $k[C_2] \otimes k[C_2]$, where C_2 is the cyclic group with two elements; the second one is endowed with the only non-trivial 3-cocycle.

The results mentioned above have been typed, they will be refined and new ones will be added. More specifically, we want to see if the new examples we got in dimension 8, together with the already known ones (due to Gelaki), allow us to classify the "basic" quasi-quantum groups of dimension 8. This last result, if we get it, will give "weight" to the article which, thus, can be send for publication to a journal in the Q2 or even Q1 area. For this reason we postpone its submission for publication, and we presented the central results in an international conference.

• Next, we present the results obtained so far in [11]. As we mentioned already, it is related to the objectives of **activitity 2.2** from the project implementation plan for the year 2023.

In [11] we introduced the notion of Nichols algebra in the case of quasi-quantum groups. Many of the techniques used in the classical case also work in the quasi-Hopf case. However, since the monoidal structure of ${}_H^H\mathcal{Y}D$ is not strict in the case of a qQG H, certain operators defined by the associativity constants and/or the baiding of the category ${}_H^H\mathcal{Y}D$ always intervene. This fact leads to difficult calculations, especially as they are already difficult in the classic case. With the help of the techniques offered by braided monoidal categories, we showed that for any $\mathbb{N}$-graded coalgebra $C = \bigoplus_{n\geq 0} C_n$ from ${}_H^H\mathcal{Y}D$ which is connected (its component of degree 0 is 1-dimensional) there is a largest graded braided coideal of it, denoted with I_C, such that $I_C \subseteq \bigoplus_{n\geq 2} C_n$. We can thus consider the factor coalgebra $\mathcal{B}(C) := C/I_C$.

The above results apply to the tensor algebra $T(V)$ associated with an object V of ${}_H^H\mathcal{Y}D$; for simplicity, in this case we denote $\mathcal{B}(T(V))$ with $\mathcal{B}(V)$ and we call it the Nichols algebra of V, a braided Hopf algebra in ${}_H^H\mathcal{Y}D$ because $T(V)$ is more than a braided $\mathbb{N}$-graded a connected coalgebra, it is a braided $\mathbb{N}$-graded Hopf algebra. Inspired by the classical case, we shown that the n-component of the ideal I_C coincides with the kernel of an operator Δ_{1^n} defined by the comultiplication Δ of C and its projection π_1 on the component C_1. In theory, this allows us to calculate I_C and implicitly $\mathcal{B}(C)$. In practice, we were able to do this for small dimensions for V, namely 1 and 2. In the second case we reobtained certain qQGs of dimension 8 introduced by Gelaki. For larger dimensions, the things get more complicated, that's why we started to study certain particular cases.

The particular case we are referring to is the "diagonal" one; is the one for which the quasi-quantum group is the group algebra of a finite abelian group G, viewed as a qQG via a nontrivial 3-cocycle of G. In the classical case, a Yetter-Drinfeld module is a G-graded space (the grading defines the coaction and vice versa) with the action on each component defined by a linear character of the group G. In the quasi-Hopf case, as we mentioned before, the definition of a Yetter-Drinfeld module is much more complicated. In addition, for a Yetter-Drinfeld G-module, we have coactions that are not defined by G-gradings (the decomposition remains but its uniqueness is lost). So the first question that came to us was, how can we introduce Yetter-Drinfeld G-modules in the quasi-Hopf case? We came to the conclusion that there are two ways to do it, which coincide in the classic case. The first possibility is to consider an abelian 3-cocycle on G: it defines the coaction and the action remains to be defined by the characters of G, in number equal to its cyclic components; we lose in this way the graduation but the characters are preserved. The second possibility is to start with a G-graded space and to define the coaction with the help of the grading; moreover, the 3-cocycle on G considered initially produces an action in which the grading plays again an important role (here the grading can be preserved but the characters are lost). Once this problem has been clarified, our next step is to calculate the Nichols algebra in both cases, and then the quasi-Hopf biproducts associated with them. This will be done as soon as possible. This will be done as soon as possible. Since the results obtained are in an insufficiently advanced phase, we want to don't share yet the file of the article containing the results mentioned above.

• We conclude this part of the report with the description of the scientific content of [12]. The results from the paper [12] are related to the objectives assumed for the next year **(activity 3.1 from the year 2024)**, we include them here because they have been obtained during this stage of the project.

Injectivity is a categorical concept. An object I in a category $\mathcal{C}$ is injective if for any two objects $E \subseteq F$ from $\mathcal{C}$,any morphism $\varphi : E \to I$ extends to a morphism $\widetilde{\varphi} : F \to I$. Cohen [*Injective envelopes of Banach spaces*, Bull. Amer.Math. Soc., 70(1964),723-726] considered the category whose objects are Banach spaces and whose morphisms are contractive linear maps. He introduced the notion of injective envelope for a Banach space and showed that each Banach space has a unique injective envelope.

Hamana [*Injective envelopes for C^*-algebras*,J. Math. Soc. Japan, 32(1979),1, 181-196] proved a C^*-algebraic version of these results. He considered the category whose objects are unital C^*-algebras and whose morphisms are unital completely positive linear maps. A unital C^*-algebra A is injective if for any unital C^*-algebra C and sel-adjoint subspace $\mathcal{S}$ of C containing the unity, any unital completely positive linear map $\varphi : \mathcal{S} \to A$ extends to a unital completely positive linear map $\widetilde{\varphi}: C \to A$. The C^*-algebra $B(\mathcal{H})$ of all bounded linear operator on a Hilbert space $\mathcal{H}$ is injective by the Arveson's extension theorem. An extension of A is a pair (B, Φ) of a unital C^*-algebra B and a $*$-monomorphism $\Phi: A \to B$. It is injective if B is injective. According to Gelfand-Naimark theorem, for any unital C^*-algebra A there exist a Hilbert space $\mathcal{H}$ and an isometric $*$-morphism $\Phi: A \to B(\mathcal{H})$. So, any unital C^*-algebra A has an injective extension. An injective envelope for A is an injective extension (B, Φ) with the property that id_B is the unique unital completely positive linear map which fixes the elements of $\Phi(A)$. He showed that any unital C^*-algebra A has a unique injective envelope in the sense that if (B_1, Φ_1) and (B_2, Φ_2) are two injective envelopes for A, there exists a unique $*$-isomorphism $\Upsilon: B_1 \to B_2$ such that $\Upsilon \circ \Phi_1 = \Phi_2$.

In this paper, we propose to extend the Hamana's results in the context of locally C^*-algebras.

A locally C^*-algebra is a complete Hausdorff complex topological $*$-algebra A whose topology is determined by an upward filtered family of C^*-seminorms $\{p_\lambda\}_{\lambda\in\Lambda}$. A Fréchet locally C^*-algebra is a locally C^*-algebra whose topology is determined by a countable family of C^*-seminorms. An element $a \in \mathcal{A}$ is called *local positive* if $a = b^*b + c$, where $b, c \in A$ such that $p_\lambda(c) = 0$ for some $\lambda \in \Lambda$. In this case, we call a as λ-positive and write $a \geq_\lambda 0$. We write $a =_\lambda 0$ whenever $p_\lambda(a) = 0$.

For each $n \in \mathbb{N}$, $M_n(A)$ denotes the collection of all matrices of size n with elements in A. Note that $M_n(A)$ is a locally C^*-algebra where the associated family of C^*-seminorms is denoted by $\{p_\lambda^n\}_{\lambda\in\Lambda}$.

Let B a locally C^*-algebra whose topology is defined by the family of C^*-seminorms $\{q_\delta\}_{\delta\in\Delta}$. For each $n \in \mathbb{N}$, the n-amplification of a linear map $\varphi: A \to B$ is the map $\varphi^{(n)}: M_n(A) \to M_n(B)$ defined by $\varphi^{(n)}\left([a_{ij}]_{i,j=1}^n\right) = [\varphi(a_{ij})]_{i,j=1}^n$.

A linear map $\varphi: A \to B$ is called:

(1) *local completely positive* if for each $\delta \in \Delta$, there exists $\lambda \in \Lambda$ such that $\varphi^{(n)}\left([a_{ij}]_{i,j=1}^n\right) \geq_\delta 0$ whenever $[a_{ij}]_{i,j=1}^n \geq_\lambda 0$ and $\varphi^{(n)}\left([a_{ij}]_{i,j=1}^n\right) =_\delta 0$ whenever $[a_{ij}]_{i,j=1}^n =_\lambda 0$, for all $n \in \mathbb{N}$.

(2) *admissible local completely positive if* $\Delta = \Lambda$, and for each $\lambda \in \Lambda$, $\varphi^{(n)}\left([a_{ij}]_{i,j=1}^n\right) \geq_\lambda 0$ whenever $[a_{ij}]_{i,j=1}^n \geq_\lambda 0$ and $\varphi^{(n)}\left([a_{ij}]_{i,j=1}^n\right) =_\lambda 0$ whenever $[a_{ij}]_{i,j=1}^n =_\lambda 0$, for all $n \in \mathbb{N}$.

(3) *local completely contractive* if for each $\delta \in \Delta$, there exists $\lambda \in \Lambda$ such that
$$q_\delta^n\left(\varphi^{(n)}\left([a_{ij}]_{i,j=1}^n\right)\right) \leq p_\lambda^n\left([a_{ij}]_{i,j=1}^n\right)$$
for all $[a_{ij}]_{i,j=1}^n \in M_n(A)$.

(4) *admissible local completely contractive if* $\Delta = \Lambda$, and for each $\lambda \in \Lambda$,
$$p_\lambda^n\left(\varphi^{(n)}\left([a_{ij}]_{i,j=1}^n\right)\right) \leq p_\lambda^n\left([a_{ij}]_{i,j=1}^n\right)$$

for all $[a_{ij}]_{i,j=1}^{n} \in M_n(A)$.

Let $(\Delta, \leq)$ be a directed poset. A *quantized domain* in a Hilbert space $\mathcal{H}$ is a triple $\{\mathcal{H};\mathcal{E};\mathcal{D}_{\mathcal{E}}\}$, where $\mathcal{E} = \{\mathcal{H}_\delta;\delta \in \Delta\}$ is an upward filtered family of closed subspaces with dense union $\mathcal{D}_{\mathcal{E}} = \bigcup\limits_{\delta\in\Delta} \mathcal{H}_\delta$ in $\mathcal{H}$. If Δ is countable, we say that $\mathcal{D}_{\mathcal{E}}$ is a Fréchet quantized domain in $\mathcal{H}$.

Let $C^*(\mathcal{D}_{\mathcal{E}}) := \{T \in \mathcal{L}(\mathcal{D}_{\mathcal{E}}); T(\mathcal{H}_\delta) \subseteq \mathcal{H}_\delta, T(\mathcal{H}_\delta^\perp \cap \mathcal{D}_{\mathcal{E}}) \subseteq \mathcal{H}_\delta^\perp \cap \mathcal{D}_{\mathcal{E}}$ and $T|_{\mathcal{H}_\delta} \in B(\mathcal{H}_\delta)$ for all $\delta \in \Delta\}$.

$C^*(\mathcal{D}_{\mathcal{E}})$ is a locally C^*-algebra with the involution given by $T^* = T^\bigstar\big|_{\mathcal{D}_{\mathcal{E}}}$, where $T^\bigstar$ is the adjoint of $T \in C^*(\mathcal{D}_{\mathcal{E}})$, and the topology given by the family of C^*-seminorms $\{\|\cdot\|_\delta\}_{\delta\in\Delta}$, where $\|T\|_\delta = \left\| T|_{\mathcal{H}_\delta} \right\|_{B(\mathcal{H}_\delta)}$.

For every locally C^*-algebra A whose topology is defined by the family of C^*-seminorms $\{p_\lambda\}_{\lambda\in\Lambda}$, there is a quantized domain $\{\mathcal{H};\mathcal{E} =\{\mathcal{H}_\lambda\}_{\lambda\in\Lambda};\mathcal{D}_{\mathcal{E}}\}$ and a local isometric $*$-homomorphism $\pi : A \to C^*(\mathcal{D}_{\mathcal{E}})$, that is a $*$-homomorphism such that $\|\pi(a)\|_\lambda = p_\lambda(a)$ for all $a \in A$ and for all $\lambda \in \Lambda$.Therefore, a locally C^*-algebra A can be identified with a $*$-subalgebra of unbounded linear operators on a Hilbert space.

A local convex version of Arveson's extension theorem was proved by Dosiev [A. Dosiev, *Local operator spaces, unbounded operators and multinormed C^*-algebras,* J. Funct. Anal. **255**(2008), 1724–1760.] in the case of unital Fréchet locally C^*-algebras. We consider the category whose objects are unital Fréchet locally C^*-algebras and whose morphisms are admissible unital local completely positive and local completely contractive maps and show that any object from this category has a unique injective envelopes. We do not know if this result is valid in the category of unital Fréchet locally C^*-algebras with unital local completely positive and local completely positive maps as morphisms.

II. Summary of progress

- The objectives assumed within the 2 activities of this year of the project were realized in percentage of 100% and 95%, respectively. Mo late than the end of this stage (31.12.2023), the second activity will be carried out in percentage of 100%, so all the objectives of this stage will be realized in percentage of 100%. Moreover, another objective corresponding to the next stage was initiated.

- 3 articles developed in 2022 were accepted for publication, two in journals from the Q2 area and one in a journal from the Q3 area; for a 4th paper, developed and sent for publication in 2022, we are still waiting for the referee's report.

- 3 other articles were sent for publication (one elaborated mostly in 2022 but finished in 2023), all to journals in the areas Q1 and Q2.

- Another 4 articles are in an advanced stage of development, and will be sent for publication in the near future.

- An article is being developed.

- The obtained results were presented within 2 international conferences and an international seminar, all of which were appreciated. More precisely at Hopf algebras and Tensor categories, Marburg (Germany), 22–26 May 2023.
Hopf Days in Brussels (Belgium), September 4–6, 2023.
Universite de Haute Alsace, Faculty of Sciences and Techniques, Mulhouse (France), September 14, 2023.

- We have improved the equipment of the endowment by purchasing some tablets and a high-performance PC unit.

- We secured a good part of the necessary material base for the production and dissemination of the scientific articles.

- The budget allocated to this stage was spent in full.

III. Executive summary of the activities carried out îduring the implementation period

In the second stage of the project, 3 specialized articles were sent for publication (to journals rated Q1 and Q2), while another 4 are in the (advanced) development process. For short, the results obtained within this stage are the following:

- the description of the antipode of a quasi-quantum group (qQG for short) with a weak projection of and its description after a deformation by a 2-cocycle;
- a new perspective on the deformation theory for associative algebras;
- the defining of the double biproduct for quasi-Hopf groups;
- the identification of a double biproduct with a left and/or right biproduct;
- the deformations of a double biproduct by means of 2-cocycles defined by almost dual skew pairings in categories of Yetter-Drinfeld modules;
- the defining of the double bosonization process for qQGs;
- the defining of the free type Yetter-Drinfeld modules;
- the defining of the Serre relations for a qQG;
- the introduction of the Drinfeld-Jimbo quasi-quantum groups, their generalizations and connections between them and other qQGs that exist in the specialized literature;
- denying the transport of certain properties from the graded algebra to its trivial component;
- the obtaining of some new classes of qQGs of rank 2;
- the definition of Nichols algebras in the case of quasi-Hopf and their description in certain particular situations;
- a study of injectivity for local C^*-algebras.

A large part of the results mentioned above were presented within 2 international conferences and a local scientific seminar, namely: - 2 presentations at "Hopf algebras and Tensor categories", Marburg (Germany), May 22–26, 2023.
- 1 presentation at "Hopf Days in Brussels", Brussels (Belgium), September 4–6, 2023.
- 1 presentation at the Universite de Haute Alsace, Faculte des Sciences et Technique, Mulhouse (France), September 14, 2023.

Date, Project leader,

November 2022 Prof. dr. Daniel Bulacu