PARA-HYPERHERMITIAN STRUCTURES ON TANGENT BUNDLES

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Abstract. In this paper we construct a family of almost para-hyperhermitian structures on the tangent bundle of an almost para-hermitian manifold and study its integrability. Also, necessary and sufficient conditions are provided in order that these structures to become para-hyper-Kähler.

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1. Introduction

The almost paraquaternionic structures, also named almost para-hypercomplex structures or almost quaternionic structures of second kind, have been introduced by P. Libermann in 1954 under the latter name [30] as a triple of endomorphisms of the tangent bundle $\{J_1, J_2, J_3\}$, in which J_1 is almost complex and J_2, J_3 are almost product structures satisfying relations of anti-commutation. Then the paraquaternionic structures were studied under various contexts by many authors (see, e.g., [1, 2, 3, 4, 5, 8, 9, 10, 12, 13, 14, 17, 21, 23, 24, 28, 31, 42, 43]). An almost para-hyperhermitian structure on a manifold consists in an almost para-hypercomplex structure and a compatible semi-Riemannian metric necessarily of neutral signature. If all three structures involved in the definition of an almost para-hyperhermitian structure are parallel with respect to the Levi-Civita connection of the compatible metric, one arrives to the concept of para-hyper-Kähler structure, which is also referred in literature as neutral hyper-Kähler or hypersymplectic structure [8, 12, 13, 17, 21].

The paraquaternionic structures are of great interest in theoretical physics, because they arise in a natural way both in string theory and integrable systems

[6, 11, 16, 22, 32, 35] and, consequently, to find new classes of manifolds endowed with such a kind of structures is an interesting topic. Kamada [29] proves that any primary Kodaira surface admits para-hyper-Kähler structures, whose compatible metrics can be chosen to be flat or nonflat. On the other hand, an integrable para-hyperhermitian structure has been constructed in [27] on Kodaira-Thurston properly elliptic surfaces and also on the Inoe surfaces modeled on Sol_1^4. In higher dimensions, para-hyperhermitian structures on a class of compact quotients of 2-step nilpotent Lie groups was found in [19]. A procedure to construct para-hyperhermitian structures on R^{4n} with complete and not necessarily flat associated metrics is given in [5]. Also, some examples of integrable almost para-hyperhermitian structures which admit compatible linear connections with totally-skew symmetric torsion are given in [28]. Recently, in [25], a natural para-hyperhermitian structure has been constructed on the tangent bundle of an almost para-hermitian manifold and on the circle bundle over a manifold with a mixed 3-structure. The main purpose of this paper is to generalize this construction to obtain an entire class of such structures; we also investigate its integrability and obtain necessary and sufficient conditions for these structures to become para-hyper-Kähler.

2. Preliminaries on almost para-hyperhermitian manifolds

An almost product structure on a smooth manifold M is a tensor field P of type (1,1) on M, $P \neq \pm Id$, such that:

$$P^2 = Id \tag{1}$$

where Id is the identity tensor field of type (1,1) on M.

An almost para-hermitian structure on a differentiable manifold M is a pair (P, g), where P is an almost product structure on M and g is a semi-Riemannian metric on M satisfying:

$$g(PX, PY) = -g(X, Y), \tag{2}$$

for all vector fields X,Y on M.

In this case, (M, P, g) is said to be an almost para-hermitian manifold. It is easy to see that the dimension of M is even. Moreover, if $\nabla P = 0$, then (M, P, g) is said to be a para-Kähler manifold.

An almost complex structure on a smooth manifold M is a tensor field J of type (1,1) on M such that:

$$J^2 = -Id. \tag{3}$$

An almost para-hypercomplex structure on a smooth manifold M is a triple $H = (J_\alpha)_{\alpha=\overline{1,3}}$, where J_1 is an almost complex structure on M and J_2, J_3 are almost product structures on M, satisfying:

$$J_2J_1 = -J_1J_2 = J_3. \tag{4}$$

In this case (M, H) is said to be an almost para-hypercomplex manifold.

A semi-Riemannian metric g on (M, H) is said to be compatible or adapted to the almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=\overline{1,3}}$ if it satisfies:

$$g(J_\alpha X, J_\alpha Y) = \epsilon_\alpha g(X, Y), \forall \alpha = \overline{1,3} \tag{5}$$

for all vector fields X,Y on M, where $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$. Moreover, the pair (g, H) is called an almost para-hyperhermitian structure on M and the triple (M, g, H) is said to be an almost para-hyperhermitian manifold. It is clear that any almost para-hyperhermitian manifold is of dimension $4m,\ m \geq 1$, and any adapted metric is necessarily of neutral signature $(2m, 2m)$. If $\{J_1, J_2, J_3\}$ are parallel in respect to the Levi-Civita connection of g, then the manifold is called para-hyper-Kähler.

An almost para-hypercomplex manifold (M, H) is called a para-hypercomplex manifold if each J_α, $\alpha = 1, 2, 3$, is integrable, that is, if the corresponding Nijenhuis tensors:

$$N_\alpha(X, Y) = [J_\alpha X, J_\alpha Y] - J_\alpha[X, J_\alpha Y] - J_\alpha[J_\alpha X, Y] - \epsilon_\alpha[X, Y] \tag{6}$$

$\alpha = 1, 2, 3$, vanish for all vector fields X,Y on M. In this case H is said to be a para-hypercomplex structure on M. Moreover, if g is a semi-Riemannian metric adapted to the para-hypercomplex structure H, then the pair (g, H) is said to be a para-hyperhermitian structure on M and (M, g, H) is called a para-hyperhermitian manifold. We note that the existence of para-hyperhermitian structures on compact complex surfaces has been recently investigated in [15].

3. A natural almost para-hyperhermitian structure on the tangent bundle of an almost para-hermitian manifold

Let (M, P, g) be an almost para-hermitian manifold and TM be the tangent bundle, endowed with the Sasakian metric:

$$G(X, Y) = (g(KX, KY) + g(\pi_* X, \pi_* Y)) \circ \pi$$

for all vector fields X, Y on TM, where π is the natural projection of TM onto M and K is the connection map (see [18]).

We remark that for each $u \in T_x M$, $x \in M$, we have a direct sum decomposition:

$$T_u TM = T_u^h TM \oplus T_u^v TM$$

where $T_u^h TM = KerK|_{T_u TM}$ is called the horizontal subspace of $T_u TM$ and $T_u^v TM = Ker\pi_*|_{T_u TM}$ is called the vertical subspace of $T_u TM$. Moreover, the elements of $T_u^h TM$ are called horizontal vectors at u and the elements of $T_u^v TM$ are said to be vertical vectors at u.

We can see that if $u, X \in T_x M$ and X_u^h (resp. X_u^v) denotes the horizontal lift (resp. vertical lift) of X to $T_u TM$ then:

$$\pi_* X_u^h = X,\ \pi_* X_u^v = 0,\ KX_u^h = 0,\ KX_u^v = X.$$

Remark 3.1. If A is a vector field along π (i.e. a map $A : TM \to TM$ such that $\pi \circ A = \pi$) and A^h (resp. A^v) denotes the horizontal lift (resp. vertical lift) of A (that is: $A^h : u \mapsto A_u^h = A(u)_u^h$ and $A^v : u \mapsto A_u^v = A(u)_u^v$), then any horizontal (vertical) vector field X on TM can be written as $X = A^h$ ($X = A^v$) for a unique vector A along π. If A, B are vector fields along π, then, by generalizing the well-known Dombrowski's lemma [18], Ii [26] has shown that the brackets of the horizontal and vertical lifts are given by:

$$[A^h, B^h]_u = (\nabla_{A^h} B)_u^h - (\nabla_{B^h} A)_u^h - (R(A(u), B(u))u)_u^v, \tag{7}$$

$$[A^h, B^v] = (\nabla_{A^h} B)^v - (\nabla_{B^v} A)^h, \tag{8}$$

$$[A^v, B^v] = (\nabla_{A^v} B)^v - (\nabla_{B^v} A)^v \tag{9}$$

where the covariant derivative of a vector field C along π in the direction of $\xi \in T_u TM$, $u \in TM$, is defined as the tangent vector to M at $x = \pi(u)$ given by:

$$\nabla_\xi C = (K \circ dC)(\xi).$$

We can also remark that every tensor field T of type (1,1) on M is a vector field along π. Moreover we have:

$$\nabla_{A^v} T = T \circ A, \tag{10}$$

and, if T is parallel,

$$\nabla_{A^h} T = 0. \tag{11}$$

We note that the identity map $id : u \mapsto u$ on TM is a parallel tensor field of type (1,1) on M. Moreover, if $||\cdot||^2$ is the function $u \mapsto ||u||^2 = g(u,u)$ on TM, then we have:

$$A^h ||\cdot||^2 = 0, \ A^v ||\cdot||^2 = 2g(A, id). \tag{12}$$

Remark 3.2. If (M, P, g) is an almost para-hermitian manifold, then we can define three tensor fields J_1, J_2, J_3 on TM by the equalities:

$$\begin{cases} J_1 X^h = X^v \\ J_1 X^v = -X^h \end{cases}, \tag{13}$$

$$\begin{cases} J_2 X^h = (PX)^v \\ J_2 X^v = (PX)^h \end{cases}, \tag{14}$$

$$\begin{cases} J_3 X^h = (PX)^h \\ J_3 X^v = -(PX)^v \end{cases}. \tag{15}$$

It is easy to see that J_1 is an almost complex structure and J_2, J_3 are almost product structures. We also have the following result (see [25]).

Theorem 3.3. *Let (M, P, g) be an almost para-hermitian manifold. Then $H = (J_\alpha)_{\alpha=\overline{1,3}}$ is an almost para-hypercomplex structure on TM which is para-hyperhermitian with respect to the Sasakian metric G. Moreover H is integrable if and only if (M, P) is a flat para-Kähler manifold.*

In the next Section, following the same techniques as in [7, 33, 34, 36, 37, 38, 39, 40, 41], we deform the almost para-hyperhermitian structure given above in order to obtain an entire family of structures of this kind on the tangent bundle of an almost para-hermitian manifold.

4. A family of almost para-hyperhermitian structure on the tangent bundle of a para-hermitian manifold

Lemma 4.1. *Let (M, P, g) be an almost para-hermitian manifold and let J_1 be a tensor field of type (1,1) on TM, defined by:*

$$\begin{cases} J_1 X_u^h = a(t)X_u^v + b(t)g(X,u)u^v + c(t)g(X,Pu)(Pu)^v \\ J_1 X_u^v = m(t)X_u^h + n(t)g(X,u)u^h + p(t)g(X,Pu)(Pu)^h \end{cases} \tag{16}$$

for all vectors $X \in T_{\pi(u)}M$, $u \in T_xM$, $x \in M$, where $t = \|u\|^2$ and a, b, c, m, n, p are differentiable real functions. Then J_1 defines an almost complex structure if and only if

$$am + 1 = 0,\ an(a + tb) - b = 0,\ ap(a - tc) - c = 0\ . \tag{17}$$

Proof. The conditions follows from the property $J_1^2 = -Id$. □

Lemma 4.2. *Let (M, P, g) be an almost para-hermitian manifold and let J_2 be a tensor field of type (1,1) on TM, defined by:*

$$\begin{cases} J_2 X_u^h = a(t)(PX)_u^v - b(t)g(X,Pu)u^v - c(t)g(X,u)(Pu)^v \\ J_2 X_u^v = q(t)(PX)_u^h - r(t)g(X,Pu)u^h - s(t)g(X,u)(Pu)^h \end{cases} \tag{18}$$

for all vectors $X \in T_{\pi(u)}M$, $u \in T_xM$, $x \in M$, where $t = \|u\|^2$ and a, b, c, q, r, s are differentiable real functions. Then J_2 defines an almost product structure if and only if

$$aq - 1 = 0,\ ar(a - tc) - c = 0,\ as(a + tb) - b = 0. \tag{19}$$

Proof. The conditions follows from the property $J_2^2 = Id$. □

Proposition 4.3. *Let (M, P, g) be an almost para-hermitian manifold. Then there exists an infinite class of almost para-hypercomplex structures on TM.*

Proof. We define a tensor field J_3 of type (1,1) on TM by $J_3 = J_2 J_1$, where J_1, J_2 are given by (16) and (18), such that (17) and (19) are satisfied. We can easily see now that $H = (J_\alpha)_{\alpha=\overline{1,3}}$ is an almost para-hypercomplex structure on TM. □

Proposition 4.4. *Let (M, P, g) be an almost para-hermitian manifold, let $H = (J_\alpha)_{\alpha=\overline{1,3}}$ be the almost para-hypercomplex structure on TM given above and let $\widetilde{G}$ be a semi-Riemannian metric on TM defined by:*

$$(20)\quad \begin{cases} \widetilde{G}(X^h, Y^h) = \alpha(t)g(X, Y) + \beta(t)g(X, u)g(Y, u) + \gamma(t)g(X, Pu)g(Y, Pu) \\ \widetilde{G}(X^v, Y^v) = \delta(t)g(X, Y) + \varepsilon(t)g(X, u)g(Y, u) + \theta(t)g(X, Pu)g(Y, Pu) \\ \widetilde{G}(X^h, Y^v) = 0 \end{cases}$$

for all vectors $X, Y \in T_{\pi(u)}M$, $u \in T_xM$, $x \in M$, where $t = \|u\|^2$ and $\alpha, \beta, \gamma, \delta, \varepsilon, \theta$ are smooth real functions. Then $\widetilde{G}$ is adapted to the almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=\overline{1,3}}$ if only if:

$$(21)\qquad \begin{cases} \beta + \gamma = 0 \\ \delta a^2 - \alpha = 0 \\ a^2\varepsilon(a + tb)^2 + b\alpha(2a + tb) - \beta a^2 = 0 \\ a^2\theta(a - tc)^2 + c\alpha(2a - tc) + \beta a^2 = 0 \end{cases}.$$

Proof. The conditions (21) are obtained by direct computations using the property (5). □

Corollary 4.5. *There exists an infinite class of almost para-hyperhermitian structures on the tangent bundle of an almost para-hermitian manifold.*

5. The study of the integrability

Let (M, P, g) be a para-Kähler manifold. A plane $\Pi \subset T_pM$, $p \in M$, is called para-holomorphic if it is left invariant by the action of P, that is $P\Pi \subset \Pi$. The para-holomorphic sectional curvature is defined as the restriction of the sectional curvature to para-holomorphic non-degenerate planes. A para-Kähler manifold is said to be a paracomplex space form if its para-holomorphic sectional curvatures are equal to a constant, say k. It is well-known that a para-Kähler manifold (M,P,g) is a paracomplex space form, denoted $M(k)$, if and only if its curvature tensor is:

$$\begin{aligned} R(X,Y)Z &= \frac{k}{4}\{g(Y,Z)X - g(X,Z)Y + g(Y,PZ)PX \\ &\quad -g(X,PZ)PY - 2g(X,PY)PZ]\} \end{aligned} \tag{22}$$

for all vector fields X,Y,Z on M.

From the above section we deduce that the tangent bundle of a paracomplex space form $M(k)$ can be endowed with a class of almost para-hypercomplex structure $H=(J_\alpha)_{\alpha=\overline{1,3}}$ given by:

$$\begin{cases} J_1X_u^h = aX_u^v + bg(X,u)u^v + cg(X,Pu)(Pu)^v \\ J_1X_u^v = -\frac{1}{a}X_u^h + \frac{b}{a(a+tb)}g(X,u)u^h + \frac{c}{a(a-tc)}g(X,Pu)(Pu)^h \end{cases}, \tag{23}$$

$$\begin{cases} J_2X_u^h = a(PX)_u^v - bg(X,Pu)u^v - cg(X,u)(Pu)^v \\ J_2X_u^v = \frac{1}{a}(PX)_u^h - \frac{c}{a(a-tc)}g(X,Pu)u^h - \frac{b}{a(a+tb)}g(X,u)(Pu)^h \end{cases}, \tag{24}$$

$$\begin{cases} J_3X_u^h = (PX)_u^h \\ J_3X_u^v = -(PX)_u^v + \frac{b+c}{a-tc}g(X,Pu)u^v + \frac{b+c}{a+tb}g(X,u)(Pu)^v, \end{cases} \tag{25}$$

where a,b,c are differentiable real functions such that $\frac{1}{a}, \frac{b}{a(a+tb)}, \frac{c}{a(a-tc)}$ are well defined and also differentiable real functions.

Theorem 5.1. *Let $M(k)$ be a paracomplex space form. Then the almost para-hypercomplex structure $H=(J_\alpha)_{\alpha=\overline{1,3}}$ given above is integrable if and only if*

$$4b(a-2ta') - 8aa' + k = 0,\ 4ac + k = 0. \tag{26}$$

Proof. First of all we remark that if two of the structures J_1, J_2, J_3 are integrable, then the third structure is also integrable because the corresponding Nijenhuis tensors are related by:

$$
\begin{aligned}
2N_\alpha(X,Y) &= N_\beta(J_\gamma X, J_\gamma Y) + N_\gamma(J_\beta X, J_\beta Y) - J_\beta N_\gamma(J_\beta X, Y) \\
&\quad - J_\beta N_\gamma(X, J_\beta Y) - J_\gamma N_\beta(J_\gamma X, Y) - J_\gamma N_\beta(X, J_\gamma Y) \\
&\quad + \epsilon_\gamma N_\beta(X,Y) + \epsilon_\beta N_\gamma(X,Y) \qquad (27)
\end{aligned}
$$

for any even permutation (α,β,γ) of (1,2,3), where $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$.

Secondly, is well-known that an almost complex structure on a manifold is integrable if and only if the distribution of the complex tangent vector fields of type (1,0), denoted by $\mathcal{X}^{(1,0)}$, is involutive, i.e. it satisfies $[\mathcal{X}^{(1,0)}, \mathcal{X}^{(1,0)}] \subset \mathcal{X}^{(1,0)}$. Now, using (6), (7)-(12) and (22), we obtain for any vector fields A, B along π, satisfying $g(A, id) = g(B, id) = g(A, P \circ id) = g(B, P \circ id) = 0$ on TM:

$$
[A^h - iaA^v, B^h - iaB^v]_u = (\nabla_{A^h} B - \nabla_{B^h} A)^h_u - ia(\nabla_{A^h} B - \nabla_{B^h} A)^v_u, \qquad (28)
$$

$$
\begin{aligned}
[A^h - iaA^v, (id)^h &- i(a+tb)(id)^v]_u = -[(\nabla_{(id)^h} A)^h_u - ia(\nabla_{(id)^h} A)^v_u] \\
&+ i(a+tb)[(\nabla_{(id)^v} A)^h_u - ia(\nabla_{(id)^v} A)^v_u] \\
&- ia[A^h_u - i\frac{\frac{kt}{4} + a(a+tb) - 2a't(a+tb))}{a} A^v_u], \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
[A^h - iaA^v, (P\circ id)^h &- i(a-tc)(P\circ id)^v]_u = -[(\nabla_{(P\circ id)^h} A)^h_u - ia(\nabla_{(P\circ id)^h} A)^v_u] \\
&+ i(a-tc)[(\nabla_{(P\circ id)^v} A)^h_u - ia(\nabla_{(P\circ id)^v} A)^v_u] \\
&- ia[(PA)^h_u - i\frac{-\frac{kt}{4} + a(a-tc))}{a}(PA)^v_u], \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
[(id)^h &- i(a+tb)(id)^v, (P\circ id)^h - i(a-tc)(P\circ id)^v]_u = \\
&i(a-tc)[(P\circ id)^h_u - i\frac{kt + (a+tb)(a-tc) - 2t(a+tb)(a-tc)'}{a-tc}(P\circ id)^v_u] \\
&- i(a+tb)[(\nabla_{(id)^v} P\circ id)^h - i(a-tc)(\nabla_{(id)^v} P\circ id)^v], \qquad (31)
\end{aligned}
$$

$$[(id)^h - i(a+tb)(id)^v, (id)^h - i(a+tb)(id)^v]_u = 0, \tag{32}$$

$$[(P\circ id)^h - i(a-tc)(P\circ id)^v, (P\circ id)^h - i(a-tc)(P\circ id)^v]_u = 0. \tag{33}$$

Consequently, J_1 is integrable if and only if the next three relations are satisfied:

$$\frac{\frac{k}{4}t + a(a+tb) - 2a't(a+tb)}{a} = a, \tag{34}$$

$$\frac{-\frac{k}{4}t + a(a-tc)}{a} = a, \tag{35}$$

$$\frac{kt + (a+tb)(a-tc) - 2t(a+tb)(a-tc)'}{a-tc} = a - tc. \tag{36}$$

Thirdly, an almost product structure on a manifold is integrable if and only the eigendistributions $\mathcal{X}^+$ and $\mathcal{X}^-$ corresponding to the eigenvalues 1 and -1 respectively, are integrable. We similarly obtain that J_2 is integrable if and only if the same three relations hold.

Finally, because the last of the above three relations is involved by the first two, the conclusion follows immediately. □

Example 5.2. If M is a flat paracomplex space form, then we set

$$a = A,\ b = 0,\ c = 0,$$

where A is an arbitrary non-zero real constant, and we can easily see that the conditions (26) are satisfied and $a, b, c, \frac{1}{a}, \frac{b}{a(a+tb)}, \frac{c}{a(a-tc)}$ are clearly differentiable, being constants. Consequently, the almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=\overline{1,3}}$ given above is integrable.

Example 5.3. If $M(k)$ is a non-flat paracomplex space form, then we set

$$a = \sqrt[4]{k^2(t^2 + A^2 + 1)},$$

$$b = \frac{\sqrt[4]{k^2(t^2 + A^2 + 1)}(4t - sgn\{k\}\sqrt{t^2 + A^2 + 1})}{4(A^2 + 1)},$$

$$c = \frac{-k}{4\sqrt[4]{k^2(t^2+A^2+1)}},$$

where A is an arbitrary real constant, and we can easily verify that the conditions (26) are satisfied and the functions $a, b, c, \frac{1}{a}, \frac{b}{a(a+tb)}, \frac{c}{a(a-tc)}$ are differentiable. Consequently, the almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=\overline{1,3}}$ given above is integrable.

Remark 5.4. From Proposition 4.4 we have a class of compatible semi-Riemannian metric on the tangent bundle of a paracomplex space form $M(k)$, given by

$$\left\{\begin{array}{l} \widetilde{G}(X^h,Y^h) = \alpha g(X,Y) + \beta g(X,u)g(Y,u) - \beta g(X,Pu)g(Y,Pu) \\ \widetilde{G}(X^v,Y^v) = \frac{\alpha}{a^2} g(X,Y) + \frac{\beta a^2 - b\alpha(2a+tb)}{a^2(a+tb)^2} g(X,u)g(Y,u) \\ \qquad\qquad\qquad + \frac{-\beta a^2 - c\alpha(2a-tc)}{a^2(a-tc)^2} g(X,Pu)g(Y,Pu) \\ \widetilde{G}(X^h,Y^v) = 0 \end{array}\right. , \tag{37}$$

where α, β are differentiable real functions. From Theorem 5.1 we may state now the following result.

Corollary 5.5. *There exists an infinite class of para-hyperhermitian structures on the tangent bundle of a paracomplex space form.*

Theorem 5.6. *Let $M(k)$ be a paracomplex space form. Then the almost para-hyperhermitian structure $(\widetilde{G}, H = (J_\alpha)_{\alpha=\overline{1,3}})$ on TM defined by (23), (24), (25) and (37) is para-hyper-Kähler if and only if M is flat and*

$$a = C_1,\ \alpha = C_2,\ b = c = \beta = 0, \tag{38}$$

where C_1, C_2 are real constants, $C_1 \neq 0$.

Proof. If M is flat and the relations (38) hold, then it is clear that $(TM, \widetilde{G}, H = (J_\alpha)_{\alpha=\overline{1,3}})$ is a para-hyper-Kähler manifold.

Conversely, if $(TM, \widetilde{G}, H = (J_\alpha)_{\alpha=\overline{1,3}})$ is a para-hyper-Kähler manifold then each J_α is integrable and the fundamentals 2-forms ω_α, given by:

$$\omega_\alpha(X,Y) = \widetilde{G}(J_\alpha X, Y), \tag{39}$$

for all vector fields X, Y on TM, are closed for all $\alpha \in \{1,2,3\}$.

For any vector fields A and B along π, satisfying $g(A, id) = g(B, id) = g(A, P \circ id) = g(B, P \circ id) = 0$ on TM, using (7)-(12), we obtain:

$$\begin{aligned}(d\omega_1)(A^h, B^v, id^v) &= A^h\widetilde{G}(J_1B^v, id^v) - B^v\widetilde{G}(J_1A^h, id^v) \\ &\quad + id^v\widetilde{G}(J_1A^h, B^v) + \widetilde{G}(J_1id^v, [A^h, B^v]) \\ &\quad - \widetilde{G}(J_1B^v, [A^h, id^v]) + \widetilde{G}(J_1A^h, [B^v, id^v]) \\ &= \left[2\left(\frac{\alpha}{a}\right)' t + \left(\frac{\alpha}{a} - \frac{\alpha + t\beta}{a + tb}\right)\right] g(X, Y) \qquad (40)\end{aligned}$$

and

$$\begin{aligned}(d\omega_1)(A^h, B^v, (P \circ id)^v) &= A^h\widetilde{G}(J_1B^v, (P \circ id)^v) - B^v\widetilde{G}(J_1A^h, (P \circ id)^v) \\ &\quad + (P \circ id)^v\widetilde{G}(J_1A^h, B^v) + \widetilde{G}(J_1(P \circ id)^v, [A^h, B^v]) \\ &\quad - \widetilde{G}(J_1B^v, [A^h, (P \circ id)^v]) + \widetilde{G}(J_1A^h, [B^v, (P \circ id)^v]) \\ &= \left(\frac{\alpha}{a} - \frac{\alpha + t\beta}{a - tc}\right) g(X, PY). \qquad (41)\end{aligned}$$

But, because ω_1 is closed, from (40) and (41) we obtain:

$$\frac{\alpha}{a} = \frac{\alpha + t\beta}{a + tb} = \frac{\alpha + t\beta}{a - tc} = C, \qquad (42)$$

where C is a real constant.

On another hand J_1, J_2, J_3 being integrable, from Theorem 5.1 we deduce that the functions a, b, c also satisfy the conditions (26). The conclusion follows now easily since $a, b, c, \alpha, \beta, \frac{1}{a}, \frac{b}{a(a+tb)}, \frac{c}{a(a-tc)}$ must be differentiable functions satisfying (26) and (42). □

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