Ricci curvature properties and stability on 3-dimensional Kenmotsu manifolds

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Abstract. In this paper we characterize the Ricci curvature and the stability of a harmonic map on a compact domain of a 3-dimensional Kenmotsu manifold.

1. Introduction

The study of harmonic maps on contact metric manifolds was initiated by S. Ianus and A. M. Pastore ([**IP**]). In this article we give some new results on harmonic maps (see [**BW, Pa**]) and holomorphic submersions (see [**FIP**]) between manifolds endowed with special geometric structures (see [**Bla, BS, Ken, GIP, Ghe, Pit**]). The paper is organized as follows.

In the next section we recall some definitions and properties of almost contact metric manifolds. In Section 3, we study the Ricci curvature of a horizontally conformal map (see [**BD**]) from a 3-dimensional Kenmotsu manifold and obtain a characterization of the stability of harmonic maps from a compact domain of a 3-dimensional Kenmotsu manifold.

Throughout the paper, all manifolds and structures on them are differentiable and of class C^∞ (smooth).

2. Riemannian manifolds endowed with almost contact structures

Let M be a manifold with odd dimension $(2n+1)$. An *almost contact structure* on M is a triple (φ, ξ, η) where ξ is a vector field, η a 1-form and φ a (1,1)-tensor field satisfying:

$$\varphi^2 = -Id + \eta \otimes \xi,\ \eta(\xi) = 1$$

where Id is the identity endomorphism on TM . Then, we have $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$. If g is a Riemannian metric on M such that $g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y)$ for any X and Y on $\Gamma(TM)$, we say that (φ, ξ, η, g) is an almost contact metric structure on M. A manifold equipped with such structure is called an

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almost contact metric manifold . The second fundamental form Φ on M is given by $\Phi(X,Y) = g(X,\varphi Y)$ for any X and Y on $\Gamma(TM)$.

An *almost contact metric structure* (φ,ξ,η,g) is *normal* if the Nijenhuis tensor N^φ satisfies $N^\varphi + 2d\eta\otimes\xi = 0$.

A Riemannian manifold (M,g) of dimension $(2n+1)$ endowed with an almost contact metric structure (φ,ξ,η,g) is an *almost Kenmotsu manifold* if the conditions $d\eta = 0$ and $d\Phi = 2\eta\wedge\Phi$ are satisfied. An almost Kenmotsu manifold is said to be a *Kenmotsu manifold* if the almost contact structure is normal.

A Riemannian manifold (M,g) of dimension $(2n+1)$ endowed with an almost contact metric structure (φ,ξ,η,g) is a Kenmotsu manifold if and only if $(\nabla_X\varphi)Y = -g(X,\varphi Y)\xi - \eta(Y)\varphi X$, for any vector fields X,Y on M. Hence

$$\nabla_X\xi = X - \eta(X)\xi. \tag{2.1}$$

On a Kenmotsu manifold M of dimension $(2n+1)$ we have also ([**Ken, Pit**])

$$\nabla_\xi\xi = 0,\ R(X,Y)\xi = \eta(X)Y - \eta(Y)X,\ Ric(X,\xi) = -2n\eta(X) \tag{2.2}$$

for any $X,Y\in\Gamma(TM)$.

Kenmotsu manifolds are interesting examples of almost contact metric manifolds which are not K-contact. Recall the following local characterization of Kenmotsu manifolds (cf. [**Ken**]).

Theorem 2.1. *Let M be a Kenmotsu manifold. Any point of M has a neighborhood isometric to the warped product $(-\epsilon,\epsilon)\times_f V$, where $(-\epsilon,\epsilon)$ is an open interval from $\mathbb{R}$, $f(t) = ce^t$, $c>0$ and V is a Kähler manifold.*

3. Horizontally conformal submersions on Kenmotsu manifolds of dimension 3

We recall some definitions on horizontally weakly conformal maps and harmonic morphisms. Let $\psi : M^m \to N^n$ be a submersion between Riemannian manifolds. We recall that the tangent bundle of M splits as the Whitney sum of two distributions, the *vertical* one $\mathcal{V} = Ker(d\psi)$ and the orthogonal complementary distribution $\mathcal{H} = \mathcal{V}^\perp$ called *horizontal*: $TM = \mathcal{V}\oplus\mathcal{H}$. As usually, we denote by v and h the projections on the vertical and horizontal distributions. The sections of $\mathcal{V}$ (respectively $\mathcal{H}$) will be called *vertical (*respectively *horizontal) vector fields.* For any vector field E, vE and hE denote the vertical and the horizontal components of E, respectively. We will use the following notations for the second fundamental forms of the horizontal and vertical distributions (see [**BW**]): $A_EF = A^h_EF = v(\nabla_{hE}hF)$, $B_EF = B^v_EF = h(\nabla_{vE}vF)$ and for the integrability tensor of $\mathcal{H}$, $I(E,F) = I^h(E,F) = v[hE,hF]$ where $E,F\in\Gamma(TM)$.

Definition 3.1. Let $\psi : (M^m,g)\to(N^n,h)$ be a map between Riemannian manifolds and let $x\in M$. Then ψ is called horizontally weakly conformal at x if either

(1) $d\psi_x = 0$, or
(2) $d\psi_x$ is surjective and there exists a number $\Lambda(x)>0$ such that

$$h(d\psi_x(X),d\psi_x(Y)) = \Lambda(x)g(X,Y)$$

where $X,Y\in\mathcal{H}_x$.

$\Lambda(x)$ is called the square dilation of ψ at x and $\lambda(x) = \sqrt{\Lambda(x)}$ is called the dilation of ψ at x. The map ψ is called horizontally weakly conformal on M if it is horizontally weakly conformal at every point of M. If ψ has rank $n = \dim N$ at every point of M we say that ψ is a *horizontally conformal submersion.* If $d\psi_x \neq 0$ and $\lambda = 1$, then ψ is a Riemannian submersion.

Let $\psi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds of dimension m and n, respectively. Its differential $d\psi$ can be viewed as a section of the bundle $T^*M \otimes \psi^{-1}(TN) \to M$ endowed with the Hilbert-Schmidt norm $\|\cdot\|$.

If $\{e_1, \ldots, e_m\}$ is an orthonormal local frame on M, the norm of $d\psi$ is given by $\|d\psi\|^2 := Tr_g(\psi^* h) = \sum_{i=1}^m h(d\psi(e_i), d\psi(e_i))$. The energy density of ψ is a smooth function $e(\psi) : M \to [0, \infty)$ defined by $e(\psi)_x = \frac{1}{2}\|d\psi_x\|^2, \quad x \in M$. For any compact domain $\Omega \subseteq M$, the energy of ψ over Ω is the integral of its energy density $E(\psi; \Omega) = \int_\Omega e(\psi)\vartheta_g$ where ϑ_g is the volume measure associated to the Riemannian metric g.

A smooth map $\psi : M \to N$ is said to be a *harmonic map* if

$$\frac{d}{dt}|_{t=0} E(\psi_t; \Omega) = 0$$

for all compact domains Ω and for all variations $\{\psi_t\}$ of ψ supported in Ω.

Definition 3.2. ([**IP**]) A smooth map $\psi : (M^{2m+1}, \varphi, \xi, \eta, g) \to (N^{2n}, h, J)$ from an almost contact metric manifold to a Kähler manifold is called a $(\varphi, J)-$holomorphic map if $d\psi \circ \varphi = J \circ d\psi$.

Theorem 3.3. *(*[**Ghe**]*) Let M be a Kenmotsu manifold with the almost contact metric structure (φ, ξ, η, g) and N with the Kähler structure (J, h). If $\psi : M \to N$ is a $(\varphi, J)-$holomorphic map then it is harmonic.*

Definition 3.4. (cf. [**BW**]) Let $\psi : M \to N$ be a smooth mapping between Riemannian manifolds. Then ψ is called a harmonic morphism if, for every harmonic function $h : V \to R$, defined on an open subset $V \subset N$ with $\psi^{-1}(V) \neq \emptyset$ the composition $h \circ \psi$ is harmonic on $\psi^{-1}(V)$.

We recall a result by Fuglede and Ishihara: A smooth map $\psi : M \to N$ between Riemannian manifolds is a harmonic morphism if and only if ψ is both harmonic and horizontally weakly conformal (cf. [**BW**]).

For the Ricci curvature of a harmonic morphism between Riemannian manifolds we recall the following result.

Theorem 3.5. *(*[**BW**]*) Let $\psi : M^m \to N^n$ $(n \geq 1)$ be a submersive harmonic morphism with dilation $\lambda : M \to (0, \infty)$. Let $x \in M$ and $\{e_a\}_{a=1,\ldots,n}$ and $\{e_r\}_{r=n+1,\ldots,m}$ be bases for the horizontal and vertical spaces at x, respectively. Let X, Y be horizontal vectors at x and U, V vertical vectors at x, then*

(1) $Ric^M(U, V) = Ric^v(U, V) + \sum_{a=1}^n \langle(\nabla_{e_a} B^*)_U e_a, V\rangle + 2(n-1)d\ln\lambda(B_U V)$

$$+n\nabla d\ln\lambda(U, V) - nU(\ln\lambda)V(\ln\lambda) + \frac{1}{4}\sum_{a,b=1}^n \langle U, I(e_a, e_b)\rangle\langle V, I(e_a, e_b)\rangle;$$

(2) $Ric^M(X, U) = 2\nabla d\ln\lambda(U, X) + (n-2)d\ln\lambda(B_U^* X) - nd\ln\lambda(A_X^* U)$

$$-\sum_{r=n+1}^m \langle(\nabla_{e_r} B)_U e_r, X\rangle + \sum_{a=1}^n \langle B_U^* e_a, I(X, e_a)\rangle - \sum_{a=1}^n \langle(\nabla_{e_a} A)_X e_a, U\rangle;$$

(3) $Ric^M(X,Y) = Ric^N(d\psi(X), d\psi(Y)) + \langle X,Y\rangle \Delta \ln\lambda - (n-2)X(\ln\lambda)Y(\ln\lambda)$

$$- \sum_{r=n+1}^{m} \langle B^*_{e_r}X, B^*_{e_r}Y\rangle - \frac{1}{2}\sum_{a=1}^{n} \langle I(X,e_a)I(Y,e_a)\rangle.$$

Definition 3.6. ([**BW**]) A horizontally weakly conformal map $\psi : M^m \to N^n$ between Riemannian manifolds is said to be horizontally homothetic if the gradient of its dilation λ is vertical (i.e. the dilation is constant along horizontal curves).

Let now ψ be a horizontally conformal $(\varphi, J)-$holomorphic submersion $\psi : (M^3, g) \to (N^2, h)$ with dilation λ, where M is a Kenmotsu 3-manifold with the almost contact metric structure (φ, ξ, η, g) and N is a Kähler 2-manifold with the structure (J, h).

Let $\{e_1, e_2, e_3\} = \{e, \varphi e, \xi\}$ be an orthonormal local frame on the Kenmotsu manifold M, where $\{e, \varphi e\}$ is the orthonormal frame for the horizontal space. The horizontal distribution $\mathcal{H} = \mathcal{V}^{\perp}$ is the contact distribution which in the case of Kenmotsu manifolds is always integrable.

Proposition 3.7. Let $\psi : (M^3, \varphi, \xi, \eta, g) \to (N^2, J, h)$ be a $(\varphi, J)-$holomorphic horizontally conformal submersion with dilation λ where M is a Kenmotsu 3-manifold and N is a Kähler 2-manifold. Then

(1) $\xi(\ln\lambda) = -1$.
(2) $Ric(g) = \{\lambda^2 K^N + \Delta \ln\lambda\}(g - \eta\otimes\eta) - 2\eta\otimes\eta$, where K^N is the Gauss curvature on manifold N.
(3) Moreover, if ψ is also horizontally homothetic then $\Delta\ln\lambda = -2$. Hence M cannot be compact.

Proof. Let $\{e_1, e_2, e_3\} = \{e, \varphi e, \xi\}$ be an orthonormal local frame of the Kenmotsu manifold M. Recall that the tensor field A satisfies, for any $X, Y \in \Gamma(\mathcal{H})$

$$A_X Y = \frac{1}{2} v[X,Y] + g(X,Y) v(\text{grad } \ln\ \lambda). \tag{3.1}$$

1) From (2.1) and (3.1) we obtain

$$\begin{aligned} g(I(e,e), I(e,e)) &= g(v[e,e], v[e,e]) = \{g(2A_e e - 2v(\text{grad } \ln\ \lambda), \xi)\}^2 \\ &= 4\{g(v\nabla_e e, \xi) - g(v(\text{grad } \ln\ \lambda), \xi)\}^2 \\ &= 4\{-g(e, \nabla_e \xi) - \xi(\ln\lambda)\}^2 = 4(\xi(\ln\lambda)+1)^2. \end{aligned}$$

But $g(I(e,e), I(e,e)) = 0$.

2) Using Theorem 3.3 and the result by Fuglede and Ishihara we obtain that ψ is a submersive harmonic morphism. From Theorem 3.5(3), (2.1), (3.1), (1) we derive that $g(B^*_\xi X, B^*_\xi Y) = 0$ and $\sum_{a=1}^{2} g(I(X,e_a), I(Y,e_a)) = 0$ for any horizontal vector fields X, Y and so, we conclude that, for any $X, Y \in \Gamma(\mathcal{H})$

$$Ric(X,Y) = \{\lambda^2 K^N + \Delta\ln\lambda\} g(X,Y). \tag{3.2}$$

Using now (2.2) and (3.2) we obtain (2).

3) We have from (1) that $\xi(\ln\lambda) = -1$ and from the condition of horizontally homothetic maps that $h(\text{grad } \ln\ \lambda) = 0$ which implies that $X(\ln\lambda) = 0$ for any

horizontal vector field X. Using the definition ([**BW**]) for Δ we obtain :

$$\begin{aligned}\Delta(\ln\lambda) &= \sum_{i=1}^{3}\{e_i(e_i(\ln\lambda)) - (\nabla^M_{e_i}e_i)\ln\lambda\} \\ &= e(e(\ln\lambda)) - (\nabla^M_e e)\ln\lambda + \varphi e(\varphi e(\ln\lambda)) - (\nabla^M_{\varphi e}\varphi e)\ln\lambda \\ &+ \xi(\xi(\ln\lambda)) - (\nabla^M_\xi \xi)\ln\lambda \\ &= -(\nabla^M_e e)\ln\lambda - (\nabla^M_{\varphi e}\varphi e)\ln\lambda = -d\ln\lambda(A_e e) - d\ln\lambda(A_{\varphi e}\varphi e) \\ &= -2d\ln\lambda(v(\text{grad }\ln\ \lambda)) = -2|v(\text{grad }\ln\ \lambda)|^2 = -2(\xi(\ln\lambda))^2 = -2.\end{aligned}$$

From the Divergence theorem ([**BW**]) we obtain that M cannot be compact. □

We recall that a Kenmotsu manifold M^{2n+1} is said to be η-Einstein if the Ricci operator Q is given by $Q = aId + b\eta\otimes\xi$ for some functions a and b on M^{2n+1}.

Corollary 3.8. Let ψ be a $(\varphi, J)-$holomorphic horizontally conformal submersion with dilation λ, $\psi : (M^3, \xi, \eta, \varphi, g) \to (N^2, J, h)$ where M is a Kenmotsu 3-manifold and N is a Kähler 2-manifold. Then M is a η-Einstein manifold.

Theorem 3.9 (Weitzenböck formula). *(see* [**BW**]*) Let* $\psi : (M^m, g) \to (N^n, h)$, $n \geq 1$ *be a submersive harmonic morphism and* X *a horizontal vector field. Then*

(1) $\Delta(\ln\lambda)g(X,X) = Ric^M(X,X) - Ric^N(d\psi(X), d\psi(X)) + (n-2)\{X(\ln\lambda)\}^2 +$
$$+\sum_{r=n+1}^{m}\left|B^*_{e_r}X\right|^2 + \frac{1}{2}\sum_{a=1}^{n}|I(X,e_a)|^2$$

(2) $n\Delta(\ln\lambda) = Tr^h Ric^M - \lambda^2 Scal^N + (n-2)\,|h(\text{grad }\ln\ \lambda)|^2 + \|B\|^2 + \frac{1}{2}\|I\|^2$

Proposition 3.10. Let ψ be a $(\varphi, J)-$holomorphic horizontally conformal submersion of dilation λ from a Kenmotsu 3-manifold $(M^3, \varphi, \xi, \eta, g)$ to a Kähler 2-manifold (N^2, h, J). Let X be a horizontal vector field, then

(1) $\|X\|^2\Delta(\ln\lambda) = Ric^M(X,X) - Ric^N(d\psi(X), d\psi(X))$
(2) $\Delta(\ln\lambda) = K^h - \lambda^2 K^N - 1$ where K^h is the Gauss curvature of the horizontal space.

Proof. Computing the components in Theorem 3.9 using (2.2) and the definition of the adjoint of B, we obtain $|B^*_\xi X|^2 = 0$ and $\sum_{a=1}^2 |I(X,e_a)|^2 = 0$. By the definition of the Ricci curvature for horizontal vector fields and Theorem 3.9 we deduce that $\|X\|^2\Delta(\ln\lambda) = (K^h - 1)g(X,X) - \lambda^2 K^N g(X,X)$. □

Example 3.11. A natural example of a Kenmotsu manifold derives from the local characterization of Theorem 2.1. Let $\psi : \mathbb{R}\times_{f^2} N^2 \to N^2$ be the projection of a warped product on N, a Kähler 2-manifold, with the function $f(t) = ce^t$, where $c \in \mathbb{R}, c > 0$. Then the warped product $M^3 = \mathbb{R}\times_{f^2} N^2$ with $\xi = \frac{d}{dt}, \eta(X) = g(X,\xi)$ for any point $(t,x) \in \mathbb{R}\times N^2$ and any vector field X tangent to M^3 is a Kenmotsu manifold ([**Ken**]). Clearly ψ is $(\varphi, J)-$holomorphic and so harmonic. From the characterization of warped products such maps are horizontally homothetic, hence horizontally weakly conformal and thus we obtain that ψ is a harmonic morphism of warped product type.

Let $N = \mathbb{C}$ with the usual Kähler structure, then the projection $\mathbb{R}\times_{f^2}\mathbb{C} \to \mathbb{C}$ is an example of horizontally homothetic submersive harmonic morphism from a

Kenmotsu space form. In this case $K^N = 0$ and $K^h = -1$ ([**Ken, Pit**]) and is easy to check that formulas from Proposition 3.10 and Proposition 3.7 are satisfied.

THEOREM 3.12. *Let M^3 be a Kenmotsu 3-manifold and $D \subset M^3$ be a compact domain. Let $\psi : (M^3, g) \to (N, h, J)$ be a submersive harmonic morphism where (N, h, J) is a Kähler 2-manifold.*

If $\lambda^2 K^N + \Delta \ln \lambda \leq 1$, then ψ is a stable harmonic map on D. Moreover, if ψ is horizontally homothetic then ψ is stable if $\lambda^2 K^N \leq 3$.

PROOF. From Proposition 3.7 (2) and [**GIP**], the Hessian of the harmonic map ψ becomes

$$\operatorname{Hess}_{\psi}(d\psi(X), d\psi(X)) = -\{\lambda^2 K^N + \Delta \ln \lambda - 1\} \int_D h(d\psi(X), d\psi(X))\vartheta_g \, .$$

for any $X \in \Gamma(TD)$. □

REMARK 3.13. The projection $\mathbb{R} \times_{f^2} \mathbb{C} \to \mathbb{C}$ of Example 3.11 is a stable harmonic map on any compact domain of $\mathbb{R} \times_{f^2} \mathbb{C}$.

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