Conformal submersions of locally conformally hyperkähler manifolds

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To the memory of Professor S. Ianuş.

Abstract

In this paper we study almost hypercomplex conformal submersions with total space a locally conformally hyperkähler (lchK) manifold, i.e. lchK τ-conformal submersions. We derive necessary and sufficient conditions for the base space to be a hyperkähler manifold and we study the harmonicity and stability of such maps.

2010 *Mathematics Subject Classification*: 53C26, 53C43.

Keywords: locally conformally Kähler, quaternionic Hermitian, harmonic map, conformal submersion.

1 Introduction

The quaternionic analogues of complex Kähler manifolds are two well-known manifolds, namely hyperkähler manifolds and quaternion Kähler manifolds, which have as locally conformal correspondents the two classes: locally conformally hyperkähler manifolds and the larger class of locally conformally quaternion Kähler manifolds. These classes are defined by requesting the compatibility of some quaternion Hermitian or hyperhermitian structure with a Weyl structure. As locally conformally Kähler structures naturally appear in the classification of Hermitian structures according to the irreducible representations of $\mathrm{O}(n)$, lchK and lcqK structures appear when studying the representations of $\mathrm{SP}(n)$ and $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$, see *e.g.* [CS]. Since 1993, when these classes were considered in [PPS], a rather rich literature on the subject appeared, see [OP], [OD], [Or] and the references therein.

*Partially supported by CNCSIS - UEFISCSU, project PNII - IDEI code 8/2008, contract no. 525/2009.

The aim of this paper is to discuss conformal submersions satisfying an appropriate condition of holomorphicity, from lchK manifolds. In the first section, we give the necessary definitions and show that the base manifold, if is hyperhermitian is forced to be lchK too. We study necessary and sufficient conditions for the base space to be hyperkähler manifold.

The next section gives necessary and sufficient conditions for a lchK conformal submersion to be a harmonic map. When is harmonic, the stability of such a map is also studied.

Throughout the paper, all manifolds and structures on them are differentiable, of class C^∞. We assume that all manifolds we work with are connected.

2 Locally conformally hyperkähler τ-conformal submersions

Let M be a $4m$ dimensional C^∞ manifold. A triple J_1, J_2, J_3 of global integrable complex structures on M satisfying the quaternionic identities: $J_\alpha J_\beta = J_\gamma$ for $(\alpha, \beta, \gamma) = (1, 2, 3)$ and cyclic permutations, defines a hypercomplex structure on M. If a Riemannian metric g is added, assumed to be Hermitian with respect to J_1, J_2, J_3, one gets a hyperhermitian manifold (M, g, J_1, J_2, J_3). If the global complex structures J_1, J_2, J_3 on M are not integrable we say that (M, g, J_1, J_2, J_3) is an almost hyperhermitian manifold.

More generally, by (M, g, H) we denote a quaternion Hermitian manifold. Here H is a rank 3 subbundle of $End(TM)$, locally spanned by (not necessarily integrable) almost complex structures J_1, J_2, J_3, again satisfying the quaternionic identities and related on the intersections of trivializing open sets by matrices of SO(3). H defines on M a structure of quaternionic manifold and the local almost complex structures J_1, J_2, J_3 are said to be compatible with the quaternionic structure H. The additional datum of a metric g, Hermitian with respect to the local compatible almost complex structures, defines the quaternion Hermitian manifold (M, g, H).

Recall that the hyperhermitian or quaternion Hermitian metric g is said to be hyperkähler or quaternion Kähler if its Levi-Civita connection ∇ satisfies respectively $\nabla J_\alpha = 0 \quad (\alpha = 1, 2, 3)$ or $\nabla H \subset H$.

We shall always assume that the real dimension of our manifolds is at least 8, as quaternion Hermitian geometry in dimension 4 is less interesting.

We recall the definition of the two classes of manifolds that form the subject of this paper:

Definition 2.1 *([OD])*

(*i*) *A hyperhermitian manifold* (M, g, J_1, J_2, J_3) *is* locally conformally hyperkähler (lchK) *if, over open neighbourhoods* U_i *covering* M, $g|_{U_i} = e^{f_i} g'_i$ *with* g'_i *quaternion Kähler on* U_i.

(*ii*) *A quaternion Hermitian manifold* (M, g, H) *is* locally conformally quaternion Kähler (lcqK) *if, over open neighbourhoods* U_i *covering* M, $g|_{U_i} = e^{f_i} g'_i$ *with* g'_i *quaternion Kähler on* U_i.

In both cases, the *Lee form* ω, locally defined by $\omega|_{U_i} = df_i$, satisfies:

$$d\Omega = \omega \wedge \Omega, \qquad d\omega = 0, \tag{2.1}$$

where $\Omega = \sum_{\alpha=1,2,3} \Omega_\alpha \wedge \Omega_\alpha$ is the (global) Kähler 4-form. Properties (2.1) for Ω are also sufficient for a hyperhermitian or quaternion Hermitian metric to be lchK or lcqK, respectively. We shall denote by B the Lee vector field $\omega^\sharp$ g-metrically equivalent with ω..

Let now (M, g, J_1, J_2, J_3) and $(M', g', J'_1, J'_2, J'_3)$ be almost hyperhermitian compact manifolds.

Definition 2.2 *A* C^∞ *surjective mapping* $\pi : (M, g, J_1, J_2, J_3) \to (M', g', J'_1, J'_2, J'_3)$ *is called an* almost hypercomplex τ-conformal submersion, *where* τ *is a real differentiable function on* M, *if* π *is a submersion,* $\pi_* \circ J_\alpha = J'_\alpha \circ \pi_*$ *for* $\alpha = 1, 2, 3$ *and for all* $x \in M$ *and for all* $X, Y \in T_xM$ *orthogonal to the vertical space at* x *(i.e. orthogonal to* $Ker(d\pi_x)$*),*

$$g'(d\pi_x(X), d\pi_x(Y)) = e^{\tau(x)} g(X, Y). \tag{2.2}$$

If the total space (M, g, J_1, J_2, J_3) *is a lchK manifold, then* π *is called a* lchK τ-conformal submersion.

If $\tau = 0$ then π is a Riemannian submersion and π is a *lchK submersion.*

Vectors which are in $Ker(d\pi_x)$ are tangent to the fiber over x and are called *vertical vectors* at x. Vectors which are in $(Ker(d\pi_x))^\perp = \mathcal{H}$ are said to be *horizontal.* A vector field X on M is said to be vertical (respectively horizontal) if X_x is vertical (respectively horizontal) for all $x \in M$. If X is a vector field on M it may be written uniquely as a sum $X = v(X) + h(X)$, where $v(X)$ is a vertical vector field, the projection of X on the vertical space, and $h(X)$ is a horizontal vector field, the projection of X on the horizontal space. A *basic vector field* is a horizontal vector field X which is π-related to a vector field X' on M'.

Proposition 2.3 *[OP] Let (M, g, J_1, J_2, J_3) be a compact lchK manifold. Then*

$$(\nabla_X J_\alpha)Y = \frac{1}{2}\{\omega(J_\alpha Y)X - \omega(Y)J_\alpha X - g(X, J_\alpha Y)B + g(X,Y)J_\alpha B\} \quad (2.3)$$

for $\alpha = 1, 2, 3$ where X, Y are vector fields on M.

Proposition 2.4 *If $\pi : (M, g, J_1, J_2, J_3) \rightarrow (M', g', J'_1, J'_2, J'_3)$ is lchK τ-conformal submersion then the base space $(M', g', J'_1, J'_2, J'_3)$ is a lchK manifold.*

If B and B' are the Lee vector fields of M and M', respectively, then $e^{-\tau}(2\mathrm{grad}\,\tau + h(B))$ is the basic vector field associated to B'.

Proof From (2.1) it is enough to show that $d\Omega' = \omega' \wedge \Omega'$ (note that, as $\dim M' > 4$, from this equation also follows $d\omega' = 0$), and that J_1, J_2, J_3 are integrable.

For the lchK τ-conformal submersion π we have that $[X', Y'] = d\pi([X,Y])$ and $J'_\alpha X' = J'_\alpha d\pi(X) = d\pi(J_\alpha X)$ for $\alpha = 1, 2, 3$ where X, Y are basic vector field on M which are π-related to the vector fields X', Y' on M'. Also the horizontal and the vertical distributions determined by π are J_α-invariant (see [Wa]). Therefore, since the Nijenhuis tensor fields $N_{J_\alpha} = 0$, we obtain that $N'_{J'_\alpha} = 0$ for $\alpha = 1, 2, 3$. By definition

$$\begin{aligned}\Omega' &= \sum_{\alpha=1,2,3} \Omega'_\alpha \wedge \Omega'_\alpha = \sum_{\alpha=1,2,3} g'(\cdot, J'_\alpha \cdot) \wedge g'(\cdot, J'_\alpha \cdot) = \sum_{\alpha=1,2,3} e^\tau \Omega_\alpha \wedge e^\tau \Omega_\alpha \\ &= e^{2\tau}(\sum_{\alpha=1,2,3} \Omega_\alpha \wedge \Omega_\alpha) = e^{2\tau}\Omega.\end{aligned} \quad (2.4)$$

Then

$$\begin{aligned} d\Omega' &= d(e^{2\tau}\Omega) = de^{2\tau} \wedge \Omega + e^{2\tau} d\Omega \\ &= 2e^{2\tau}(d\tau) \wedge \Omega + e^{2\tau}(\omega \wedge \Omega) \\ &= 2d\tau \wedge (e^{2\tau}\Omega) + \omega \wedge (e^{2\tau}\Omega) \\ &= (2d\tau + \omega) \wedge \Omega' \\ &= \omega' \wedge \Omega'. \end{aligned} \quad (2.5)$$

We obtained $d\Omega' = \omega' \wedge \Omega'$ where

$$\omega' = 2d\tau + \omega. \tag{2.6}$$

Moreover, the Lee vector field B' of M' is defined by

$$B' = e^{-\tau} d\pi(2\text{grad}\,\tau\, + B). \tag{2.7}$$

Now, if C is the basic vector field associated to B' then, using (2.6), we have:

$$\begin{aligned} g'(B', X') = \omega'(X') &= 2d\tau(X) + \omega(X) = g(2\text{grad}\,\tau\, + h(B), X) \\ &= e^{-\tau} g'(d\pi(2\text{grad}\,\tau\, + h(B)), d\pi(X)) \end{aligned}$$

and

$$g(C, X) = g(e^{-\tau}(2\text{grad}\,\tau\, + h(B)), X),$$

for all basic vector field X on M. Therefore $C = e^{-\tau}(2\text{grad}\,\tau\, + h(B))$.

□

We now obtain necessary and sufficient conditions for the base space of a lchK τ-conformal submersion to be hyperkähler manifold.

Proposition 2.5 *If $\pi : (M, g, J_1, J_2, J_3) \to (M', g', J_1', J_2', J_3')$ is a lchK τ-conformal submersion, then the following conditions are equivalent:*

1. *The vector field $2\text{grad}\,\tau\, + B$ is vertical where B is the Lee vector field of M .*
2. *If F is a fiber of submersion π then the mean curvature vector field of F is $\mu^F = \frac{1}{2} h(\text{grad}\,\tau\,)|_F$.*
3. *The lchK manifold M' is a hyperkähler manifold.*

Proof Let F be a hypercomplex submanifold of M. It is known from [Vai] that its mean curvature vector field is the normal component (with respect to F) B^n of the Lee vector field.

Let now F be a fiber of submersion π. Then the mean curvature vector field μ^F of F is given by [MR]

$$\overline{\mu}^F = e^{-\tau}(\mu^F - \frac{1}{2}(\text{grad}\,\tau\,)^n\,|_F \tag{2.8}$$

where $\overline{\mu}^F$ is the mean curvature vector field of F with respect to the metric $\overline{g} = e^{\tau} g$ and $(\text{grad}\,\tau\,)^n = h(\text{grad}\,\tau\,)$ is the normal component to F of $\text{grad}\,\tau$.

But the submersion π defines a lchK submersion between manifold $(M,\overline{g},J_1,J_2,J_3)$ and (M',g',J_1',J_2',J_3'). Since this submersion is Riemannian we obtain that $\overline{\mu}^F = \overline{B}^n = h(\overline{B}) = h(e^{-\tau}(2\mathrm{grad}\,\tau + B) = e^{-\tau}(2h(\mathrm{grad}\,\tau) + h(B))$.

We now prove the equivalence of (1) and (2). If the vector field $2\mathrm{grad}\,\tau + B$ is vertical then $\overline{\mu}^F = 0$ and this implies (2). If (2) is true then from (2.8) we obtain that $\overline{\mu}^F = 0$ and (1) follows by the definition of $\bar{\mu}^F$.

Assuming (1), we now want to show that $(\nabla'_{X'}J'_\alpha)Y' = 0$ for $\alpha = 1,2,3$ where X', Y' are vector fields on M'. If $2\mathrm{grad}\,\tau + B$ is vertical, then $2h(\mathrm{grad}\,\tau) + h(B) = 0$. Then $B' = e^{-\tau}d\pi(2\mathrm{grad}\,\tau + B) = 0$ and so $\omega' = 0$. Hence, from (2.3) and Proposition 2.4 we obtain (3).

Conversely, if (3) is true then $(\nabla'_{X'}J'_\alpha)Y' = 0$ for $\alpha = 1,2,3$ where X', Y' are vector fields on M'. By a simple calculation with the definition we obtain that

$$(\nabla'_{X'}\Omega'_\alpha)(Y',Z') = -g((\nabla'_{X'}J'_\alpha)Y',Z') \tag{2.9}$$

From here and from ([KN], p.148) we have:

$$\begin{aligned} d\Omega'_\alpha(X',Y',Z') &= \circlearrowleft_{X',Y',Z'} (\nabla'_{X'}\Omega'_\alpha)(Y',Z') \\ &= -\circlearrowleft_{X',Y',Z'} g((\nabla'_{X'}J'_\alpha)Y',Z'), \end{aligned} \tag{2.10}$$

where $\circlearrowleft$ is the cyclic sum. Hence $d\Omega'_\alpha = 0$ and so $d\Omega' = 0$. But $d\Omega' = \omega' \wedge \Omega'$ so that we derive $\omega' = 0$ and (1) follows.

□

Corollary 2.6 *If $\pi : (M,g,J_1,J_2,J_3) \to (M',g',J_1',J_2',J_3')$ is a lchK submersion, then the following conditions are equivalent:*

1. *The Lee vector field B of M is vertical.*
2. *The fibers of π are minimal submanifolds of M.*
3. *M' is a hyperkähler manifold.*

It is known, see [OP], that the Lee form pf a compact lchk manifold is parallel (the manifold thus having three nested Vaisman structures). On the other hand, it is proven in [Ve] that the Lee form of a compact Vaisman manifold restricts to any compact complex submanifold (the latter one being Vaisman too). As, by holomorphicity, the fibers of the vertical distribution are trivially complex submanifolds, we obtain:

Corollary 2.7 *If $\pi : (M, g, J_1, J_2, J_3) \to (M', g', J'_1, J'_2, J'_3)$ is a lchK τ-conformal submersion with compact fibers from a compact lchk manifold, then the mean curvature vector field of a fiber F is $\mu^F = \frac{1}{2}h(\mathrm{grad}\,\tau\,)|_F$. So M' is a hyperkähler manifold and $2\mathrm{grad}\,\tau\, + B$ is vertical.*

3 Harmonic maps on locally conformally hyperkäler manifolds

The second fundamental form α_π of a map $\pi : (M^m, g) \to (M'^{m'}, g')$ between two Riemannian manifolds of dimensions m, respectively m', is defined by $\alpha_\pi(X, Y) = \nabla^\pi_X \pi_* Y - \pi_* \nabla_X Y$, for any vector fields X, Y on M, where ∇ is the Levi-Civita connection of M and ∇^π is the pullback of the connection ∇' of M' to the induced vector bundle $\pi^{-1}(TM')$, $\nabla^\pi_X \pi_* Y = \nabla'_{\pi_* X} \pi_* Y$.

The tension field $\tau(\pi)$ of π is defined as the trace of second fundamental form of π:

$$\tau(\pi)_x = \sum_{i=1}^{m} \alpha_\pi(e_i, e_i), \tag{3.1}$$

where $\{e_1, \dots, e_m\}$ is a local orthonormal frame of $T_x M$, $x \in M$. We say that π is a harmonic map if and only if $\tau(\pi)$ vanishes at each point $x \in M$.

Let (M^m, g) be a compact Riemannian manifold and let $\pi : (M^m, g) \to (M'^{m'}, g')$ be a harmonic map. Let $\pi_{s,t}$ be a smooth variation, with parameters $s, t \in (-\varepsilon, -\varepsilon)$, and with $\pi_{0,0} = \pi$. The corresponding variational vector fields are denoted by V and W.

The Hessian of a harmonic map π is defined by: $Hess_\pi(V, W) = \frac{\partial^2}{\partial s \partial t}|_{(s,t)=(0,0)} E(\pi_{s,t})$ where $E(\pi_{s,t}) = \frac{1}{2}\int_M |d\pi_{s,t}|^2 \vartheta_g$ is the energy of π over M.

From the second variation of the energy, the Hessian of π for any V and W vector fields along π is given by ([BW], p. 91):

$$\begin{aligned} Hess_\pi(V, W) &= \int_M g'(J_\pi(V), W)\vartheta_g \\ &= \int_M g'(-trace(\nabla^\pi)^2 V - traceR^{M'}(V, d\pi)d\pi, W)\vartheta_g. \end{aligned} \tag{3.2}$$

where J_π, called the Jacobi operator, is acting on the space of variational vector fields along π, $\Gamma(\pi^{-1}(TM'))$.

Let $J_\pi := \overline{\Delta}_\pi - \mathcal{R}_\pi$, where $\overline{\Delta}_\pi$ is the rough Laplacian operator defined by

$$\overline{\Delta}_\pi V := -\sum_{i=1}^{m} (\nabla^\pi_{e_i} \nabla^\pi_{e_i} - \nabla^\pi_{\nabla_{e_i} e_i})V, \quad V \in \Gamma(\pi^{-1}TM')). \tag{3.3}$$

By definition (see [BW], p. 92), a harmonic map defined on a compact manifold is energy-stable if the Hessian for the energy is positive semi-definite, *i.e.* $Hess_\pi(V,V) \geq 0$ for $V \in \Gamma(\pi^{-1}TM')$. Otherwise, it is called unstable. One the other hand we can define this using the notion of index. The index of a harmonic map $\pi : (M,g) \to (M',g')$ is defined as the dimension of the largest subspace of $\Gamma(\pi^{-1}(TM'))$ on which the Hessian $Hess_\pi$ is negative definite. A harmonic map π is said to be stable if the index of π is zero and otherwise, is said to be unstable.

We recall a result by Fuglede and Ishihara: A smooth map between Riemannian manifolds is a harmonic morphism if and only if it is both harmonic and horizontally weakly conformal (see [BW], p. 108, Theorem 4.2.2).

Recall now a property of the harmonic maps:

Theorem 3.1 *([BE]) Suppose that $\pi : (M^m, g) \to (M'^{m'}, g')$ is a non-constant horizontally weakly conformal map between Riemannian manifolds with $\dim M' > 3$. Then any two of the following assertions imply the third:*

(i) π is a harmonic map (and so a harmonic morphism);

(ii) π is horizontally homothetic, i.e. the gradient of the dilation is vertical;

(iii) The fibers of π are minimal submanifolds of M.

Now we consider a lchK τ-conformal submersion and we find necessary and sufficient conditions for it to be a harmonic map.

The following technical fact can be proven by direct computation:

Proposition 3.2 *Let (M,g,J_1,J_2,J_3) and (M',g',J_1',J_2',J_3') be two almost hyperhermitian compact manifolds. If $\pi : M \to M'$ is a map such that $\pi_* \circ J_\alpha = J_\alpha' \circ \pi_*$ for $\alpha = 1,2,3$, then we have*

$$\tau(\pi) = J_\alpha'(trace_g \pi^* \nabla' J_\alpha') - \pi_*(J_\alpha div J_\alpha) \tag{3.4}$$

for all $\alpha = 1,2,3$.

Theorem 3.3 *Let $\pi : (M,g,J_1,J_2,J_3) \to (M',g',J_1',J_2',J_3')$ be a lchK τ-conformal submersion. Then π is a harmonic map if and only if the gradient of the function τ is vertical. Moreover, if π is a harmonic map then π is a horizontally homothetic harmonic morphism .*

Proof Let $\{e_1, \ldots, e_m, J_1e_1, \ldots, J_1e_m, J_2e_1, \ldots, J_2e_m, J_3e_1, \ldots, J_3e_m\}$ be an orthonormal basis of T_xM, $x \in M$. Since M is a lchK manifold, calculating each term of the formula (3.4) we obtain:

$$\begin{aligned} divJ_\alpha &= trace\nabla J_\alpha = \sum_{i=1}^{m}\{(\nabla_{e_i}J_\alpha)e_i + \sum_{\beta=1}^{3}(\nabla_{J_\beta e_i}J_\alpha)J_\beta e_i\} \\ &= \frac{1}{2}\sum_{i=1}^{m}\{\omega(J_\alpha e_i)e_i - \omega(e_i)J_\alpha e_i - g(e_i, J_\alpha e_i)B + g(e_i, e_i)J_\alpha B \\ &+ \sum_{\beta=1}^{3}[\omega(J_\alpha J_\beta e_i)J_\beta e_i - \omega(J_\beta e_i)J_\alpha J_\beta e_i \\ &- g(J_\beta e_i, J_\alpha J_\beta e_i)B + g(J_\beta e_i, J_\beta e_i)J_\alpha B]\} \\ &= \sum_{i=1}^{m}\{\omega(J_\alpha e_i)e_i - \omega(e_i)J_\alpha e_i + J_\alpha B\} \\ &+ \frac{1}{2}\sum_{i=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3}\{\omega(J_\alpha J_\beta e_i)J_\beta e_i - \omega(J_\beta e_i)J_\alpha J_\beta e_i + J_\alpha B\} \end{aligned} \tag{3.5}$$

Similarly we find:

$$\begin{aligned} trace_g\pi^*\nabla' J'_\alpha &= \sum_{i=1}^{m}\{(\nabla'_{\pi_*e_i}J'_\alpha)\pi_*e_i + \sum_{\beta=1}^{3}(\nabla'_{J'_\beta\pi_*e_i}J'_\alpha)J'_\beta\pi_*e_i\} \\ &= \frac{1}{2}\sum_{i=1}^{m}\{\omega'(J'_\alpha\pi_*e_i)\pi_*e_i - \omega'(\pi_*e_i)J'_\alpha\pi_*e_i - g'(\pi_*e_i, J'_\alpha\pi_*e_i)B' \\ &+ g'(\pi_*e_i, \pi_*e_i)J'_\alpha B' + \sum_{\beta=1}^{3}[\omega'(J'_\alpha J'_\beta\pi_*e_i)J'_\beta\pi_*e_i \\ &- \omega'(J'_\beta\pi_*e_i)J'_\alpha J'_\beta\pi_*e_i - g'(J'_\beta\pi_*e_i, J'_\alpha J'_\beta\pi_*e_i)B' \\ &+ g'(J'_\beta\pi_*e_i, J'_\beta\pi_*e_i)J'_\alpha B']\} \\ &= \sum_{i=1}^{m}\{\omega'(\pi_*J_\alpha e_i)\pi_*e_i - \omega'(\pi_*e_i)\pi_*J_\alpha e_i + g'(\pi_*e_i, \pi_*e_i)J'_\alpha B'\} \\ &+ \frac{1}{2}\sum_{i=1}^{m}\sum_{\beta=1,\beta\neq\alpha}^{3}\{\omega'(\pi_*J_\alpha J_\beta e_i)\pi_*J_\beta e_i - \omega'(\pi_*J_\beta e_i)\pi_*J_\alpha J_\beta e_i \\ &+ g'(\pi_*e_i, \pi_*e_i)J'_\alpha B'\} \end{aligned} \tag{3.6}$$

We derive:

$$trace_g \pi^* \nabla' J'_\alpha = \pi_*(div J_\alpha) + 2(2m-1)\pi_*(J_\alpha \mathrm{grad}\,\tau\ ),$$

which implies $\tau(\pi) = 2(1-2m)\pi_*(\mathrm{grad}\,\tau\ )$.

Now, from the definition of a lchK τ-conformal submersion we deduce that π is a horizontally conformal submersion (in the terminology of [BW]). From the result of Fuglede and Ishihara, if π is a harmonic map, then π is a harmonic morphism and since the gradient of τ is vertical we find that π is horizontally homothetic.

□

Combining Proposition 2.5, Theorem 3.1 and Theorem 3.3, we also obtain:

Proposition 3.4 *If $\pi : (M, g, J_1, J_2, J_3) \to (M', g', J'_1, J'_2, J'_3)$ is a lchK τ-conformal submersion then π is a harmonic map if and only if the Lee vector field B of M is vertical and the lchK manifold M' is a hyperkähler manifold.*

Corollary 3.5 *If $\pi : (M, g, J_1, J_2, J_3) \to (M', g', J'_1, J'_2, J'_3)$ is a lchK submersion then π is a harmonic map, the Lee vector field B of M is vertical and the lchK manifold M' is a hyperkähler manifold.*

As regards the stability, we have the following result:

Theorem 3.6 *Let $\pi : (M, g, J_1, J_2, J_3) \to (M', g', J'_1, J'_2, J'_3)$ be a lchK τ-conformal submersion between compact locally conformally hyperkähler manifolds. If the gradient of the function τ is vertical then π is a stable map.*

Proof From Theorem 3.3, π is a harmonic map. From Proposition 3.4, we know that the lchK manifold M' is a hyperkähler manifold. It then follows that M' is Ricci flat ([Bs], p.398).

Since π is a lchK τ-conformal submersion we obtain that $g'(traceR^{M'}(V, d\pi)d\pi, V) = e^\tau Ricci^{M'}(V, V)$ where $Ricci^{M'}$ is the Ricci curvature of M' and $V \in \Gamma(\pi^{-1}(TM'))$.

From (3.2) then obtain:

$$\begin{aligned} Hess_\pi(V, V) &= \int_M g'(-trace(\nabla^\pi)^2 V - traceR^{M'}(V, d\pi)d\pi, V)\vartheta_g \\ &= \int_M g'(\nabla^\pi V, \nabla^\pi V)\vartheta_g. \end{aligned} \tag{3.7}$$

for any $V \in \Gamma(\pi^{-1}(TM'))$.

This implies that $Hess_\pi(V,V) \geq 0$, $\forall V \in \Gamma(\pi^{-1}(TM'))$ and hence π is a stable harmonic map.

$\square$.

Acknowledgments

The author wishes to thank Prof. L. Ornea for valuable discussions and remarks.

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