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HARMONIC MAPS AND RIEMANNIAN SUBMERSIONS BETWEEN MANIFOLDS ENDOWED WITH SPECIAL STRUCTURES

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Abstract. It is well known that Riemannian submersions are of interest in physics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories. In this paper we investigate some classes of Riemannian submersions between manifolds endowed with special geometric structures.

1. Introduction. The motivation for study harmonic maps and Riemannian submersions comes from theoretical physics (see e.g. Chapter 8 of [FIP]). Presently, we see an increasing interest in harmonic maps between (pseudo-)Riemannian manifolds which are endowed with certain special geometric structure (like almost Hermitian structures [Chi1, Kob, LY], almost metric contact structures [BG, BS, Bur, Fet2], f-structures [BB, Erd, FP, Fet1], (para)quaternionic structures [Man, Sah, SAS, Vil1, Vil2]). In this

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article we give a survey and some new results concerning harmonic maps and Riemannian submersions between manifolds endowed with remarkable geometric structures. The paper is organized as follows. In Section 2 we recall some definitions and properties of harmonic maps between almost contact manifolds and on a generalization of almost contact structures, namely f.pk-structures. In Section 3 we recall the notions of quaternionic manifold, quaternionic submersion and present some properties. In the last two sections of this paper we study holomorphic maps and semi-Riemannian submersions between manifolds endowed with metric mixed 3-structures.

2. Harmonic maps between almost contact manifolds.

2.1. Manifolds endowed with almost contact structures. Let M be a differentiable manifold equipped with a triple (φ, ξ, η), where φ is a field of endomorphisms of the tangent spaces, ξ is a vector field and η is a 1-form on M. If we have:

$$\varphi^2 = -Id + \eta \otimes \xi, \ \ \eta(\xi) = 1$$

then we say that (φ, ξ, η) is an almost contact structure on M (see [Bla]). Moreover, if g is a Riemannian metric associated on M, i.e. a metric satisfying, for any sections X and Y in $\Gamma(TM)$,

$$g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y)$$

then we say that (φ, ξ, η, g) is an almost contact metric structure. A manifold equipped with such structure is called an *almost contact metric manifold*.

If the Nijenhuis tensor N^φ satisfies

$$N^\varphi + 2d\eta \otimes \xi = 0$$

we say that the *almost contact metric structure* (φ, ξ, η, g) is *normal*.

A *contact manifold* is a $(2n + 1)$-dimensional manifold M together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. We say that $(M, \varphi, \xi, \eta, g)$ is a *Sasakian manifold* if it is a normal contact metric manifold such that $\Phi = d\eta$, where Φ is the second fundamental form on M defined for any $X, Y \in \Gamma(TM)$ by

$$\Phi(X, Y) = g(X, \varphi Y) \tag{1}$$

2.2. Holomorphic and harmonic maps. Let $\psi : (M^m, g) \to (N^n, h)$ be a smooth map between two (semi-)Riemannian manifolds. The *norm of* $d\psi$ is given by $\|d\psi\|^2 := Tr_g(\psi^* h)$ and the *energy density* of ψ is a smooth function $e(\psi) : M \to [0, \infty)$ defined by: $e(\psi)_x = \frac{1}{2}\|d\psi_x\|^2, \ \ x \in M$. Next we will write ψ_* instead of $d\psi$.

For any compact $\Omega \subseteq M$, the *energy of* ψ over Ω is the integral of its energy density

$$E(\psi; \Omega) = \int_\Omega e(\psi)\vartheta_g,$$

where ϑ_g is the volume measure associated to g. A smooth map $\psi : M \to N$ is said to be a *harmonic map* if $\frac{d}{dt}|_{t=0} E(\psi_t; \Omega) = 0$, for all compact domains $\Omega \subseteq M$ and for all variations $\{\psi_t\}_{t \in (-\epsilon, \epsilon)}$ of ψ supported in Ω, such that $\psi_0 = \psi$. Equivalently, the map ψ is harmonic if the tension field $\tau(\psi)$ of ψ vanishes at each point $x \in M$, where $\tau(\psi)$ is defined as the trace of the second fundamental form α_ψ of ψ, i.e.:

$$\tau(\psi)_x = \sum_{i=1}^{m} \epsilon_i \alpha_\psi(e_i, e_i),$$

where $\{e_1, e_2, ..., e_m\}$ is a local pseudo-orthonormal frame of $T_x M$, $x \in M$, with $\epsilon_i = g(e_i, e_i) \in \{\pm 1\}$. The quantity α_ψ is defined by:

$$\alpha_\psi(X, Y) = \widetilde{\nabla}_X \psi_* Y - \psi_* \nabla_X Y, \tag{2}$$

for any vector fields X, Y on M, where ∇ is the Levi-Civita connection of M and $\widetilde{\nabla}$ is the pullback of the Levi-Civita connection ∇' of N to the induced vector bundle $f^{-1}(TN)$:

$$\widetilde{\nabla}_X \psi_* Y = \nabla'_{\psi_* X} \psi_* Y.$$

We consider now $\{\psi_{s,t}\}_{s,t \in (-\epsilon, \epsilon)}$ a smooth two-parameter variation of ψ such that $\psi_{0,0} = \psi$ and let $V, W \in \Gamma(\psi^{-1}(TN))$ be the corresponding variational vector fields:

$$V = \frac{\partial}{\partial s}(\psi_{s,t})|_{(s,t)=(0,0)},\ W = \frac{\partial}{\partial t}(\psi_{s,t})|_{(s,t)=(0,0)}.$$

The Hessian of a harmonic map ψ is defined by:

$$H_\psi(V, W) = \frac{\partial^2}{\partial s \partial t}(E(\psi_{s,t}))|_{(s,t)=(0,0)}.$$

The index of a harmonic map $\psi : (M, g) \to (N, h)$ is defined as the dimension of the largest subspace of $\Gamma(\psi^{-1}(TN))$ on which the Hessian H_ψ is negative definite. A harmonic map ψ is said to be *stable* if the index of ψ is zero and otherwise, is said to be *unstable.* Concerning the stability of the identity map on Sasakian manifolds we have the following result.

THEOREM 2.1. (see [GIP]) *Let $M(\varphi, \xi, \eta, g)$ be a Sasakian compact manifold of constant φ - sectional curvature c, such that $c \leq 1$. If the first eigenvalue of the Laplacian $\triangle_g$ acting on $C^\infty(M, \mathbf{R})$ satisfies*

$$\lambda_1 < c(n+1) + 3n - 1,$$

then the identity map $1_{|M}$ is a harmonic unstable map.

DEFINITION 2.2. (see [GIP]) A smooth map $\psi : (M^{2m+1}, \varphi, \xi, \eta, g) \to (N^{2n}, h, J)$ from an almost contact manifold to an almost Hermitian manifold is called $(\varphi, J)-$ *holomorphic map* if $\psi_* \circ \varphi = J \circ \psi_*$.

THEOREM 2.3. (see [IP2]) *Let $M(\varphi, \xi, \eta, g)$ be a $(2n+1)$ - dimensional Sasakian compact manifold of constant φ - sectional curvature c and $N(J, h)$ a Kähler manifold. Then any nonconstant (φ, J) - holomorphic map from M to N is an unstable harmonic map if*

$$c > -\frac{3(n-1)}{n+1},\ n \geq 1.$$

DEFINITION 2.4. (see [FIP]) A map $\psi : (M, \varphi, \xi, \eta, g) \to (M', \varphi', \xi', \eta', g')$ between almost contact manifolds is said to be (φ, φ') - *holomorphic map* if $\psi_* \circ \varphi = \varphi' \circ \psi_*$.

THEOREM 2.5. (see [IP1]) *Let $M(\varphi, \xi, \eta, g)$ and $N(\varphi', \xi', \eta', g')$ be almost contact manifolds. Then any (φ, φ') - holomorphic map $\psi : M \to N$ is a harmonic map.*

2.3. A generalization of almost contact structures: *f*.pk-structures. An f - *structure* on a manifold M is a non-vanishing endomorphism of TM, that satisfies $f^3 + f = 0$ and which has constant rank $2n$. We have the split of the tangent bundle

$$TM = D \oplus D^{'} = Imf \oplus Kerf$$

in two complementary subbundle. When $D^{'}$ is parallelizable f is called an f *-structure with parallelizable kernel (*f.*pk-structure)* and $dimM = 2n+s$. Then, there exists a global frame ξ_i, $1 \leq i \leq s$, for the subbundle $D^{'}$ and 1 - forms η^i, $1 \leq i \leq s$ such that:

$$f^2 = -Id + \eta^i \otimes \xi_i, \ \eta^i(\xi_j) = \delta^i_j.$$

We say that an f-structure is *normal* if

$$N^f + 2d\eta^i \otimes \xi_i = 0.$$

A metric *f*.pk-structure (f, η^i, ξ_i, g), where the Riemannian metric g satisfies

$$g(X, Y) = g(fX, fY) + \eta^i(X)\eta^i(Y)$$

is called a *K-structure* if the corresponding 2-form Φ, defined by $\Phi(X, Y) = g(X, fY)$, for any vector fields X and Y on M, is closed and the normality condition holds.

An *almost C-manifold* is a manifold endowed with a metric f.pk-structure with $d\Phi = 0$ and $d\eta^i = 0$, for any $i \in \{1, ..., s\}$. An *almost C-manifold with Kählerian leaves* is an almost C-manifold with any leaf of canonical foliation Kählerian (see also [Ols] for the case of almost cosymplectic manifolds with Kählerian leaves). Concerning (f, J)-holomorphic maps between an almost C-manifold with Kählerian leaves and a Kähler manifold, we have the following.

Theorem 2.6. (see [IP2]) *Let $M(f, \eta^i, \xi_i, g)$ be an almost C-manifold with Kählerian leaves and $N(J, h)$ a Kähler manifold. Then, any (f, J) - holomorphic map $\psi : M \longrightarrow N$ is a harmonic map. Moreover, if M is a compact manifold then ψ is stable.*

3. Harmonic maps and submersions between quaternionic manifolds. If (M, g) and (N, g') are two Riemannian manifolds, then a surjective C^∞-map $\pi : M \to N$ is said to be a C^∞*-submersion* if it has maximal rank at any point of M. Putting $\mathcal{V}_x = Ker\ \pi_{*x}$, for any $x \in M$, we obtain an integrable distribution $\mathcal{V}$, which is called *vertical* distribution and corresponds to the foliation of M determined by the fibres of π. The complementary distribution $\mathcal{H}$ of $\mathcal{V}$, determined by the Riemannian metric g, is called *horizontal* distribution. A C^∞-submersion $\pi : M \to N$ between two Riemannian manifolds (M, g) and (N, g') is called a *Riemannian submersion* if, at each point x of M, π_{*x} preserves the length of the horizontal vectors (see [ON1]). We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields.

Definition 3.1. (see [Ish]) An *almost quaternionic structure* on a differentiable manifold M of dimension n is a rank 3-subbundle σ of $End(TM)$ such that a local basis $\{J_1, J_2, J_3\}$ exists on sections of σ satisfying for all $\alpha \in \{1, 2, 3\}$

$$J^2_\alpha = -Id, \ J_\alpha J_{\alpha+1} = -J_{\alpha+1}J_\alpha = J_{\alpha+2} \tag{3}$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3. Moreover, (M, σ) is said to be an *almost quaternionic manifold.* A Riemannian metric g is said to be *adapted* to an almost quaternionic structure σ on a manifold M if

$$g(J_\alpha X, J_\alpha Y) = g(X, Y), \forall \alpha \in \{1, 2, 3\} \tag{4}$$

for all vector fields X,Y on M. In this case (M, σ, g) is said to be an *almost quaternionic Hermitian manifold.* Moreover, (M, σ, g) is said to be a *quaternionic Kähler manifold* if the bundle σ is parallel with respect to the Levi-Civita connection ∇ of g.

Definition 3.2. (see [IMV3]) Let (M, σ, g) and $(N, \sigma^{'}, g^{'})$ be two almost quaternionic Hermitian manifolds. A map $f : M \longrightarrow N$ is called a $(\sigma, \sigma^{'})$*-holomorphic map* at a point x of M if for any $J \in \sigma_x$ exist $J^{'} \in \sigma^{'}_{f(x)}$ such that $f_* oJ = J^{'} of_*$. Moreover, we say that f is a $(\sigma, \sigma^{'})$ - holomorphic map if f is a $(\sigma, \sigma^{'})$-holomorphic map at each point $x \in M$.

Definition 3.3. (see [IMV2]) Let (M, σ, g) and $(N, \sigma^{'}, g^{'})$ be two almost quaternionic Hermitian manifolds. A Riemannian submersion $\pi : M \longrightarrow N$ which is a $(\sigma, \sigma^{'})$ - holomorphic map is called *quaternionic submersion.* Moreover, if (M, σ, g) is a quaternionic Kähler manifold, then we say that π is a *quaternionic Kähler submersion.*

Theorem 3.4. (see [IMV3]) *Let (M, σ, g) and $(N, \sigma^{'}, g^{'})$ be two quaternionic Kähler manifolds. If $f : M \longrightarrow N$ is a $(\sigma, \sigma^{'})$-holomorphic map such that, for any local section $J \in \Gamma(\sigma)$ and corresponding $J^{'} \in \Gamma(\sigma^{'})$ one has $(\nabla^{'}_{f_* X} J^{'}) of_* = f_* o(\nabla_X J)$, for any local vector field X on M, then f is a harmonic map.*

Corollary 3.5. (see [IMV2]) *Any quaternionic Kähler submersion is a harmonic map.*

Theorem 3.6. (Stability of the $(\sigma, \sigma^{'})$ - holomorphic maps, [IMV3]) *Let (M^{4m}, σ, g) and $(N^{4n}, \sigma^{'}, g^{'})$ be two quaternionic Kähler manifolds such that M is compact, N has non positive scalar curvature and, in any point $p \in M$, exists a basis $\{J_1, J_2, J_3\}$ of σ_p such that one of J_1, J_2 or J_3 is parallel. If $f : M \longrightarrow N$ is a $(\sigma, \sigma^{'})$ - holomorphic map such that, for any local section $J \in \Gamma(\sigma)$ and corresponding $J^{'} \in \Gamma(\sigma^{'})$ one has $(\nabla^{'}_{f_* X} J^{'}) of_* = f_* o(\nabla_X J)$, for any local vector field X on M, then f is stable.*

Corollary 3.7. (see [IMV2]) *If $\pi : (M, \sigma, g) \longrightarrow (N, \sigma^{'}, g^{'})$ is a quaternionic Kähler submersion, then the fibres are totally geodesic quaternionic Kähler submanifolds.*

Example 3.8. Let (M, σ, g) be an almost quaternionic hermitian manifold and TM be the tangent bundle, endowed with the metric:

$$G(A, B) = g(KA, KB) + g(\pi_* A, \pi_* B), \ \forall A, B \in T(TM),$$

where π is the natural projection of TM onto M and K is the connection map (see [Dom]).

We remark that if $X \in \Gamma(TM)$, then there exists exactly one vector field on TM called the "horizontal lift" (resp. "vertical lift") of X such that for all $U \in TM$:

$$\pi_* X^h_U = X_{\pi(U)}, \ \pi_* X^v_U = 0_{\pi(U)}, \ KX^h_U = 0_{\pi(U)}, \ KX^v_U = X_{\pi(U)}.$$

We define three tensor fields J_1', J_2', J_3' on TM by the equalities:

$$J_\alpha' X^h = (J_\alpha X)^h, \ J_\alpha' X^v = (J_\alpha X)^v, \ \forall \alpha \in \{1, 2, 3\},$$

where $\{J_1, J_2, J_3\}$ is a canonical local basis of σ.

If we consider now the vector bundle σ' over TM generated by $\{J_1', J_2', J_3'\}$, then we have that (TM, σ', G) is an almost quaternionic hermitian manifold and the natural projection $\pi : TM \to M$ is a quaternionic submersion (see [IMV2]).

We note that the results from this section were recently extended to the class of manifolds endowed with quaternionic structures of second kind (paraquaternionic structures) and compatible metrics in [Cal]. The counterpart in odd dimension of a paraquaternionic structure, called mixed 3-structure, was introduced in [IMV1]. This concept, which arises in a natural way on lightlike hypersurfaces in paraquaternionic manifolds, has been refined in [CP], where the authors have introduced positive and negative metric mixed 3-structures. Next we study holomorphic maps from manifolds endowed with such kind of structures.

4. Holomorphic maps between manifolds endowed with mixed 3-structures and compatible metrics. Let M be a smooth manifold equipped with a triple (φ, ξ, η), where ϕ is a field of endomorphisms of the tangent spaces, ξ is a vector field and η is a 1-form on M. If we have:

$$\varphi^2 = Id - \eta \otimes \xi, \quad \eta(\xi) = 1 \tag{5}$$

then we say that (φ, ξ, η) is an almost paracontact structure on M (cf. [Sat]).

Definition 4.1. [CP] A *mixed 3-structure* on a smooth manifold M is a triple of structures $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha \in \{1, 2, 3\}$, which are almost paracontact structures for $\alpha = 1, 2$ and almost contact structure for $\alpha = 3$, satisfying the following conditions:

$$\eta_\alpha(\xi_\beta) = 0, \tag{6}$$

$$\varphi_\alpha(\xi_\beta) = \tau_\beta \xi_\gamma, \quad \varphi_\beta(\xi_\alpha) = -\tau_\alpha \xi_\gamma, \tag{7}$$

$$\eta_\alpha \circ \varphi_\beta = -\eta_\beta \circ \varphi_\alpha = \tau_\gamma \eta_\gamma\,, \tag{8}$$

$$\varphi_\alpha \varphi_\beta - \tau_\alpha \eta_\beta \otimes \xi_\alpha = -\varphi_\beta \varphi_\alpha + \tau_\beta \eta_\alpha \otimes \xi_\beta = \tau_\gamma \varphi_\gamma\,, \tag{9}$$

where (α, β, γ) is an even permutation of (1,2,3) and $\tau_1 = \tau_2 = -\tau_3 = -1$.

Moreover, if a manifold M with a mixed 3-structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$ admits a semi-Riemannian metric g such that:

$$g(\varphi_\alpha X, \varphi_\alpha Y) = \tau_\alpha [g(X, Y) - \varepsilon_\alpha \eta_\alpha(X) \eta_\alpha(Y)], \tag{10}$$

for all $X, Y \in \Gamma(TM)$ and $\alpha \in \{1, 2, 3\}$, where $\varepsilon_\alpha = g(\xi_\alpha, \xi_\alpha) = \pm 1$, then we say that M has a *metric mixed 3-structure* and g is called a *compatible* metric.

Remark 4.2. If $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ is a manifold with a metric mixed 3-structure then from (7) and (10) we can easily obtain

$$\eta_\alpha(X) = \varepsilon_\alpha g(X, \xi_\alpha), \; g(\varphi_\alpha X, Y) = -g(X, \varphi_\alpha Y) \tag{11}$$

and

$$g(\xi_1, \xi_1) = g(\xi_2, \xi_2) = -g(\xi_3, \xi_3).$$

Hence the vector fields ξ_1 and ξ_2 are both either space-like or time-like and these force the causal character of the third vector field ξ_3. We may therefore distinguish between positive and negative metric mixed 3-structures, according as ξ_1 and ξ_2 are both space-like, or both time-like vector fields. Because at each point of M, there always exists a pseudo-orthonormal frame field given by $\{(E_i, \varphi_1 E_i, \varphi_2 E_i, \varphi_3 E_i)_{i=\overline{1,n}}, \xi_1, \xi_2, \xi_3\}$ we conclude that the dimension of the manifold is $4n+3$ and the signature of $\overline{g}$ is $(2n+1, 2n+2)$, if the metric mixed 3-structure is positive (i.e. $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1$), or the signature of g is $(2n+2, 2n+1)$, if the metric mixed 3-structure is negative (i.e. $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -1$).

Definition 4.3. [CP] Let $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ be a manifold with a metric mixed 3-structure.
(i) If $(\varphi_1, \xi_1, \eta_1, g)$, $(\varphi_2, \xi_2, \eta_2, g)$ are para-cosymplectic structures and $(\varphi_3, \xi_3, \eta_3, g)$ is a cosymplectic structure, i.e. the Levi-Civita connection ∇ of g satisfies

$$\nabla \varphi_\alpha = 0 \tag{12}$$

for all $\alpha \in \{1,2,3\}$, then $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ is said to be a mixed 3-cosymplectic structure on M.
(ii) If $(\varphi_1, \xi_1, \eta_1, g)$, $(\varphi_2, \xi_2, \eta_2, g)$ are para-Sasakian structures and $(\varphi_3, \xi_3, \eta_3, g)$ is a Sasakian structure, i.e.

$$(\nabla_X \varphi_\alpha)Y = \tau_\alpha[g(X,Y)\xi_\alpha - \epsilon_\alpha \eta_\alpha(Y)X] \tag{13}$$

for all $X, Y \in \Gamma(TM)$ and $\alpha \in \{1,2,3\}$, then $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ is said to be a mixed 3-Sasakian structure on M.

Remark that from (12) it follows

$$\nabla \xi_\alpha = 0, \ \nabla \eta_\alpha = 0 \tag{14}$$

and from (13) we obtain

$$\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi_\alpha X, \tag{15}$$

for all $\alpha \in \{1,2,3\}$ and $X \in \Gamma(TM)$.

We also note that the main property of a manifold endowed with a mixed 3-Sasakian structure is given by the following theorem (see [CP, IV]).

Theorem 4.4. *Any $(4n+3)-$dimensional manifold endowed with a mixed 3-Sasakian structure is an Einstein space with Einstein constant $\lambda = (4n+2)\varepsilon$, with $\varepsilon = \mp 1$, according as the metric mixed 3-structure is positive or negative, respectively.*

Definition 4.5. Let $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ and $(N, (\varphi'_\alpha, \xi'_\alpha, \eta'_\alpha)_{\alpha=\overline{1,3}}, g')$ be two manifolds endowed with metric mixed 3-structures. We say that a smooth map $f : M \to N$ is *holomorphic* if the equation

$$f_* \circ \varphi_\alpha = \varphi'_\alpha \circ f_* \tag{16}$$

holds for all $\alpha \in \{1,2,3\}$.

Example 4.6. If $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ is a manifold endowed with a metric mixed 3-structure and M' is an invariant submanifold of M (i.e. a non-degenerate submanifold of M such that $\varphi_\alpha(T_pM') \subset T_pM'$, for all $p \in M'$ and $\alpha = 1,2,3$), tangent to the structure vector fields, then the restriction of $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ to M' is a metric mixed 3-structure and the inclusion map $i : M' \to M$ is holomorphic.

Now we are able to state the following.

Theorem 4.7. *Let* $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ *and* $(N, (\varphi'_\alpha, \xi'_\alpha, \eta'_\alpha)_{\alpha=\overline{1,3}}, g')$ *be two mixed 3-cosymplectic or mixed 3-Sasakian manifolds. If* $f : M \to N$ *is a holomorphic map, then* f *is a harmonic map.*

Proof. Let $\{(E_i, \varphi_1 E_i, \varphi_2 E_i, \varphi_3 E_i)_{i=\overline{1,n}}, \xi_1, \xi_2, \xi_3\}$ be a local pseudo-orthonormal basis of vector fields tangent to M, with $\epsilon_i = g(E_i, E_i) \in \{\pm 1\}$. Then we have from (10) and (11) that

$$g(\varphi_j E_i, \varphi_j E_i) = \tau_j \epsilon_i, \ j \in \{1, 2, 3\}$$

and we deduce that the tension field of f is given by

$$\tau(f) = \sum_{i=1}^{n} \epsilon_i[\alpha_f(E_i, E_i) + \sum_{j=1}^{3} \tau_j \alpha_f(\varphi_j E_i, \varphi_j E_i)] + \sum_{j=1}^{3} \varepsilon_j \alpha_f(\xi_j, \xi_j). \tag{17}$$

We remark now that in both cases (mixed 3-cosymplectic and mixed 3-Sasakian), we obtain from (14) or (15) that

$$\nabla_{\xi_j} \xi_j = 0, \ \nabla'_{\xi'_j} \xi'_j = 0,$$

since $\varphi_j \xi_j = 0$, for all $j \in \{1, 2, 3\}$ (see e.g. [Bla]). Taking into account now that there exists a positive real number r such that $f_* \xi_j = r \xi'_j$ (see [IP1]), we deduce

$$\alpha_f(\xi_j, \xi_j) = 0. \tag{18}$$

Using after case (12) or (13), we can easily obtain for all $j \in \{1, 2, 3\}$ and $i \in \{1, ..., n\}$:

$$\nabla_{E_i} E_i = -\tau_j \varphi_j \nabla_{E_i} \varphi_j E_i \tag{19}$$

and

$$\nabla_{\varphi_j E_i} \varphi_j E_i = \varphi_j \nabla_{\varphi_j E_i} E_i. \tag{20}$$

From (19) and (20) we derive

$$\nabla_{E_i} E_i + \tau_j \nabla_{\varphi_j E_i} \varphi_j E_i = \tau_j \varphi_j [\varphi_j E_i, E_i]. \tag{21}$$

Similarly, since f is holomorphic, we obtain

$$\widetilde{\nabla}_{E_i} f_* E_i + \tau_j \widetilde{\nabla}_{\varphi_j E_i} f_* \varphi_j E_i = \tau_j \varphi'_j [f_* \varphi_j E_i, f_* E_i]. \tag{22}$$

From (2), (21) and (22), taking account of (16), we derive

$$\alpha_f(E_i, E_i) + \tau_j \alpha_f(\varphi_j E_i, \varphi_j E_i) = 0, \tag{23}$$

for all $j \in \{1, 2, 3\}$.

Applying repeatedly (23) and making use of (9) and (11), we obtain:

$$\begin{aligned} \alpha_f(E_i, E_i) &= -\alpha_f(\varphi_3 E_i, \varphi_3 E_i) = -\alpha_f(\varphi_2 \varphi_3 E_i, \varphi_2 \varphi_3 E_i) \\ &= -\alpha_f(-\varphi_1 E_i - \eta_3(E_i)\xi_2, -\varphi_1 E_i - \eta_3(E_i)\xi_2) \\ &= -\alpha_f(\varphi_1 E_i, \varphi_1 E_i) = -\alpha_f(E_i, E_i) \end{aligned}$$

and so we conclude that

$$\alpha_f(E_i, E_i) = 0. \tag{24}$$

From (23) and (24) we obtain

$$\alpha_f(\varphi_j E_i, \varphi_j E_i) = 0, \ j \in \{1, 2, 3\}. \tag{25}$$

Using now (18), (24) and (25) in (17) we derive $\tau(f) = 0$ and the conclusion follows. ■

5. Semi-Riemannian submersions from manifolds endowed with metric mixed 3-structures. An almost para-hypercomplex structure on a smooth manifold M is a triple $H = (J_\alpha)_{\alpha=\overline{1,3}}$, where J_1, J_2 are almost product structures on M and J_3 is an almost complex structure on M, satisfying:

$$J_\alpha J_\beta = -J_\beta J_\alpha = \tau_\gamma J_\gamma,$$

for every even permutation (α, β, γ) of (1,2,3), where $\tau_1 = \tau_2 = -\tau_3 = -1$.

A semi-Riemannian metric g on (M, H) is said to be compatible or adapted to the almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=\overline{1,3}}$ if it satisfies:

$$g(J_\alpha X, J_\alpha Y) = \tau_\alpha g(X, Y)$$

for all vector fields X,Y on M. Moreover, the triple (M, H, g) is said to be an almost para-hyperhermitian manifold. If $\{J_1, J_2, J_3\}$ are parallel with respect to the Levi-Civita connection of g, then the manifold is called para-hyper-Kähler.

Let (M, g) and (M', g') be two connected semi-Riemannian manifold of index s ($0 \leq s \leq dimM$) and s' ($0 \leq s' \leq dimM'$) respectively, with $s' \leq s$. The concept of semi-Riemannian submersion was introduced by O'Neill (see [ON2]) as a smooth map $\pi : M \to M'$ which is onto and satisfies the following conditions:

- (i) $\pi_*|_p$ is onto for all $p \in M$;
- (ii) The fibres $\pi^{-1}(p')$, $p' \in M'$, are semi-Riemannian submanifolds of M;
- (iii) π_* preserves scalar products of vectors normal to fibres.

A semi-Riemannian submersion $\pi : M \to M'$ determines, as well as in the Riemannian case (see [ON1]), two (1,2) tensor field T and A on M, by the formulas:

$$T(E, F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{26}$$

and respectively:

$$A(E, F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{27}$$

for any $E, F \in \Gamma(TM)$, where v and h are the vertical and horizontal projection. We remark that for $U, V \in \Gamma(\mathcal{V})$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds.

An horizontal vector field X on M is said to be *basic* if X is π-related to a vector field X' on M'. It is clear that every vector field X' on M' has a unique horizontal lift X to M and X is basic.

Remark 5.1. If $\pi : M \to M'$ is a semi-Riemannian submersion and X, Y are basic vector fields on M, π-related to X' and Y' on M', then we have the next properties (see [ON2]):

- (i) $h[X, Y]$ is a basic vector field and $\pi_* h[X, Y] = [X', Y'] \circ \pi$;
- (ii) $h(\nabla_X Y)$ is a basic vector field π-related to $\nabla'_{X'} Y'$, where ∇ and ∇' are the Levi-Civita connections on M and M';
- (iii) $[E, U] \in \Gamma(\mathcal{V}), \forall U \in \Gamma(\mathcal{V})$ and $\forall E \in \Gamma(TM)$.

Definition 5.2. Let $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ and $(N, (\varphi'_\alpha, \xi'_\alpha, \eta'_\alpha)_{\alpha=\overline{1,3}}, g')$ be two manifolds endowed with metric mixed 3-structures. A semi-Riemannian submersion $\pi : M \to N$ is said to be *a mixed 3-submersion* if it is holomorphic map and the structure vector field ξ_α on M is a basic vector field π-related to the structure vector field ξ'_α on N, for all $\alpha \in \{1, 2, 3\}$.

Using the same techniques as in [Wat] (see also [Chi2, IMV2, TM, Vil2]), we can prove the following.

Theorem 5.3. *Let $\pi : M \to N$ be a mixed 3-submersion. Then:*

(*i*) *The vertical and horizontal distributions induced by π are invariant under each φ_α, $\alpha \in \{1, 2, 3\}$.*
(*ii*) *The fibres of the submersion are almost para-hyperhermitian manifolds.*
(*iii*) *If M is a mixed 3-cosymplectic manifold, then the base space N is also a mixed 3-cosymplectic manifold. Moreover, the fibres are totaly geodesic para-hyper-Kähler submanifolds.*
(*iv*) *If M is a mixed 3-Sasakian manifold, then π is a semi-Riemannian covering map.*

Proof.

(*i*) Let $V \in \Gamma(\mathcal{V})$. Then, we have $\pi_*\varphi_\alpha V = \varphi'_\alpha \pi_* V = 0$, and so we conclude that $\varphi_\alpha(\mathcal{V}) \subset \mathcal{V}$. On another hand, for any $X \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$, we derive from (11) that $g(\varphi_\alpha X, V) = -g(X, \varphi_\alpha V) = 0$ and thus we obtain $\phi_\alpha(\mathcal{H}) \subset \mathcal{H}$.
(*ii*) If we denote by J_α the restriction of φ_α to $\mathcal{V}$, then we have for any vertical vector field V:
$$J_\alpha^2 V = \varphi_\alpha^2 V = \tau_\alpha[-V + \eta_\alpha(V)\xi_\alpha] = -\tau_\alpha V,$$
and from (9) we obtain
$$J_\alpha J_\beta V = -J_\beta J_\alpha V = \tau_\gamma J_\gamma V,$$
since ξ_α is horizontal. On another hand, from (10) we deduce that the restriction of g to any fibre F is compatible with $\{J_\alpha\}_{\alpha=\overline{1,3}}$ defined above and so we conclude that $(F, \{J_\alpha\}_{\alpha=\overline{1,3}}, g_{|F})$ is an almost para-hyperhermitian manifold.
(*iii*) For any basic vector fields X, Y on M, π-related with X' and Y' on N, we deduce from (12):
$$\pi_*(\nabla_X \varphi_\alpha Y) - \pi_* \varphi_\alpha \nabla_X Y = 0,\ \alpha \in \{1, 2, 3\}. \tag{28}$$
Since $\varphi_\alpha Y$ is a basic vector field π-related with $\varphi'_\alpha Y'$, using (16) and Remark 5.1 we obtain:
$$\nabla'_{X'} \varphi'_\alpha Y' - \varphi'_\alpha \nabla'_{X'} Y' = 0,\ \alpha \in \{1, 2, 3\}$$
and thus N is a mixed 3-cosymplectic manifold.
Using the Gauss's formula and (12), by identifying the tangential and normal components to a fibre F, we obtain for any vector fields U, V tangent to F:
$$(\nabla_U J_\alpha)V = 0 \tag{29}$$
and
$$T_U J_\alpha V = \varphi_\alpha(T_U V). \tag{30}$$

From (29) it follows that $(F, \{J_\alpha\}_{\alpha=\overline{1,3}}, g_{|F})$ is a para-hyper-Kähler manifold and applying repeatedly (30) we obtain $T = 0$. So F is a totaly geodesic submanifold.

(iv) Because $d\eta(V, \varphi_\alpha V) = 0$, for any vector field V tangent to a fibre F and $\Phi_\alpha = d\eta_\alpha$, it follows from (1) that $g(V, V) = 0$. Therefore, since $g_{|F}$ is non-degenerate, we obtain that the fibres are discrete and the conclusion follows.

The proof is now complete. ∎

Corollary 5.4. *Any mixed 3-submersion from a mixed 3-cosymplectic manifold is a harmonic map.*

Proof. The statement is obvious since it is well known that a semi-Riemannian submersion is a harmonic map if and only if each fibre is a minimal submanifold (see e.g. [FIP]). ∎

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