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Essential points of conformal vector fields ☆

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ABSTRACT

An essential point of a conformal vector field ξ on a conformal manifold (M, c) is a point around which the local flow of ξ preserves no metric in the conformal class c. It is well-known that a conformal vector field vanishes at each essential point. In this note we show that essential points are isolated. This is a generalization to higher dimensions of the fact that the zeros of a holomorphic function are isolated. As an application, we show that every connected component of the zero set of a conformal vector field is totally umbilical.

1. Introduction

It is a classical result by Alekseevskii [1] that the group of conformal automorphisms of a Riemannian manifold either fixes a conformally equivalent metric or the manifold is conformally flat. A gap in his proof, found by Zimmer and Gutschera in 1992, was filled by Ferrand [2].

From an infinitesimal point of view, if a conformal manifold admits a complete and *essential* conformal vector field (whose global flow acts by conformal transformations, but not by isometries with respect to any compatible metric), then it is conformally flat, [3]. More recently, Frances proved a local version of this result [4], see also [5,6] for the pseudo-Riemannian setting.

For a given conformal vector field ξ (whose flow is not globally defined in general), there might exist local metrics in the conformal class preserved by the local flow of ξ. The union of the definition domains of these ξ-invariant local metrics is the open set of *non-essential* points of the conformal vector field ξ. This motivates the following:

Definition 1.1. Let (M, c) be a conformal manifold and let ξ be a conformal vector field on M. A point $x \in M$ is called *essential* for ξ if there is no local metric in c preserved by the local flow of ξ around x.

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If ξ does not vanish at some point p, it is easy to see that p is not essential. Indeed, ξ is rectifiable in a neighborhood U of p, so there exists a system of coordinates $(x_1, \ldots, x_n)$ near p such that $x_i(p) = 0$ for all i and $\xi = \partial/\partial x_1$. Take any metric in c, restrict it to the hypersurface $U \cap \{x_1 = 0\}$ and extend it to U as to be constant with respect to x_1. Then ξ preserves the new metric, which moreover belongs to the conformal class c because ξ is conformal.

In this note, we prove

Theorem 1.2. *The essential set of a conformal vector field ξ on a Riemannian manifold M of dimension $n \geq 2$ consists of isolated zeros of ξ.*

Note that this is equivalent for $n = 2$ to the fact that the zeros of a non-constant holomorphic function are isolated. The theorem above may thus be seen as an extension to higher dimensions of this classical fact.

For the proof, we focus on the zero set of ξ and show that if a zero point x of ξ is not isolated then an algebraic condition (Theorem 2.1) is satisfied by the derivative of ξ at x, which implies, via a result by Beig [7] and Capocci [8], that ξ is Killing with respect to a local metric around x.

As an application, we show in Theorem 3.1 that every connected component of the zero set of ξ is a totally umbilical submanifold, thus generalizing a well-known result of Kobayashi [9], that states that the connected components of the set of zeros of a Killing vector field on a Riemannian manifold are totally geodesic submanifolds of even codimension.

A proof of Theorem 3.1 was previously given by Blair [10] under the additional restriction (required by his proof based on the Obata theorem) that the manifold is compact. Instead, our proof is purely local, as is the proof of the above mentioned Kobayashi's result.

Very recently, results similar to Theorem 1.2 were obtained independently by Lampe in his Ph.D. thesis [6].

2. Proof of the main result

Let (M^n, c) $(n \geq 3)$ be a conformal manifold. We choose a background Riemannian metric $g \in c$ and make the usual identifications between vectors and 1-forms, 2-forms and skew-symmetric endomorphisms etc., induced by this metric.

A vector field ξ is called *conformal* if the symmetric part of its covariant derivative with respect to the Levi–Civita connection of g is reduced to its trace:

$$\nabla_Y \xi = \frac{1}{2} Y \lrcorner\, d\xi + \varphi Y, \quad \forall Y \in TM. \tag{1}$$

The function φ is equal to $-\delta^g \xi / n$, but we will not need this in the sequel.

Let Z_ξ and E_ξ denote the zero set and the essential set of ξ respectively. We have seen that $E_\xi \subset Z_\xi$, so in order to prove Theorem 1.2, it will be enough to show that every limit point[1] x of Z_ξ is not essential, *i.e.* there exists a ξ-invariant metric defined locally around x.

To do that, we make use of the following criterion by Capocci [8, Theorem 2.1], (cf. also [7]):

Theorem 2.1 ([8]). *Let $x \in Z_\xi$ be a zero of the conformal field ξ on a Riemannian manifold (M, g) of dimension at least 3 and let φ be the function defined by (1). Then ξ is a homothetic vector field with respect to some conformal metric $\tilde{g}$ defined on a neighborhood of x if and only if the gradient of φ with respect to g belongs to the image of $\nabla\xi$ at x; moreover, ξ is Killing with respect to $\tilde{g}$ if, in addition, $\varphi(x) = 0$.*

Theorem 1.2 will follow directly from Theorem 2.1, together with:

Theorem 2.2. *Let $x \in Z'_\xi$ be a limit point of the zero set Z_ξ of a conformal vector field ξ. Then the function φ defined in (1) vanishes at x and its gradient with respect to g belongs to the image of $\nabla\xi$ at x.*

Proof. Let $x_k \neq x$ be a sequence of zeros of ξ converging to x. In a geodesic chart around x, we connect x with x_k by uniquely defined minimizing geodesics, denoted by $c_k : [0, t_k] \to M$, $c_k(0) = x$, $c_k(t_k) = x_k$, $\|\dot{c_k}\| = 1$.

The function $f_k := g(\xi, \dot{c_k})$ admits the following Taylor–Lagrange expansion:

$$f_k(t_k) = f_k(0) + t_k f_k'(\tau_k), \quad \text{for } \tau_k \in (0, t_k). \tag{2}$$

Since $f_k(t_k) = f_k(0) = 0$ and $t_k \neq 0$, we have found a sequence of points $c_k(\tau_k)$, converging to x, such that $f_k'(\tau_k) = 0$.

On the other hand, Eq. (1) implies that $f_k'(t) = \varphi(c_k(t))$, therefore φ vanishes on a sequence converging to x, thus $\varphi(x) = 0$.

For the second part of the theorem, we need to show that $d\varphi_x$ lies in the image of the skew-symmetric endomorphism $d\xi_x$ of T_xM for every $x \in Z'_\xi$.

[1] A *limit point* of a set S in a topological space X is a point $x \in S$ such that every neighborhood of x intersects $S \setminus \{x\}$. The set of limit points of S is denoted by S'.

For a geodesic $c : [0, T] \to M$ with $c(0) = x$ and $\|\dot{c}\| = 1$, we denote by $\xi'(t)$ and $\xi''(t)$ the derivatives $\nabla_{\dot{c(t)}}\xi$, respectively $\nabla_{\dot{c(t)}}\nabla_{\dot{c(t)}}\xi$. From (1) we have

$$\xi'(t) = \frac{1}{2}d\xi(\dot{c}(t)) + \varphi(c(t))\dot{c}(t), \tag{3}$$

in particular $\xi'(0) = \frac{1}{2}d\xi(\dot{c}(0))$.

Lemma 2.3. *For a conformal vector field ξ on a Riemannian manifold (M, g), such that $\mathcal{L}_\xi g = 2\varphi g$, the following relation holds:*

$$\nabla_X d\xi = 2R_{X,\xi} + 2d\varphi \wedge X, \quad \forall X \in TM.$$

Proof. The notations are those from (1), which we will use in the following equivalent form

$$g(\nabla_A\xi, B) + g(\nabla_B\xi, A) = 2\varphi g(A, B), \quad \forall A, B \in TM. \tag{4}$$

Let $x \in M$ and let X, Y, Z be vector fields parallel at x. Since $d\xi(Y, Z) = g(\nabla_Y\xi, Z) - g(\nabla_Z\xi, Y)$, we have at x:

$$\begin{aligned}(\nabla_X d\xi)(Y, Z) &= \nabla_X (d\xi(Y, Z)) = g(\nabla_X\nabla_Y\xi, Z) - g(\nabla_X\nabla_Z\xi, Y)\\ &= g(R_{X,Y}\xi, Z) - g(R_{X,Z}\xi, Y) + g(\nabla_Y\nabla_X\xi, Z) - g(\nabla_Z\nabla_X\xi, Y).\end{aligned}$$

We use (4) to substitute the last two terms, and the Bianchi identity for the first two and get

$$\begin{aligned}(\nabla_X d\xi)(Y, Z) &= g(R_{X,\xi}Y, Z) - g(\nabla_Y\nabla_Z\xi, X) + \nabla_Y(2\varphi g(X, Z)) + g(\nabla_Z\nabla_Y\xi, X) - \nabla_Z(2\varphi g(Y, X))\\ &= 2g(R_{X,\xi}Y, Z) + 2d\varphi(Y)g(X, Z) - 2d\varphi(Z)g(X, Y). \quad \square\end{aligned}$$

We compute then (for clarity, we omit the argument t)

$$\xi'' = R_{\dot{c},\xi}\dot{c} + (d\varphi \wedge \dot{c})(\dot{c}) + d\varphi(\dot{c})\dot{c}, \tag{5}$$

in particular, since $\xi_x = 0$, we have

$$\xi''(0) = 2d\varphi_x(\dot{c}(0))\dot{c}(0) - d\varphi_x. \tag{6}$$

We come now to the core of our argument. For an arbitrary geodesic c generated by a unit vector in T_xM, we estimate the function $f(t) := g(\xi_{c(t)}, \dot{c}(t))$ using the Taylor expansion of order 2:

$$f(t) = f(0) + tf'(0) + \frac{t^2}{2}f''(0) + O(t^3). \tag{7}$$

Here the function f depends on the chosen geodesic, and the error term $O(t^3)$ is locally bounded (around x) by t^3K, where K is a positive constant depending only on the derivatives of ξ but not on the geodesic c. Note that $\xi_x = 0$ implies $f(0) = 0$, and that $f'(0) = \varphi(x) = 0$ for any geodesic c.

We choose $c := c_k$ and $t := t_k$, and denote by $f_k(t) := g(\xi_{c_k(t)}, \dot{c}(t))$. From (7) we infer

$$f_k''(0) \longrightarrow 0 \quad \text{for } k \longrightarrow \infty. \tag{8}$$

Restricting c_k to a subsequence for which $\dot{c}_k(0)$ converges to a unit vector $V \in T_xM$ yields

$$d\varphi_x(V) = 0. \tag{9}$$

In the next step we estimate $\xi(t) := \xi_{c(t)}$ using a version of the Taylor expansion for vector-valued functions (here we consider $\xi(.)$ as a function on an interval with values in $\mathbb{R}^n$, whose components are the components of ξ with respect to a chosen orthonormal basis parallel along c):

$$\xi(t) = \xi(0) + t\xi'(0) + \frac{t^2}{2}\xi''(0) + O(t^3). \tag{10}$$

Note that the function $t \mapsto \xi(t)$ depends on the chosen geodesic c, but the norm of the error term $O(t^3)$ is bounded by Mt^3 for a constant M depending only on the derivatives of ξ. We now set $c := c_k$ and $t = t_k$ and denote the corresponding function $\xi_{c_k(.)}$ by ξ_k. We get

$$0 = t_k\xi_k'(0) + \frac{t_k^2}{2}\xi_k''(0) + O(t_k^3),$$

which implies, after taking the quotient by t_k^2 and using (3) and (6):

$$\left\| \frac{1}{2}d\xi\left(\frac{\dot{c}_k(0)}{t_k}\right) + 2d\varphi_x(\dot{c}_k(0))\dot{c}_k(0) - d\varphi_x \right\| \le Mt_k \quad \forall k \in \mathbb{N}. \tag{11}$$

Denote now by V_k the vector $\dot{c}_k(0)/2t_k$. The sequence $\{V_k\}$ is unbounded, but in our relation (11) only counts the projection of V_k on the image of $d\xi_x$, denoted by $W_k := \pi V_k$.

Since $\dot{c}_k(0) \longrightarrow V \in T_xM$ and $d\varphi_x(V) = 0$ by (9), we conclude that the middle term in (11) tends to zero, thus

$$\|d\xi_x(W_k) - d\varphi_x\| \longrightarrow 0, \quad \text{for } k \longrightarrow \infty. \tag{12}$$

Since $W_k \perp \ker(d\xi_x)$, it follows that the norm of W_k is bounded, and we can restrict to a subsequence that converges to $W \in \mathrm{Im}(d\xi_x)$ (Here we consider the kernel and the image of $d\xi_x$ as the kernel and the image of a skew-symmetric endomorphism of T_xM). It thus follows

$$d\varphi_x = d\xi_x(W),$$

which finishes the proof. □

Remark. We have shown that all zeros of ξ which are not isolated are not essential. Nothing can be said, in general, about isolated zeros of a conformal vector field, as the following classical example shows:

Let ξ be the conformal vector field on the round sphere $S^n \subset \mathbb{R}^{n+1}$, which is sent through the stereographic projection from one pole P to a translation vector field on $\mathbb{R}^n$. Then both ξ and its derivative vanish at P, which is the only zero of ξ. Therefore, in the notations of the Eq. (1), $d\xi_P = 0$ and $\varphi_P = 0$. On the other hand, $d\varphi_P \neq 0$ (which is implied by the very fact that ξ is not trivial), so the condition in Theorem 2.1 is violated, which is a proof that P is an essential point for ξ.

3. The zero set of a conformal vector field

As an application to our main result, we prove

Theorem 3.1. *Let ξ be a conformal vector field on a Riemannian manifold (M, g) of dimension at least 2, and let Z_ξ be the zero set of ξ on M. Then Z_ξ is a disjoint union of embedded connected totally umbilical submanifolds of M, of even codimension when not reduced to a point.*

Let $N \subset M$ be an isometrically embedded submanifold and denote with the same letter g both the metric of M and the induced metric on N. Let then ∇^M, ∇^N be the associated Levi–Civita connections on M and N. The $(0, 2)$ tensor field $B : T^*N \otimes T^*N \to \nu(N)$, defined by $B(X, Y) := \nabla^M_X Y - \nabla^N_X Y$, where $\nu(N)$, the normal bundle of N in M, is called the *second fundamental form* of the embedding.

By definition, the submanifold N is called *totally umbilical* if its second fundamental form is proportional to the metric tensor induced on N in the following sense:

$$B(X, Y) = Hg(X, Y), \quad \forall X, Y \in \mathfrak{X}(N).$$

The normal vector field H is called the *mean curvature* vector field. One-dimensional submanifolds and totally geodesic submanifolds (*i.e.* with vanishing second fundamental form) are totally umbilical.

By definition, every point is considered to be a totally umbilical submanifold of dimension 0.

Using the relation between the Levi–Civita connections ∇, ∇' of two conformal metrics $g' = e^{2f}g$:

$$\nabla' = \nabla + df \otimes Id + Id \otimes df - g \cdot \mathrm{grad}_g f,$$

one easily sees that if N is totally umbilical with respect to a metric then it is totally umbilical with respect to any other conformally equivalent metric. In particular, N is totally umbilical if there exists a metric $\tilde{g}$ in the conformal class, for which N is totally geodesic.

Proof of Theorem 3.1. First, we apply Theorem 1.2 to decompose the zero set Z_ξ as the disjoint union of its isolated points Z^{iso} and non-isolated points $Z' := Z'_\xi$, which are inessential, and conclude that every $x \in Z'$ admits a metric $g_x \in [g]$, defined locally around x, such that ξ is Killing with respect to g_x, and $\xi_x = 0$.

The classical result of Kobayashi [9] implies then that there is a neighborhood U_x of x such that $Z_\xi \cap U_x = \exp_x(\ker d\xi_x) \cap U_x = \exp_y(\ker d\xi_y) \cap U_x$, for any $y \in Z_\xi \cap U_x$ (here, we consider the dual 1-form to ξ with respect to g_x, and compute its exterior derivative $d\xi$ at x; also, the exponential map $\exp_x$ is taken with respect to g_x).

In conclusion, every point in Z_ξ admits a neighborhood U_x such that $Z_\xi \cap U_x$ is a totally umbilical submanifold of even codimension. By usual connectedness arguments, these local totally umbilical submanifolds have a common dimension and glue together to a global submanifold. □

Note that a classification of totally umbilical submanifolds exists only for space forms and, more generally, for locally symmetric spaces (see *e.g.* [11]).

The above result can then be a useful tool for producing examples of totally umbilical submanifolds or obstructions to the existence of conformal vector fields on certain Riemannian manifolds.

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