Contemporary Mathematics

Cayley 4-Frames and a Quaternion Kähler Reduction Related to Spin(7)

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Abstract. The object of this note is CAYLEY, the Grassmannian of the oriented 4-planes in $\mathbb{R}^8$ that are closed under the three-fold cross product. We describe an action of $\mathrm{U}(1) \times \mathrm{Sp}(1)$ on the quaternionic projective space $\mathbb{H}P^7$, that allows to obtain a $\mathbb{Z}_2$-quotient of CAYLEY by quaternion Kähler reduction.

1. Introduction

The existence of only two exceptional cross products - in $\mathbb{R}^7$ and in $\mathbb{R}^8$, with two and three factors respectively - attracted the interest of Alfred Gray in the sixties, and this was one of his approaches to the study of holonomies G_2 and Spin(7) on Riemannian manifolds [**5**], [**10**]. About one decade later, the symmetric space structure of the Grassmannians of those planes in $\mathbb{R}^7$ or $\mathbb{R}^8$ that are closed under such cross products was recognized in the positive quaternion Kähler manifolds $\frac{\mathrm{G}_2}{\mathrm{SO}(4)}$ and $\frac{\mathrm{Spin}(7)}{(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1))/\mathbb{Z}_2}$ [**14**], [**11**]. The latter of these manifolds is in fact isometric to the Grassmannian $\mathrm{Gr}_4(\mathbb{R}^7) = \frac{\mathrm{SO}(7)}{\mathrm{SO}(3)\times\mathrm{SO}(4)}$ of oriented 4-planes in $\mathbb{R}^7$, but its rôle as "exceptional Grassmannian" of some distinguished 4-planes in $\mathbb{R}^8$ is so interesting to make it deserving of notations like CAY or CAYLEY in papers on calibrations [**6**], [**9**].

In this note we describe how CAYLEY can be obtained – up to a $\mathbb{Z}_2$-quotient – through a quaternion Kähler reduction of the projective space $\mathbb{H}P^7$, acted on by a group isomorphic to $\mathrm{U}(1) \times \mathrm{Sp}(1)$. This action is similar to that used by P. Kobak and A. Swann in $\mathbb{H}P^6$, obtaining a $\mathbb{Z}_3$-quotient of $\frac{\mathrm{G}_2}{\mathrm{SO}(4)}$ by quaternion Kähler reduction [**12**]. In the later note [**13**] a different action of the same group in $\mathbb{H}P^7$ is described, obtaining this time as a reduction a $\mathbb{Z}_2$-quotient of $\mathrm{Gr}_4(\mathbb{R}^7)$. Our reduction is in fact equivalent to this latter, via the isometry CAYLEY $\cong \mathrm{Gr}_4(\mathbb{R}^7)$, although our definition of the action is much closer to the point of view of the former article [**12**]. We observe also that, through the same isometry with this

2000 *Mathematics Subject Classification.* Primary 53C26; Secondary 53C27, 53C38.

Key words and phrases. Quaternion Kähler manifold, 3-Sasakian manifold, moment map, reduction, Spin(7), Cayley numbers.

The authors are members of EDGE, Research Training Network HRPN-CT-2000-00101, supported by The European Human Potential Programme.

real Grassmannian, one can obtain CAYLEY as a reduction of $\mathbb{H}P^6$ acted on by $\mathrm{Sp}(1)$, following one of the classical procedures described in [**1**] (and in this way no finite quotient is involved). However, the reduction we are going to describe here – without making use of the isometry CAYLEY $\cong \mathrm{Gr}_4(\mathbb{R}^7)$ – has the advantage of admitting two interesting generalizations.

One of them is a consequence of the possibility of introducing weights in the action of the $\mathrm{U}(1)$-factor. This allows to obtain a family of 12-dimensional quaternion Kähler orbifolds, some of them admitting smooth 3-Sasakian manifolds over them. These smooth 15-dimensional manifolds are, together with similar 11-dimensional manifolds related to G_2, the first examples of 3-Sasakian manifolds which are neither homogeneous nor toric [**3**].

The other possibility of extending the present reduction is obtained by looking at two other quaternion Kähler Wolf spaces. Since CAYLEY can be regarded as the manifold of the hypercomplex 4-planes in $\mathbb{R}^8$ identified with the real vector space of Cayley numbers $\mathbb{C}\mathrm{a}$ (Proposition 3.1 below), higher dimensional analogues of it are the manifolds $\frac{\mathrm{Spin}(9)}{(\mathrm{Sp}(2)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1))/\mathbb{Z}_2} \cong \mathrm{Gr}_4(\mathbb{R}^9)$ and the exceptional Wolf space $\frac{\mathrm{F}_4}{\mathrm{Sp}(3)\cdot\mathrm{Sp}(1)}$, geometrically the manifolds of the $\mathbb{H}P^1 \subset \mathbb{C}\mathrm{a}P^1$ and of the $\mathbb{H}P^2 \subset \mathbb{C}\mathrm{a}P^2$, respectively. Some issues related to this second generalization will be studied in a future work [**4**].

Acknowledgements. The first author acknowledges financial support by C.N.R. of Italy and by the Cultural Agreement between Università di Roma "La Sapienza" and University of Bucharest. Both authors thank Kris Galicki for helpful and stimulating conversations and Robert Bryant for a kind observation.

2. Preliminaries on Cayley numbers

Let $\mathbb{C}\mathrm{a}$ be the algebra of Cayley numbers, and let $\{1, \mathrm{i}, \mathrm{j}, \mathrm{k}, \mathrm{e}, \mathrm{f} = \mathrm{ie}, \mathrm{g} = \mathrm{je}, \mathrm{h} = \mathrm{ke}\}$ be its canonical basis over $\mathbb{R}$. The multiplication is given by

$$xy = (ac - \overline{d}b) + (b\overline{c} + da)\mathrm{e},$$

where $x = a + b\mathrm{e}, y = c + d\mathrm{e} \in \mathbb{C}\mathrm{a}$ are written through the identification $\mathbb{C}\mathrm{a} \cong \mathbb{H}^2$ with pairs of quaternions. The quaternionic conjugation (already used in the previous formula) induces a conjugation in $\mathbb{C}\mathrm{a}$: $\overline{x} = \overline{a} - b\mathrm{e}$, allowing to write the non-commutativity rule: $\overline{xy} = \overline{y}\,\overline{x}$.

The non-associativity of $\mathbb{C}\mathrm{a}$ gives rise to the associator $[x, y, z] = (xy)z - x(yz)$, alternating form that vanishes whenever two of its arguments are either equal or conjugate. Geometrically, the associator defines the class of *associative 3-planes* in $\mathbb{R}^7 \cong \mathrm{Im}\mathbb{C}\mathrm{a}$, defined in orthonormal bases by $[x, y, z] = 0$. They are characterized as the 3-planes of $\mathbb{R}^7$ closed with respect to the *two-fold cross product*:

$$x \times y = \mathrm{Im}(\overline{y}x),$$

that is so more generally defined for any $x, y \in \mathbb{R}^8 \cong \mathbb{C}\mathrm{a}$. Note that if x, y are orthogonal and in $\mathbb{R}^7 \cong \mathrm{Im}\mathbb{C}\mathrm{a}$, the cross product is simply xy.

The *three-fold cross product* in $\mathbb{C}\mathrm{a} \cong \mathbb{R}^8$ is defined by the formula

$$x \times y \times z = \frac{1}{2}(x(\overline{y}z) - z(\overline{y}x)),$$

reducing to $x(\overline{y}z)$ for x, y, z orthogonal.

The following properties hold whenever x, y are orthogonal and for any $w \in \mathbb{C}\mathrm{a}$:

$$x(\overline{y}w) = -y(\overline{x}w), \qquad (w\overline{y})x = -(w\overline{x})y. \tag{2.1}$$

Moreover, for any $x, y, z \in \mathbb{C}\mathrm{a}$:

$$(xy)(zx) = x(yz)x. \tag{2.2}$$

(A reference for these preliminaries is [**11**], Appendix IV).

3. The Stiefel manifold of Cayley 4-frames

A 4-plane ζ of $\mathbb{R}^8$ that is closed under the three-fold cross product is called a *Cayley 4-plane* and it is oriented by choices of bases $\{w = x \times y \times z, x, y, z\}$. The manifold of the Cayley 4-planes in $\mathbb{R}^8$ is CAYLEY $= \frac{\mathrm{Spin}(7)}{(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1)/\mathbb{Z}_2)}$, 12-dimensional quaternionic submanifold of the Grassmannian $\mathrm{Gr}_4(\mathbb{R}^8)$ of oriented 4-planes in $\mathbb{R}^8$ ([**14**], p. 262 or [**11**], p. 123).

Proposition IV,1.27 in [**11**] states that a 4-plane ζ in $\mathbb{R}^8$ is Cayley if and only if $-\zeta$ is closed under the complex structures defined by the 2-planes $\alpha \subset \zeta$. This fact can be reformulated as follows.

PROPOSITION 3.1. *A 4-plane ζ in $\mathbb{R}^8$ is Cayley if and only if any triple of mutually orthogonal 2-planes $\alpha, \beta, \gamma \subset \zeta$, all intersecting in a line, defines a hypercomplex structure on ζ.*

PROOF. For any ζ, $\dim(\zeta \cap \mathrm{Im}\mathbb{C}\mathrm{a})$ is either 3 or 4. Thus, if $\zeta \in$ CAYLEY we may select orthonormal imaginary octonions $x, y, z \in \zeta$ such that $\{x \times y \times z, x, y, z\}$ is an oriented basis of ζ. If $u = x \times (x \times y \times z) = y \times z$, $v = y \times (x \times y \times z) = z \times x$, $w = z \times (x \times y \times z) = x \times y$ we have $u, v, w \in S^6$, and their corresponding complex structures J_u, J_v, J_w are associated to the 2-planes $\alpha = \mathrm{span}\{x \times y \times z, x\}$, $\beta = \mathrm{span}\{x \times y \times z, y\}$,$\gamma = \mathrm{span}\{x \times y \times z, z\}$. Since $J_u y = -z$, $J_v x = z$, $J_w x = -y$, then (J_u, J_v, J_w) satisfy $J_v \circ J_u = -J_u \circ J_v = J_w$, i.e. it is a hypercomplex structure on ζ. The converse follows from the aforementioned characterization in [**11**], p. 119. □

Our construction of (u, v, w) out of $\zeta = \mathrm{span}\ \{x \times y \times z, x, y, z\}$ corresponds to the isometry $\sim$: CAYLEY $\to \mathrm{Gr}_3(\mathbb{R}^7)$ of [**6**], p. 11. The image under $\sim$ of the $\zeta \in$ CAYLEY can be interpreted as a *tricomplex section* of S^6, oriented orthonormal bases of the $\zeta^\sim$ being triples u, v, w of unit octonions non necessarily satisfying the hypercomplex relations. The non-associativity of $\mathbb{C}\mathrm{a}$ allows and ensures that such triples define a hypercomplex structure on ζ. An example is the Cayley 4-plane $\zeta = \mathrm{span}\ \{1 - \mathrm{h}, \mathrm{i} + \mathrm{g}, \mathrm{j} - \mathrm{f}, \mathrm{k} + \mathrm{e}\}$: our procedure gives the tricomplex triple $(u, v, w) = (\mathrm{i}, \mathrm{j}, \mathrm{e})$, whose associated $(J_\mathrm{i}, J_\mathrm{j}, J_\mathrm{e})$ is hypercomplex on ζ.

This discussion permits to describe the inverse of the isometry $\sim$, as follows.

COROLLARY 3.1. *Given a tricomplex section of S^6 with oriented orthonormal basis (u, v, w), there is a unique Cayley 4-plane ζ in $\mathbb{R}^8$ on which (J_u, J_v, J_w) is hypercomplex.*

DEFINITION 3.1. A *Cayley 4-frame* in $\mathbb{R}^8$ is an oriented orthonormal 4-frame in a Cayley 4-plane ζ, hence a frame $\{x, I_1 x, I_2 x, I_3 x\}$, where (I_1, I_2, I_3) is the hypercomplex structure of ζ.

By the action of $\mathrm{Spin}(7) \supset \mathrm{G}_2 \supset \mathrm{SU}(3)$ on the spheres $S^7 \supset S^6 \supset S^5$, the latter with isotropy $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$, we have:

Proposition 3.2. *The Stiefel manifold of Cayley 4-frames in* $\mathbb{R}^8$ *is the homogeneous space* $V = \frac{\mathrm{Spin}(7)}{\mathrm{Sp}(1)}$.

Observe finally that:

Proposition 3.3. *An orthonormal frame* $\{f_1, f_2, f_3, f_4\}$ *in* $\mathbb{R}^8$ *is a Cayley 4-frame if and only if* $\overline{f}_2 f_1 = \overline{f}_3 f_4$.

4. CAYLEY and a reduction of $\mathbb{H}P^7$.

We now show how a $\mathbb{Z}_2$-quotient of CAYLEY can be obtained as quaternion Kähler reduction of $\mathbb{H}P^7$ by the action of $\mathrm{U}(1) \times \mathrm{Sp}(1)$. According to [**2**], we first reduce (by the same group) the 3-Sasakian manifold which stands over $\mathbb{H}P^7$, namely the sphere S^{31}. Then we interpret the quotient of S^{31} by $\mathrm{U}(1) \times \mathrm{Sp}(1)$ as the total space of an SO(3)-bundle over a quaternion Kähler orbifold which is the quotient of $\mathbb{H}P^7$ by the same group.

The 3-Sasakian sphere S^{31} is acted on by $\mathrm{U}(1) \times \mathrm{Sp}(1)$ as follows. The factor $\mathrm{Sp}(1)$ acts by right multiplication on $\vec{h} = (h_\alpha) \in S^{31} \subset \mathbb{H}^8$, and the moment map $\mu : S^{31} \to \mathbb{R}^9$ of the action reads:

$$\mu(\vec{h}) = (\sum_{\alpha=1}^{8} \overline{h}_\alpha \mathrm{i} h_\alpha, \quad \sum_{\alpha=1}^{8} \overline{h}_\alpha \mathrm{j} h_\alpha, \quad \sum_{\alpha=1}^{8} \overline{h}_\alpha \mathrm{k} h_\alpha).$$

By writing $\vec{h} = \vec{a} + \vec{b}\mathrm{i} + \vec{c}\mathrm{j} + \vec{d}\mathrm{k}$, it is easy to see that $\mu^{-1}(0)$ coincides with the Stiefel manifold of oriented (renormalized) orthonormal 4-frames in $\mathbb{R}^8$ [**2**].

We then act by the factor U(1), rotating pairs of coordinates. This is explicitly described by $\vec{h} \mapsto \mathrm{diag}\,(A(\theta), A(\theta), A(\theta), A(\theta)) \cdot \vec{h}$, where $A(\theta) = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$, $\theta \in \mathbb{R}$. The associated moment map $\nu : S^{31} \to \mathbb{R}^3$ is now:

$$\nu(\vec{h}) = \sum_{\beta=1}^{4} (\overline{h}_{2\beta-1} h_{2\beta} - \overline{h}_{2\beta} h_{2\beta-1}).$$

We are interested in the common zero set $\mathcal{N} = \mu^{-1}(0) \cap \nu^{-1}(0)$.

Proposition 4.1. $\mathcal{N} = \mathrm{U}(1) \cdot V$, *where* $\cdot$ *is the action of* $\mathrm{U}(1)$ *on Cayley 4-frames.*

Proof. The inclusion $V \subset \mathcal{N}$ can be checked either by direct computation, using Proposition 3.3, or by a standard choice of the frame, like $(1, \mathrm{i}, \mathrm{j}, \mathrm{k})$, and the observation that $\nu(\vec{h}) = \nu(\vec{a} + \vec{b}\mathrm{i} + \vec{c}\mathrm{j} + \vec{d}\mathrm{k}) = 0$ is invariant under right multiplication of $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ by any $u \in S^6$, and hence by Spin(7) (cf. [**11**], p. 121). It follows also $\mathrm{U}(1) \cdot V \subset \mathcal{N}$, by the U(1)-equivariance of ν.

Conversely, to see that $\mathcal{N} \subset \mathrm{U}(1) \cdot V$, refer to a standard choice of three vectors to be substituted in the moment map equation $\nu(\vec{h}) = \nu(\vec{a} + \vec{b}\mathrm{i} + \vec{c}\mathrm{j} + \vec{d}\mathrm{k}) = 0$, assuming $f_2 = \vec{b} = \mathrm{j}, f_3 = \vec{c} = \mathrm{e}, f_4 = \vec{d} = \mathrm{g}$, (cf. the similar proof of the G_2-case in [**12**]). Then the equation $\nu(\vec{h}) = 0$ and the orthonormality of the frame give $\vec{f_1} = \vec{a} = \cos\theta + \sin\theta \mathrm{i}$. Then it is easy to check that the element $\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}$ of U(1) transforms $(\cos\theta + \sin\theta\mathrm{i}, \mathrm{j}, \mathrm{e}, \mathrm{g})$ into a Cayley 4-frame. □

Observe now that $\mathrm{U}(1) \cap \mathrm{Spin}(7) = \mathrm{U}(1) \cap \mathrm{SU}(4) = \mathbb{Z}_4$ with generator $\tau = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$, and that under the action of τ on V, a Cayley 4-frame (f_1, f_2, f_3, f_4) is transformed into another frame of the same Cayley 4-plane if and only if (f_1, f_2, f_3, f_4) is complex unitary, i.e. an element of $\frac{\mathrm{SU}(4)}{\mathrm{Sp}(1)}$. Also, of course $\tau^2 = -1$, so that any Cayley 4-plane is fixed under it. This explains the following description of the orbits of the $\mathrm{U}(1) \times \mathrm{Sp}(1)$-action on $\mathcal{N}$: points in $\frac{\mathrm{SU}(4)}{\mathrm{Sp}(1)} \subset V$ generate orbits that are the fixed points of an induced action of $\mathbb{Z}_2$ on all the orbits of $\mathcal{N}$, and a 3-Sasakian orbifold $\mathbb{Z}_2 \backslash \frac{\mathrm{Spin}(7)}{\mathrm{Spin}(4)}$ is obtained as quotient. We state the corresponding quaternion Kähler reduction.

THEOREM 4.1. *The quaternion Kähler quotient of* $\mathbb{H}P^7$ *by the described action of* $\mathrm{U}(1) \times \mathrm{Sp}(1)$ *is an orbifold* $\mathbb{Z}_2 \backslash \mathrm{CAYLEY}$*, with a singular stratum isometric to the complex Grassmannian* $\frac{\mathrm{SU}(4)}{S(\mathrm{U}(2)\times\mathrm{U}(2))}$.

REMARK 4.2. By identifying any ζ with its orthogonal complement $\zeta^\perp$, one obtains a *smooth* $\mathbb{Z}_2$-quotient of CAYLEY. Since $\perp$ corresponds to the change of orientation on 4-planes in $\mathbb{R}^7$ ([**6**], p. 11), this smooth $\mathbb{Z}_2$-quotient of CAYLEY is the locally quaternion Kähler Grassmannian of *unoriented* 4-planes in $\mathbb{R}^7$.

The $\mathbb{Z}_2\backslash$ CAYLEY given by Theorem 4.1 is not smooth, its construction yielding the stratified space $\mathcal{M}_{\mathrm{reg}} \cup \mathrm{Gr}_2(\mathbb{C}^4)$. The singular stratum $\mathrm{Gr}_2(\mathbb{C}^4)$ corresponds, under the isometry CAYLEY $\cong \mathrm{Gr}_4(\mathbb{R}^7)$, to the standard $\mathrm{Gr}_4(\mathbb{R}^6) \subset \mathrm{Gr}_4(\mathbb{R}^7)$. Thus the orbifold $\mathbb{Z}_2\backslash$CAYLEY in Theorem 4.1 is isometric to the singular quotient $\mathrm{Gr}_4(\mathbb{R}^7)/\sigma_{\mathbb{R}^6}$ by the symmetry $\sigma_{\mathbb{R}^6}$ with respect to $\mathbb{R}^6 \subset \mathbb{R}^7$.

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