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An integral invariant from the view point of locally conformally Kähler geometry

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Abstract. In this article we study an integral invariant which obstructs the existence on a compact complex manifold of a volume form with the determinant of its Ricci form proportional to itself, in particular obstructs the existence of a Kähler-Einstein metric, and has been studied since 1980s. We study this invariant from the view point of locally conformally Kähler geometry. We first see that we can define an integral invariant for coverings of compact complex manifolds with automorphic volume forms. This situation typically occurs for locally conformally Kähler manifolds. Secondly, we see that this invariant coincides with the former one. We also show that the invariant vanishes for any compact Vaisman manifold.

1. Introduction

In [6], the first author introduced an integral invariant defined on Fano manifolds and showed that it obstructs the existence of Kähler-Einstein metrics. More precisely, if M is a Fano manifold of dimension n and

$$\omega = i g_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

is a Kähler form representing $2\pi c_1(M)$, there exists a real smooth function $F \in C^\infty(M)$ such that the Ricci form

$$\rho(\omega) = -i\partial\bar{\partial} \log \det g$$

is written as

$$\rho(\omega) - \omega = i\partial\bar{\partial} F$$

since both $\rho(\omega)$ and ω represent $2\pi c_1(M)$. Then the invariant is defined as a character f of the Lie algebra $\mathfrak{h}(M)$ of all holomorphic vector fields on M into \mathbb{C} and is expressed for $X \in \mathfrak{h}(M)$ by

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$$f(X) = \int_M XF \omega^n. \quad (1)$$

This invariant was later extended in various ways. We first briefly review the various extensions.

The first line of extension is as invariants for compact Kähler manifolds with fixed Kähler class. Let $(M, [\omega])$ be a compact Kähler manifold M with a Kähler class $[\omega]$. Then we can extend f as an obstruction to the existence of a Kähler form in $[\omega]$ of constant scalar curvature ([7], [2]). This is defined by the same formula (1) if we replace the condition of F by

$$\sigma - \int_M \sigma \omega^n / \text{vol}(M) = \Delta F$$

where σ denotes the scalar curvature of ω . When $[\omega]$ is an integral class this was further reformulated by Donaldson [3] as an invariant for polarized schemes, and was used to define the notion of K-stability. In a guise the reformulated invariant was expressed as slopes for subschemes by Ross and Thomas [17]. For Fano manifolds with the anticanonical class, the invariant f has recently been extended to an obstruction to the existence of Kähler–Einstein metrics with cone singularities along a divisor (Donaldson [4], Li [14]), and it is used to define logarithmic K-stability. Around the same time as the work [7] and [2], the invariant f obstructing the constant scalar curvature Kähler metric was extended further by Bando [1] to a family of invariants $f_k, k = 1, \dots, n$, where f_k obstructs the existence of a Kähler form in $[\omega]$ such that the k th Chern form $c_k(\omega)$ is harmonic. Notice that the scalar curvature is constant if and only if the first Chern form is harmonic by the second Bianchi identity. Thus f_1 coincides with f . Bando’s idea can be further extended to transverse Kähler geometry of compact Sasaki manifolds [12].

The second line of extension was obtained in [10], but this extension is obtained by relating the invariant f for Fano manifolds to invariants classically known in the theory of the equivariant cohomology. Again, let M be a Fano manifold and $\omega = i g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$ is a Kähler form representing $2\pi c_1(M)$. By the solution by Yau [20] to the Calabi conjecture, there exists a Kähler form η representing $2\pi c_1(M)$ such that $\rho(\eta) = \omega$. Then we can rewrite f as in (1) in terms of η and obtain

$$f(X) = \int_M \text{div} X \rho(\eta)^n \quad (2)$$

where

$$\text{div} X \cdot \eta^n = \partial i(X)\eta^n \quad (3)$$

and $i(X)$ denotes the interior product by X , see [10] or [8] for the detail. Note that, instead of the Kähler form η , we may use any volume form Ω and its Ricci form

$$\rho(\Omega) = -i\partial\bar{\partial} \log \Omega.$$

Then we may write (2) as

$$f(X) = \int_M \operatorname{div} X \rho(\Omega)^n \tag{4}$$

where

$$\operatorname{div} X \cdot \Omega = \partial i(X)\Omega. \tag{5}$$

We can prove that f is then independent of the choice of Ω , and thus we do not need to assume that M is Fano or Kähler. Thus we obtained an invariant for (possibly non-Kähler) complex manifolds. This last invariant is the one we wish to study in this paper. Note also that we can rewrite (4) as

$$f(X) = - \int_M X \left(\frac{\rho(\Omega)^n}{\Omega} \right) \Omega. \tag{6}$$

Therefore the invariant f is an obstruction to the existence of a volume form Ω such that $\rho(\Omega)^n / \Omega$ is constant.

We remark in passing that there is a larger family of invariants including these two lines of extension ([9]). Among them we have a family of invariants which obstructs asymptotic Chow semistability of polarized manifolds ([9], [11]). By computing them for a 7-dimensional toric Fano manifold suggested by Nill and Paffenholz [15], Ono, Sano and Yotsutani [16] showed that there is a Kähler-Einstein Fano manifold which is asymptotically unstable.

Now let us turn to the study of the invariant defined by (4) or (6). Let M be a compact connected complex manifold of dimension n . Consider a covering space $\pi : \tilde{M} \rightarrow M$ with the group Γ of the deck transformations, and let $\chi : \Gamma \rightarrow \mathbb{R}^+$ be a homomorphism. A volume form Ω on \tilde{M} is said to be automorphic with respect to χ if, for any $\gamma \in \Gamma$,

$$\gamma^* \Omega = \chi(\gamma) \Omega. \tag{7}$$

Such a covering with automorphic volume form naturally occurs for locally conformally Kähler manifolds as we shall see in the next section.

Given such a covering \tilde{M} with automorphic volume form with respect to χ we have a Ricci form ρ_Ω of Ω defined on \tilde{M} by

$$\rho_\Omega = -i \partial \bar{\partial} \log \Omega. \tag{8}$$

Since Ω is automorphic, ρ_Ω is invariant under the action of Γ , and thus descends to a 2-form on M which is denoted by the same notation ρ_Ω . This represents the first Chern class $2\pi c_1(M)$, and also $2\pi c_1(\tilde{M})$ upstairs if \tilde{M} is compact.

Denote by $\mathfrak{h}(M)$ and $\mathfrak{h}(\tilde{M})$ the Lie algebras of all holomorphic vector fields on M and \tilde{M} respectively. Denote also by $\mathfrak{h}_\Gamma(\tilde{M})$ the Lie subalgebra of $\mathfrak{h}(\tilde{M})$ consisting of all holomorphic vector fields on \tilde{M} which are invariant under the action of Γ . Then a holomorphic vector field in $\mathfrak{h}_\Gamma(\tilde{M})$ descends to a holomorphic vector field

on M , and thus $\mathfrak{h}_\Gamma(\tilde{M})$ can be naturally regarded as a Lie subalgebra of $\mathfrak{h}(M)$. For an X in $\mathfrak{h}_\Gamma(\tilde{M})$, its divergence $\operatorname{div} X$ is defined by

$$\operatorname{div} X \cdot \Omega = \partial i(X)\Omega \quad (9)$$

where $i(X)$ denotes the interior product by X . Since Ω is automorphic and X is invariant under Γ it follows that $\operatorname{div} X$ is invariant under Γ and that $\operatorname{div} X$ descends to a smooth function on M .

We define a linear function $f : \mathfrak{h}_\Gamma(\tilde{M}) \rightarrow \mathbb{C}$ by

$$f(X) = \int_M \operatorname{div} X \rho_\Omega^n. \quad (10)$$

The main theorem of this article is the following.

Theorem 1.1. (a) *Let M be a compact connected complex manifold and \tilde{M} its covering space with the group Γ of deck transformations. Suppose that we are given a character $\chi : \Gamma \rightarrow \mathbb{R}^+$. Then f is independent of the choice of the volume form automorphic with respect to χ .* (b) *The invariants defined by (4) and (10) coincide. This implies, in particular, that the invariant defined by (10) is independent of choice of χ .*

A locally conformally Kähler manifold (M, J, g) is said to be a Vaisman manifold if there is a metric in the conformal class of g for which the Lee form is parallel, see Sect. 2 for more detail.

Theorem 1.2. *The invariant in the previous theorem vanishes for any compact Vaisman manifold.*

This article is organized as follows. In Sect. 2 we summarize the basics of locally conformally Kähler geometry, and give a proof of Theorem 1.2. In Sect. 3 we give a proof of Theorem 1.1. In Sect. 4 we compute the invariant for the one point blow-up of the Hopf surface, and see that this surface gives an example of nontrivial invariant.

2. Locally conformally Kähler manifolds

Let (M, J) be a connected complex manifold of complex dimension $n \geq 2$ with J a complex structure. A locally conformally Kähler structure (LCK structure for short) on (M, J) is a covering

$$\Gamma \rightarrow (\tilde{M}, \tilde{J}, \tilde{\omega}) \rightarrow (M, J)$$

where \tilde{M} is a covering space of M , $\tilde{\omega}$ a Kähler form on \tilde{M} , and Γ the group of deck transformations acting on \tilde{M} as holomorphic homotheties. Thus there is a homomorphism $\chi : \Gamma \rightarrow \mathbb{R}^+$ satisfying

$$\gamma^* \tilde{\omega} = \chi(\gamma) \tilde{\omega}.$$

A p -form α on \tilde{M} is said to be automorphic with respect to λ if $\gamma^*\alpha = \lambda(\gamma)\alpha$ for some character $\lambda : \Gamma \rightarrow \mathbb{R}^+$. The above Kähler form $\tilde{\omega}$ is an example of an automorphic 2-form.

There is an equivalent definition of an LCK structure described as follows. An LCK structure is a collection of an open covering $M = \cup_{\alpha \in \Lambda} U_\alpha$ and Kähler metrics g_α on U_α satisfying

$$g_\alpha = c_{\alpha\beta} g_\beta$$

on $U_\alpha \cap U_\beta$ with $c_{\alpha\beta} \in \mathbb{R}_+$. Then $\{c_{\alpha\beta}\}$ gives a cocycle. Let θ be a representative as a closed one form defining the same cohomology class as $\{\log c_{\alpha\beta}\}$. Thus we have $d\theta = 0$, and locally $\theta|_{U_\alpha} = df_\alpha$ for a smooth function f_α on U_α with $f_\beta - f_\alpha = \log c_{\alpha\beta}$ and $e^{f_\alpha} g_\alpha = e^{f_\beta} g_\beta$ on $U_\alpha \cap U_\beta$. Therefore $g := e^{f_\alpha} g_\alpha$ defines a global Hermitian metric locally conformal to a Kähler metric. The 1-form θ is called the Lee form. Let ω be the fundamental 2-form defined by

$$\omega(X, Y) = g(JX, Y).$$

Then one easily shows that

$$d\omega = \theta \wedge \omega \tag{11}$$

$$d\theta = 0. \tag{12}$$

As an equivalent third definition we may say that an Hermitian manifold (M, J, g) is a locally conformally Kähler manifold if the fundamental 2-form ω of g satisfies (11) and (12) (see e.g. [5]).

Remark 2.1. When we say (M, J, g) is an LCK manifold we assume that $\theta \neq 0$, that is, (M, J, g) is not globally Kähler.

The equivalence between the second and the third definitions is obvious. To see that first implies the third, suppose that we are in a situation of the first definition. Let L be the \mathbb{R} -bundle given by $\tilde{M} \times_\chi \mathbb{R}$. Since χ is \mathbb{R}^+ -valued, L is oriented and thus is a trivial bundle. It follows that L has a nowhere zero section which defines a positive χ -equivariant function ϕ on \tilde{M} . Then $\omega := \phi^{-1} \tilde{\omega}$ is a Γ -invariant positive 2-form. This ω satisfies the third definition with $\theta = -\log \phi$.

We need only to show that the second implies the first. Suppose that we have Kähler forms ω_α on U_α such that $\omega_\alpha = c_{\alpha\beta} \omega_\beta$ with $c_{\alpha\beta} \in \mathbb{R}^+$. Then $\{c_{\alpha\beta}\} \in H^1(M, \mathbb{R}^{+\delta})$ defines a flat principal \mathbb{R}^+ -bundle. The holonomy gives a character $\chi : \Gamma = \pi_1(M) \rightarrow \mathbb{R}^+$. Let $L = \tilde{M} \times_\chi \mathbb{R}$ be the associated \mathbb{R} -bundle. We may regard $\{\omega_\alpha\}$ as a section of

$$L \otimes \Lambda^2 T^*M = (\tilde{M} \times_\chi \mathbb{R}) \otimes \Lambda^2 T^*M = \tilde{M} \times_\chi (\mathbb{R} \otimes p^* \Lambda^2 T^*M)$$

where $p : \tilde{M} \rightarrow M$ is the projection. Thus $\{\omega_\alpha\}$ defines a χ -equivariant closed 2-form $\tilde{\omega}$ on \tilde{M} . This completes the equivalence of the three definitions of LCK structures.

Proof of Theorem 1.2. Recall that a locally conformally Kähler manifold (M, J, g) is said to be a Vaisman manifold if there is a metric in the conformal class of g for which the Lee form is parallel. It is shown in [13] that a Vaisman manifold is obtained as a quotient of the Kähler cone $C(S)$ of a Sasakian manifold S by a subgroup Γ of the homotheties acting freely and properly discontinuously. Then the proof follows from Lemma 2.2 below since for the Reeb vector field $\xi = Jr \frac{\partial}{\partial r}$ on $C(S)$, $\xi - iJ\xi$ generates a holomorphic flow and the Ricci tensor degenerates on this orbit. See the arguments below for the notations.

Recall that a Riemannian manifold (S, g) of dimension $2m + 1$ is a Sasakian manifold if the cone $(C(S), \bar{g}) = (\mathbb{R}^+ \times S, dr^2 + r^2g)$ is a Kähler manifold. Here r is the standard coordinate on \mathbb{R}^+ . The metric \bar{g} is a warped product metric, and the Riemannian geometry of $C(S)$ is easily studied from that of S . Let $\bar{\nabla}$ and ∇ be the Levi-Civita connections on $C(S)$ and S respectively. Let X, Y be tangent vector fields on S , which are naturally regarded as vector fields on $C(S)$ by the product structure $C(S) = \mathbb{R}^+ \times S$. Consider a vector field $\Psi := r \frac{\partial}{\partial r}$ on $C(S)$. It is generally true for cone manifolds that

$$\bar{\nabla}_X \Psi = \bar{\nabla}_\Psi X = X \tag{13}$$

and that

$$\bar{\nabla}_X Y = \nabla_X Y - g(X, Y)\Psi. \tag{14}$$

Let \bar{R} be the curvature tensors on $C(S)$. Then using (13) and (14) we obtain

$$\begin{aligned} \bar{R}(X, \Psi, Y, \Psi) &= \bar{g}(-\bar{\nabla}_X \bar{\nabla}_\Psi Y + \bar{\nabla}_\Psi \bar{\nabla}_X Y + \bar{\nabla}_{[X, \Psi]} Y, \Psi) \\ &= \bar{g}(-\bar{\nabla}_X Y + \bar{\nabla}_\Psi(\nabla_X Y - g(X, Y)\Psi), \Psi) \\ &= -\bar{g}(\nabla_X Y - g(X, Y)\Psi, \Psi) + \bar{g}(\nabla_X Y - g(X, Y)\Psi, \Psi) \\ &= 0. \end{aligned}$$

This implies that

$$\bar{R}ic(\Psi, \Psi) = 0 \tag{15}$$

where $\bar{R}ic$ denotes the Ricci tensor on $C(S)$.

Let J be the complex structure on $C(S)$. The vector field $\xi = Jr \frac{\partial}{\partial r}$ on $C(S)$ is called the Reeb vector field, and it is a standard fact in Sasakian geometry that $\xi - iJ\xi$ is a holomorphic vector field. Since the Ricci tensor on a Kähler manifold is J -invariant, (15) implies

$$\bar{R}ic(\xi, \xi) = 0. \tag{16}$$

From (15) and (16) we have proved the following.

Lemma 2.2. *The Ricci tensor on $C(S)$ vanishes on the plane spanned by ξ and $J\xi = -r \frac{\partial}{\partial r}$.*

For a Vaisman manifold M we can also find an LCK metric g for which the Ricci form $\rho(g)$ satisfies $\rho(g)^n = 0$, showing that the integrand of (4) and (6) vanishes. Recall that the Kähler form $\tilde{\omega}$ on $C(S)$ is written as $\tilde{\omega} = dd^c r^2$, see for example [12]. As we have seen in Lemma 2.2, we have $\rho(\tilde{\omega})^{m+1} = 0$. Note that $n = m + 1$ here. Then $\tilde{\omega}/r^2 = \frac{1}{r^2} dd^c r^2$ defines a Γ -invariant 2-form and descends to M . Since $dd^c \log r$ is the transverse Kähler form ω^T on the Sasaki manifold S (but regarded as lifted to $C(S)$), the Ricci form of $\tilde{\omega}/r^2$ is equal to $\rho(\tilde{\omega}) + 2(m + 1)\omega^T$. This also degenerates on the orbit of the flow generated by $\xi - iJ\xi$. Hence we have $\rho(\tilde{\omega}/r^2)^{m+1} = 0$ on the Vaisman manifold M .

3. Proof of Theorem 1.1

In this section we prove the following result.

Theorem 3.1. *Let M be a compact connected complex manifold and \tilde{M} its covering space with the group Γ of deck transformations. Suppose that we are given a character $\chi : \Gamma \rightarrow \mathbb{R}^+$. Then f defined by (10) is independent of the choice of the automorphic volume form Ω and its character χ .*

Theorem 1.1 follows from Theorem 3.1 because (a) in Theorem 1.1 is obtained by comparing (Ω_1, χ) and (Ω_2, χ) , and (b) in Theorem 1.1 is obtained by comparing (Ω_1, χ) and $(\Omega_2, 1)$.

Let M be a compact connected complex manifold and \tilde{M} be a covering space of M with the group Γ of deck transformations. Let Ω be a smooth volume form on \tilde{M} automorphic with respect to $\chi : \Gamma \rightarrow \mathbb{R}^+$. If z^1, \dots, z^n are local holomorphic coordinates on \tilde{M} , the volume form Ω can be expressed as

$$\Omega = a \, idz^1 \wedge d\bar{z}^1 \wedge \dots \wedge idz^n \wedge d\bar{z}^n \tag{17}$$

where a is a local positive smooth function. The Ricci form ρ_Ω and the divergence $\operatorname{div} X$ can be expressed using a as

$$\rho_\Omega = -i\partial\bar{\partial} \log a, \tag{18}$$

and

$$\operatorname{div} X = \sum_{i=1}^n \frac{\partial X^i}{\partial z^i} + X(\log a). \tag{19}$$

From (19) we obtain

$$\bar{\partial} \operatorname{div} X = i(X)\partial\bar{\partial} \log a. \tag{20}$$

Proof of Theorem 3.1. Let Ω_0, Ω_1 be volume forms automorphic with respect to $\chi_0, \chi_1 : \Gamma \rightarrow \mathbb{R}^+$, respectively. Then Ω_i can be expressed as

$$\begin{aligned} \Omega_0 &= a \, idz^1 \wedge d\bar{z}^1 \wedge \dots \wedge idz^n \wedge d\bar{z}^n, \\ \Omega_1 &= \varphi a \, idz^1 \wedge d\bar{z}^1 \wedge \dots \wedge idz^n \wedge d\bar{z}^n, \end{aligned}$$

where the positive real valued function φ on \tilde{M} is given by $\Omega_1 = \varphi\Omega_0$. Then we have

$$\gamma^*\varphi = \frac{\chi_1(\gamma)}{\chi_0(\gamma)}\varphi \quad (21)$$

for all $\gamma \in \Gamma$.

Let Ω_t be

$$\Omega_t = \varphi^t a idz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge idz^n \wedge d\bar{z}^n, \quad (22)$$

for each $0 \leq t \leq 1$. Then each Ω_t is automorphic with respect to a character $\chi_t := \chi_0^{1-t}\chi_1^t$. Thus a smooth family of linear maps $f_t : \mathfrak{h}_\Gamma(\tilde{M}) \rightarrow \mathbb{C}$ is defined by

$$f_t(X) = \int_M \operatorname{div}_t X \rho_{\Omega_t}^n, \quad (23)$$

where $\operatorname{div}_t X$ is the divergence determined by Ω_t . Then it suffices to show that

$$\frac{d}{dt} f_t(X) = 0 \quad (24)$$

for all $X \in \mathfrak{h}_\Gamma(\tilde{M})$.

It is easy to see

$$\frac{d}{dt} (\operatorname{div}_t X) = X(\log \varphi) \quad (25)$$

and

$$\frac{d}{dt} \rho_{\Omega_t} = -i\partial\bar{\partial} \log \varphi. \quad (26)$$

Then we have

$$\begin{aligned} & \frac{d}{dt} \int_M \operatorname{div}_t X \cdot \rho_{\Omega_t}^n \\ &= \int_M X(\log \varphi) \rho_{\Omega_t}^n - \int_M \operatorname{div}_t X i\partial\bar{\partial}(\log \varphi) \wedge n\rho_{\Omega_t}^{n-1} \\ &= \int_M X(\log \varphi) \rho_{\Omega_t}^n + \int_M \bar{\partial}(\operatorname{div}_t X \wedge \partial(i \log \varphi) \wedge n\rho_{\Omega_t}^{n-1}) \\ & \quad - \int_M \bar{\partial}(\operatorname{div}_t X) \wedge \partial(i \log \varphi) \wedge n\rho_{\Omega_t}^{n-1}. \end{aligned}$$

Although φ may not be Γ -invariant, $\partial \log \varphi$ is Γ -invariant from (21) and descends to a 1-form on M . Since $\operatorname{div}_t X$ and ρ_{Ω_t} are also defined globally on M , we can deduce

$$\int_M \bar{\partial}(\operatorname{div}_t X \wedge \partial(i \log \varphi) \wedge n \rho_{\Omega_t}^{n-1}) = 0 \quad (27)$$

from Stokes' Theorem. Therefore we have

$$\begin{aligned} & \frac{d}{dt} \int_M \operatorname{div}_t X \cdot \rho_{\Omega_t}^n \\ &= \int_M X(\log \varphi) \rho_{\Omega_t}^n - \int_M \bar{\partial}(\operatorname{div}_t X) \wedge \partial(i \log \varphi) \wedge n \rho_{\Omega_t}^{n-1} \\ &= \int_M X(\log \varphi) \rho_{\Omega_t}^n \\ &\quad - \int_M (i(X) \partial \bar{\partial} \log(\varphi^t a)) \wedge \partial(i \log \varphi) \wedge n \rho_{\Omega_t}^{n-1} \\ &= \int_M X(\log \varphi) \rho_{\Omega_t}^n + \int_M (i(X) \rho_{\Omega_t}^n) \wedge \partial \log \varphi \\ &= \int_M i(X)(\rho_{\Omega_t}^n \wedge \partial \log \varphi) = 0 \end{aligned}$$

since $\rho_{\Omega_t}^n \wedge \partial \log \varphi \equiv 0$ because of dimension reasons. This completes the proof of Theorem 3.1. \square

Remark 3.2. In general the vanishing of f is the obstruction to the existence of an automorphic volume form on \tilde{M} with $\rho_{\Omega} = 0$. But if $\tilde{M} = M$ or χ is trivial, the vanishing of f obstructs the existence of a volume form with $\rho_{\Omega}^n = k\Omega$ for some constant k .

4. The localization formula and an example

Now that the invariant is independent of the choice of (Ω, χ) we may use an old result to compute the case when χ is trivial. This is a residue formula for holomorphic vector fields.

Let X be a holomorphic vector field in $\mathfrak{h}(M)$. Define a section $L(X)$ of the endomorphism bundle $\operatorname{End}(TM)$ of the holomorphic tangent bundle TM by

$$L(X)Y = \nabla_X Y - [X, Y]. \quad (28)$$

Suppose that the zero set $\operatorname{zero}(X)$ of X consists of smooth submanifolds $\{Z_\lambda\}_{\lambda \in \Lambda}$. Then $L(X)$ induces a section $L^\nu(X)$ of the endomorphism bundle of the normal bundle $\nu(Z_\lambda) = (TM|_{Z_\lambda})/TZ_\lambda$ of Z_λ .

Theorem 4.1 (Theorem 5.2.8 in [8]). *If $L^v(X)$ is nonsingular at every $q \in \text{zero}(X)$, we have the following localization formula*

$$\left(\frac{1}{2\pi}\right)^n (n+1)f(X) = \sum_{\lambda} \int_{Z_{\lambda}} \left((\text{div } X + c_1(M))^{n+1}|_{Z_{\lambda}} \right) / \det \left(L^v(X) + \frac{i}{2\pi} K \right)$$

where K is the curvature form of $v(Z_{\lambda})$ with respect to the induced Hermitian connection.

We provide an explicit computation of non-zero invariant on the blow-up at a point of a Hopf surface which by [18, 19] is an LCK manifold, using the localization formula.

Let H^2 be a Hopf surface that we regard as $\mathbb{C}^2 \setminus \{0\}/\mathbb{Z}$, where \mathbb{Z} is generated by the transformation $(z_1, z_2) \mapsto (2z_1, 2z_2)$. We choose the fundamental domain on $\mathbb{C}^2 \setminus \{0\}$ to be $\{(z_1, z_2) \mid 1 \leq |z_1|^2 + |z_2|^2 \leq 2\}$.

Let M be the blow-up of H^2 at the point $(0, \frac{3}{2})$. It will be convenient to change the coordinates (z_1, z_2) into (w_1, w_2) by:

$$w_1 = z_1, \quad w_2 = z_2 - \frac{3}{2}.$$

Then the blow-up takes place at the origin $(w_1, w_2) = (0, 0)$ and the exceptional divisor E is $\{(w_1 : w_2)\} \cong \mathbb{C}P^1 \subset M$.

Let $X = z_1 \frac{\partial}{\partial z_1}$ be the radial (global) vector field on \mathbb{C}^2 . Its zero set contains $(0, \frac{3}{2})$. We shall equally denote by X its lift to M . Its zero set on M will certainly contain $\{z_1 = 0\}$, but also some other point that we now determine.

Take first local coordinates on M around $(1 : 0) \in E$ to be

$$\zeta_1 = w_1, \quad \zeta_2 = \frac{w_2}{w_1}.$$

This change of coordinates is consistent with the coordinates on the exceptional divisor. In the new coordinates, X is written as

$$X = \zeta_1 \frac{\partial}{\partial \zeta_1} - \zeta_2 \frac{\partial}{\partial \zeta_2}.$$

In these coordinates (ζ_1, ζ_2) , $\text{zero}(X) = \{(0, 0)\}$. Hence, the zero is on E (as $\zeta_1 = 0$). On the other hand, $\zeta_2 = 0$ implies $w_2 = 0$. Thus, the isolated zero of X is $(w_1 : w_2) = (1 : 0)$.

Now recall that, in general, if a holomorphic vector field $Y = a \frac{\partial}{\partial \zeta_1} + b \frac{\partial}{\partial \zeta_2}$, then

$$L(Y)\left(\frac{\partial}{\partial \zeta_j}\right)|_{\text{zero}(Y)} = -\mathcal{L}_Y \frac{\partial}{\partial \zeta_j} + \nabla_X \frac{\partial}{\partial \zeta_j} = \frac{\partial a}{\partial \zeta_j} \frac{\partial}{\partial \zeta_1} + \frac{\partial b}{\partial \zeta_j} \frac{\partial}{\partial \zeta_2}, \quad (29)$$

as the $\nabla_Y = 0$ on the zero set of Y .

Hence, in our case, for the point $(1 : 0)$, the localization formula reduces to:

$$\frac{\text{tr}(L(X))^3}{\det(L^v(X))} = \frac{\left(\text{tr} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)^3}{\det \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} = 0,$$

as the normal bundle of the point equals the tangent bundle at the point, and this is trivial, hence $\Theta = 0$.

So, the isolated zero does not contribute to the value of the invariant.

For the dimension 1 component of $\text{zero}(X)$, take on M , around $(0 : 1)$, the coordinates:

$$\zeta_1 = w_2, \quad \zeta_2 = \frac{w_1}{w_2}.$$

In these coordinates X takes the form

$$X = \zeta_2 \frac{\partial}{\partial \zeta_2},$$

and $\text{zero}(X) = \{(\zeta_1, 0)\}$, a line represented by $(0 : 1)$. It is the proper transform of $\{z_1 = 0\}$. Using (29), we find now

$$L(X) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The localization formula gives:

$$\int_Z \frac{\left(\text{tr} \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sqrt{-1}}{2\pi} \Theta \right) \right)^3}{1 + \frac{\sqrt{-1}}{2\pi} \Theta^\nu} = \int_Z \frac{(1 + c_1(Z) + c_1(\nu(Z)))^3}{1 + c_1(\nu(Z))},$$

where $\nu(Z)$ is the normal bundle of the zero set $Z = \text{zero}(X)$.

Observe that $\nu(Z) = -[E]$. Indeed, if $\pi : M \rightarrow H^2$ denotes the natural projection, then:

$$[\pi(Z)] = [\pi(Z + E)],$$

and hence (as they are trivial line bundles),

$$0 = \pi^*[\pi(Z)] = [Z + E] = [Z] + [E] = [\nu(Z)] + [E].$$

On the other hand, $c_1(Z) = 0$, as Z is an elliptic curve. We obtain:

$$\begin{aligned} \int_Z \frac{(1 + c_1(Z) + c_1(\nu(Z)))^3}{1 + c_1(\nu(Z))} &= \int_Z (1 - c_1([E]))^3 (1 + c_1([E])) \\ &= \int_Z (-3c_1([E]) + c_1([E])) \\ &= Z \cdot (-2[E]) = -2. \end{aligned}$$

In conclusion, $3\left(\frac{1}{2\pi}\right)^2 f(X) = -2 \neq 0$.

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