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Sasakian structures on CR-manifolds

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Abstract A contact manifold M can be defined as a quotient of a symplectic manifold X by a proper, free action of $\mathbb{R}$, with the symplectic form homogeneous of degree 2. If X is also Kähler, and its metric is homogeneous of degree 2, M is called Sasakian. A Sasakian manifold is realized naturally as a level set of a Kähler potential on a complex manifold, hence it is equipped with a pseudoconvex CR-structure. We show that any Sasakian manifold M is CR-diffeomorphic to an S^1-bundle of unit vectors in a positive line bundle on a projective Kähler orbifold. This induces an embedding of M into an algebraic cone C. We show that this embedding is uniquely defined by the CR-structure. Additionally, we classify the Sasakian metrics on an odd-dimensional sphere equipped with a standard CR-structure.

Keywords CR-manifold · Sasakian manifold · Reeb field · Pseudo convex manifold · Vaisman manifold · Stein manifold · Deformation · Potential

Mathematics Subject Classification (2000) 53C55

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1 Introduction

1.1 Sasakian manifolds and algebraic cones

In this paper, we study existence of Sasakian metrics on strictly pseudoconvex CR-manifolds. A pseudoconvex CR-manifold is a geometric structure arising on a smooth boundary of a Stein domain X (Defintion 2.13, Remark 2.14). If M is compact and strictly pseudoconvex CR-manifold of dimension > 3, then M can always be realized as a boundary of a Stein variety X with at most isolated singularities [1,16]. In fact, the geometry of CR-structures is essentially the same as the holomorphic geometry of the corresponding Stein variety. In particular, the automorphisms of X are in a natural correspondence with the CR-diffeomorphisms of its boundary.

Strictly pseudoconvex CR-manifolds are always contact; they are sometimes called *contact pseudoconvex*.

Sasakian metrics are the special Riemannian metrics on contact pseudoconvex CR-manifolds. They are related to Kähler metrics, in the same way as the contact structures are related to symplectic structures. A contact manifold can be defined as a manifold with a symplectic structure on its cone; a Sasakian metric on a contact manifold induces a Kähler metric on its symplectic cone (Definition 3.1).

Sasakian manifolds can be defined in terms of algebraic cone spaces, as follows.

Definition 1.1 *A closed algebraic cone* is an affine variety $\mathcal{C}$ admitting a $\mathbb{C}^*$-action ρ with a unique fixed point x_0, which satisfies the following.

1. $\mathcal{C}$ is smooth outside of x_0.
2. ρ acts on the Zariski tangent space $T_{x_0}\mathcal{C}$ diagonally, with all eigenvalues $|\alpha_i| < 1$.

An open algebraic cone is $\mathcal{C}\backslash\{x_0\}$.

In Sec. 4.1 we give another, equivalent but more constructive, definition of an algebraic cone (Definition 4.2).

By definition, a Sasakian manifold M admits a CR-embedding into an algebraic cone $\mathcal{C}(M) := M \times \mathbb{R}^{>0}$, as a set $M \times \{t_0\}$. The function $\mathcal{C}(M) \longrightarrow \mathbb{R}^{>0}$, $(m, t) \longrightarrow t^2$ is a Kähler potential of $\mathcal{C}(M)$, as follows from an elementary calculation (see e.g. [21]). The converse is also true: given a Kähler potential $\varphi: \mathcal{C} \longrightarrow \mathbb{R}$ on an algebraic cone $\mathcal{C}$, satisfying $\mathrm{Lie}_v\, \varphi = 2\varphi$, for a vector field $v \in T\mathcal{C}$ inducing a holomorphic contraction on $\mathcal{C}$, we may assume that $(\mathcal{C}, \partial\overline{\partial}\varphi)$ is a Riemannian cone of M.[1]

The correspondence between algebraic cones and Sasakian manifolds is quite significant. One may argue that the algebraic cone, associated with a Sasakian manifold, gives a functor similar in many respects to the forgetful functor from the category of Kähler manifolds to the category of complex manifolds. Indeed, the moduli space of algebraic cones is finite-dimensional, as follows from Definition 4.2, and the Sasakian metrics are determined by an additional set of C^∞-data (the Kähler potential).

One could also argue that a proper analogy of a complex structure is a CR-structure underlying a Sasakian manifold. However, the CR-structure (unlike complex structure, or a structure of an algebraic cone) in many cases, e.g. in dimension 3, determines the Sasakian metric completely (up to a constant). In fact, there is only a finite-dimensional set of Sasakian metrics on a given CR-manifold ([BGS]; see also Theorem 1.11).

In this paper, we study the forgetful functor from the category of Sasakian manifolds to the category of algebraic cones. We show that it is determined by the CR-structure.

[1] Here, Lie_v denotes the Lie derivative.

Theorem 1.2 *Let M be a compact pseudoconvex contact CR-manifold. Then the following conditions are equivalent.*

(i) *M admits a Sasakian metric, compatible with the CR-structure.*
(ii) *M admits a proper, transversal CR-holomorphic S^1-action.*
(iii) *M admits a nowhere degenerate, transversal CR-holomorphic vector field.*

Theorem 1.3 *Let M be a compact, strictly pseudoconvex CR-manifold admitting a proper, transversal CR-holomorphic S^1-action. Then M admits a unique (up to an automorphism) S^1-invariant CR-embedding into an algebraic cone $\mathcal{C}$. Moreover, a Sasakian metric on M can be induced from an automorphic Kähler metric on this cone.*[2]

We prove Theorem 1.2 in Subsect. 5.1, and Theorem 1.3 in Subsect. 5.2 (when M is not a sphere). The case when M is a sphere is considered at the end of Subsect. 6.2.

Remark 1.4 The Sasakian metric is by definition induced from an embedding to its cone, which is a Kähler manifold. This cone is algebraic, as indicated above. This metric is not unique, though the embedding is unique and canonical, as follows from Theorem 1.3.

Remark 1.5 For another approach to the existence of Sasakian structures compatible with a contact pseudoconvex structure on a compact manifold, see [8]. A still different approach is the following: In the course of the proof of [14, Theorem E], Lee proves that the infinitesimal generator of a transverse CR-automorphism (of a pseudoconvex contact structure) is necessarily a Reeb vector field for a contact form underlying the given contact bundle. On the other hand, Webster proved in [22] that if the Reeb field of a pseudoconvex contact structure is a CR-automorphism, then the torsion of the Tanaka connection vanishes. But it is known (see e.g. [9]) that a pseudoconvex contact structure with zero Tanaka torsion is Sasakian. However, it seems that this result was never explicitly stated as such.

Remark 1.6 In dimension 3, Sasakian structures on CR-manifolds were completely classified [3,4,11]. For a 3-dimensional CR-manifold M, not isomorphic to a sphere, the Sasakian metric is unique, hence the corresponding cone is also unique. Theorem 1.2 and Theorem 1.3 in dimension 3 follow immediately from [3,4]. In this paper, we shall always assume that $\dim M \geq 5$.

1.2 Sasakian geometry and contact geometry

There is a way to define contact manifolds and Sasakian manifolds in a uniform manner. Let M be a smooth manifold equipped with a free, proper action of the multiplicative group $\mathbb{R}^{>0}$, and a symplectic form ω. Assume that ω is homogeneous of weight 2 with respect to ρ, that is, $\mathrm{Lie}_v\, \omega = 2\omega$, where $v \in TM$ is the tangent vector field of ρ. Then the quotient M/ρ is contact. This can be considered as a definition of a contact manifold (see Remark 2.9). Then M is called a *symplectic cone* of a contact manifold M/ρ.

Now, let (M, g, ω) be a Kähler manifold (here we consider (M, ω) as a symplectic manifold, equipped with a compatible Riemannian structure g). Assume again that ρ is a free, proper action of $\mathbb{R}^{>0}$ on M, and g and ω are homogeneous of weight 2:

$$\mathrm{Lie}_v\, \omega = 2\omega, \quad \mathrm{Lie}_v\, g = 2g.$$

[2] By automorphic Kähler metric on an open algebraic cone we understand a metric ω which satisfies $\rho^*(\lambda)(\omega) = |\lambda|^2\omega$, for all $\lambda \in \mathbb{C}^*$.

The quotient M/ρ is contact (as indicated above) and Riemannian (the Riemannian metric is obtained from g by appropriate rescaling). It easily follows from the definitions that M/ρ is Sasakian. In fact, the Sasakian manifolds can be defined this way (see Definition 3.1). The Sasakian metric is therefore a natural odd-dimensional counterpart to Kähler metrics.

In symplectic geometry, one is often asked the following question.

Question 1.7 Let M be a symplectic manifold. Is there a Kähler metric compatible with the symplectic structure?

It is natural to ask the same question for contact geometry.

Question 1.8 Let M be a contact manifold. Is there a Sasakian metric compatible with the contact structure?

A partial answer to this question is given in this paper, in the additional assumptions of existence of CR-structure, which is natural and very common in contact topology.

A set of natural examples of CR and Sasakian manifolds is provided by algebraic geometry.

Example 1.9 Let X be a projective orbifold with quotient singularities (in algebraic geometry such an object is also known under the name of "Deligne-Mumford stack"), and L an ample Hermitian line bundle on X. Assume that the curvature of L is positive. Let $\operatorname{Tot}(L^*)$ be the space of all non-zero vectors in the dual bundle, considered as a complex manifold, and φ: $\operatorname{Tot}(L^*) \longrightarrow \mathbb{R}$ map $v \in L^*$ into $|v|^2$. It is easy to check that φ is strictly plurisubharmonic, that is, $\partial\overline{\partial}\varphi$ is a Kähler form on $\operatorname{Tot}(L^*)$. Therefore, the level set $M := \varphi^{-1}(\lambda)$ of φ is a strictly pseudoconvex CR-manifold. This level set is a $U(1)$-bundle on X.

It is easy to see that the metric $\partial\overline{\partial}\varphi$ induces a Sasakian structure on M (see e.g. [21]). Such Sasakian manifolds are called *quasiregular*.

In dimension 3, Belgun has shown that all Sasakian manifolds are obtained this way (see [3,4,11]):

Theorem 1.10 *A strictly pseudoconvex, compact CR-manifold M of dimension 3 admits a Sasakian metric if and only if M is isomorphic to a $U(1)$-fibration associated with a positive line bundle on a projective orbifold (Example 1.9). Moreover, the Sasakian metric on M is unique, up to a constant multiplier, unless M is $S^3 \subset \mathbb{C}^2$.*

Using a construction of Sasakian positive cone due to [8], we shall extend this theorem to arbitrary dimension.

Theorem 1.11 *Let M be a strictly pseudoconvex, compact CR-manifold. Then M admits a Sasakian metric if and only if M is CR-isomorphic to a $U(1)$-fibration associated with a positive line bundle on a projective orbifold (Example 1.9). Moreover, the set of Sasakian structures on M is in bijective correspondence with the set of positive and transversal CR-holomorphic vector fields on M.*[3]

Proof The last claim of Theorem 1.11 is due to [8]; we give a new proof of this statement in Subsect. 6.1, and prove the rest of Theorem 1.11. □

[3] This set is called *the positive Sasakian cone of M*. For a definition of positive and transversal CR-holomorphic vector fields, see Subsect. 6.1.

2 Strictly pseudoconvex CR-manifolds

2.1 CR-manifolds and contact manifolds

We recall some definitions, which are well known.

Definition 2.1 Let M be a smooth manifold. A *CR-structure* (Cauchy-Riemann structure) on M is a subbundle $H \subset TM \otimes \mathbb{C}$ of the complexified tangent bundle, which is closed under commutator:

$$[H, H] \subset H$$

and satisfies $H \cap \overline{H} = 0$.

A complex manifold (P, I) is considered as a CR-manifold, with $H = T^{1,0}P \subset TP \otimes \mathbb{C}$.

Definition 2.2 Consider a real submanifold $M \subset P$ in a complex manifold (P, I). Suppose that $TM \cap I(TM)$ has constant rank. Clearly, $H_M := TM \otimes \mathbb{C} \cap T^{1,0}P$ is a CR-structure on M. Then (M, H_M) is called *a CR-submanifold in* M, and H_M is called *the induced CR-structure*.

Remark 2.3 Given a real hypersurface $M \subset P$ in a complex manifold, $\dim_{\mathbb{C}} P = n$, the rank of $TM \cap I(TM)$ is $n - 1$ everywhere, hence M is a CR-submanifold.

Given a CR-manifold (M, H), consider the bundle $H \oplus \overline{H} \subset TM \otimes \mathbb{C}$. This bundle is preserved by the complex conjugation, hence it is the complexification of a $H_{\mathbb{R}} \subset TM$. Since $H \cap \overline{H} = 0$, the map $\operatorname{Re}: H \longrightarrow H_{\mathbb{R}}$ is an isomorphism. The map $\sqrt{-1}\,\operatorname{Id}_H: H \longrightarrow H$ defines a complex structure operator I_H on $H_{\mathbb{R}}$, $I_H^2 = -\operatorname{Id}_{H_{\mathbb{R}}}$. Clearly, H is the $\sqrt{-1}$ -eigenspace of the I_H-action on $H_{\mathbb{R}} \otimes \mathbb{C}$.

Remark 2.4 We obtain that a CR-structure on a manifold M can be defined as a pair $(H_{\mathbb{R}}, I_H)$, where $H_{\mathbb{R}} \subset TM$ is a subbundle in TM, and $I_H \in \operatorname{End}(H_{\mathbb{R}})$ is an endomorphism, $I_H^2 = -\operatorname{Id}_{H_{\mathbb{R}}}$, such that the $\sqrt{-1}$ -eigenspace of I_H-action on $H_{\mathbb{R}} \otimes \mathbb{C}$ satisfies

$$[H, H] \subset H$$

This is the definition we shall use.

Definition 2.5 A *CR-holomorphic* function on a CR-manifold (M, H) is a function $f: M \longrightarrow \mathbb{C}$ which satisfies $D_V(f) = 0$ for any $V \in \overline{H}$ (D_V denotes the derivative). A *CR-holomorphic* map is a smooth map of CR-manifolds such that a pullback of CR-holomorphic functions is CR-holomorphic.

Definition 2.6 Let M be a smooth manifold, and $R \subset TM$ a subbundle. Consider the commutator $[R, R] \longrightarrow TM$. This map is not $C^\infty(M)$-linear. However, its composition with the projection to TM/R is linear. It is called *the Frobenius tensor of the distribution* $R \subset TM$.

Definition 2.7 A *contact manifold* is a smooth manifold M equipped with a codimension 1 subbundle $R \subset TM$ such that the Frobenius tensor $R \times R \longrightarrow TM/R$ is a nowhere degenerate skew-symmetric TM/R-valued form on R. In this case, R is called *the contact distribution on* M.

Remark 2.8 The bundle TM/R is one-dimensional, hence trivial (if oriented). A trivialization η of TM/R defines a 1-form on TM, which is called *a contact form of* M. Its differential $d\eta$ is nowhere degenerate on the contact distribution R. A choice of a trivialization η also defines a *Reeb vector field* ξ by the conditions: $\xi \rfloor \eta = 1$, $\xi \rfloor d\eta = 0$.

Remark 2.9 Let (M, R) be a contact manifold, and η a contact form. Using η, we define a trivialization of TM/R. Then the total space S of positive vectors in $(TM/R)^*$ is identified with the cone $M \times \mathbb{R}^{>0}$. We consider the contact structure as a TM/R-valued 1-form on M. This gives a canonical 1-form on $\operatorname{Tot}((TM/R)^*)$.

Let t be a unit parameter on $\mathbb{R}^{>0}$, and $t\theta$ the corresponding 1-form on S. It is easy to check that $d(t\theta)$ is a symplectic form on S. The converse is also true. Starting from a symplectic form ω on a cone $M \times \mathbb{R}^{>0}$, satisfying $\rho(q)^*\omega = q^2\omega$, where $\rho(q)(m, t) = (m, qt)$ is a dilatation map, we may reconstruct the contact structure on M and the contact form. This can be summarized by saying that a *contact form on M is the same as a conical symplectic structure on $M \times \mathbb{R}^{>0}$*. This construction is explained in greater detail in most textbooks on contact geometry, e.g. [2].

Definition 2.10 A *contact CR-manifold* is a CR-manifold $(M, H_{\mathbb{R}}, I_H)$, such that the distribution $H_{\mathbb{R}} \subset TM$ is contact.

Remark 2.11 Given a CR-manifold $(M, H_{\mathbb{R}}, I_H)$, with $H_{\mathbb{R}}$ of codimension 1, the Frobenius 2-form $H_{\mathbb{R}} \times H_{\mathbb{R}} \longrightarrow TM/R$ is of type (1, 1) with respect to the complex structure on $H_{\mathbb{R}}$. Indeed, this form vanishes on H and $\overline{H}$, because $[H, H] \subset H \subset H_{\mathbb{R}} \otimes \mathbb{C}$.

Definition 2.12 In these assumptions, the (1, 1)-form $H_{\mathbb{R}} \times H_{\mathbb{R}} \longrightarrow TM/R$ is called *the Levi form* of the CR-manifold $(M, H_{\mathbb{R}}, I_H)$.

Definition 2.13 A CR-manifold $(M, H_{\mathbb{R}}, I_H)$ with $H_{\mathbb{R}}$ of codimension 1 is called *pseudoconvex* if the Levi form ω is positive or negative, depending on the choice of orientation. If this form is also sign-definite, then $(M, H_{\mathbb{R}}, I_H)$ is called *strictly pseudoconvex*, or *contact pseudoconvex*.

Remark 2.14 Let $S \subset P$ be a Stein domain in a complex manifold, and ∂S its boundary. Assume that ∂S is smooth; then ∂S inherits a natural CR-structure from P. It is well known that in this case the Levi form on P is positive, though not always definite (see [12]).

2.2 Automorphisms of CR-manifolds

Definition 2.15 Let $\varphi : M \longrightarrow M'$ be a smooth map of CR-manifolds (M, H, I), (M', H', I'). If φ maps H to H' and commutes with the complex structure, φ is called *CR-holomorphic*. A CR-holomorphic diffeomorphism is called *a CR-diffeomorphism*.

Definition 2.16 Let (M, H, I) be a strictly pseudoconvex CR-manifold. A vector field $V \in TM$ is called *transversal* if its image in TM/H is nowhere degenerate. A diffeomorphism flow on M is called *transversal* if its tangent field is transversal.

The following result is proven in [20] (see also [8]):

Theorem 2.17 *Let M be a compact, strictly pseudoconvex CR-manifold which is not isomorphic to a sphere with a standard CR-structure, and G the group of CR-automorphisms of M. Then G is a compact Lie group.*

If $M = S^{2n-1}$ is an odd-dimensional sphere, the group of CR-diffeomorphisms of M is isomorphic to $SU(n, 1)$ (see e.g. [8]; an explicit construction is given in Subsect. 6.2).

3 Vaisman manifolds and Sasakian geometry

3.1 Sasakian manifolds

Definition 3.1 A Riemannian manifold (M, h) of odd real dimension is called *Sasakian* if the metric cone $\mathcal{C}(M) = (M \times \mathbb{R}^{>0}, t^2 h + dt^2)$ is equipped with a dilatation-invariant complex structure, which makes $\mathcal{C}(M)$ a Kähler manifold (see [5,7]).

Remark 3.2 A Sasakian manifold is naturally embedded as a real hypersurface in its cone, $M = (M \times \{1\}) \subset \mathcal{C}(M)$. This defines a CR-structure on M (Remark 2.3).

Claim 3.3 This CR-structure is contact and pseudoconvex (that is, strictly pseudoconvex).

Proof The function $\varphi(m, t) = t^2$ on $\mathcal{C}(M)$ defines a Kähler potential on $\mathcal{C}(M)$ (see e.g. [21]). The level set of a Kähler potential is strictly pseudoconvex, because its Levi form is equal to $\partial\bar{\partial}\varphi\big|_H$. □

Remark 3.4 It is easy to show that a Sasakian manifold is equipped with a canonical contact structure. Indeed, a contact form on M is the same as a conical symplectic form on $\mathcal{C}(M)$, as explained in Remark 2.9. Such a symplectic form is a part of the Kähler structure on $\mathcal{C}(M)$.

Remark 3.5 Let M be a Sasakian manifold. On $M \subset \mathcal{C}(M)$, consider the vector field $\xi = I\left(t\frac{d}{dt}\right)$, where $t\frac{d}{dt}$ is the dilatation vector field of the cone $\mathcal{C}(M) = (M \times \mathbb{R}^{>0}, t^2 h + dt^2)$, and I the complex structure operator. Then $\xi \rfloor \eta = 1$, $\xi \rfloor d\eta = 0$, hence ξ is the Reeb vector field of M.

The next result is well known, see for example [7]:

Claim 3.6 Let (M, h) be a Sasakian manifold. The Reeb field is unitary and Killing: its flow $\rho(t)$ acts on M by isometries. Moreover, it preserves the CR-structure.

If the orbits of the Reeb flow of a Sasakian manifold are compact, then the Sasakian structure is called *quasi-regular*. In this case, if compact, M fibers in circles over a compact Kähler orbifold. The construction can be reversed if one starts with a compact Hodge orbifold, cf. [6, Theorem 2.8].

Remark 3.7 Any Sasakian metric h is S^1-invariant with respect to some CR-holomorphic S^1-action. Indeed, let G be the closure of the one-parametric group generated by the Reeb field. In [18] (see also [13]) it is shown that this group is a compact torus. Clearly, h is G-invariant. Taking $S^1 \subset G$ generated by a vector field sufficiently close to the Reeb field, we may also assume that this S^1-action is proper and transversal.

3.2 Vaisman manifolds

Our method will be to constantly translate the Sasakian geometry into locally conformally Kähler and Vaisman geometry. Here we recall the basics. For details and examples we refer to [10,17–19,21].

As we shall only deal with compact manifolds, we can take as definition the following characterization:

Definition 3.8 A compact complex manifold (N, I) is called a *Vaisman manifold* if it admits a Kähler covering $\Gamma \to (\tilde{N}, I, h) \to (N, I)$ such that:

- Γ acts by holomorphic homotheties with respect to h (this says that (N, I) is equipped with a *locally conformally Kähler structure*).
- $(\tilde{N}, I, h)$ is isomorphic to a Kähler cone over a compact Sasakian manifold M. Moreover, there exists a Sasakian automorphism φ and a positive number $q > 1$ such that Γ is isomorphic to the cyclic group generated by $(x, t) \mapsto (\varphi(s), tq)$.

In particular, the product of a compact Sasakian manifold with S^1 is equipped with a natural Vaisman structure.

The Kähler metric h on $\mathcal{C}(M) = M \times \mathbb{R}^{>0}$ has a global potential ψ, which is expressed as $\psi(m, t) = t^2$. The metric $\psi^{-1} \cdot h$ projects on N into a Hermitian, locally conformally Kähler metric, say g, whose fundamental two-form ω satisfies the equation $d\omega = \theta \wedge \omega$ for a closed one-form θ called the Lee form. Then $\psi =| \theta |^2$.

Further on, N is considered as a Hermitian manifold, with the Hermitian metric g. Let $\theta^\sharp$ be the vector field on N dual to the Lee form θ. Then $\theta^\sharp$ is called *the Lee field on N*.

The Lee field $\theta^\sharp$ is Killing, parallel with respect to the Levi-Civita connection on N and holomorphic. It thus determines two foliations on N:

- $\mathcal{F}_1$, one-dimensional, tangent to $\theta^\sharp$.
- $\mathcal{F}_2$, holomorphic two-dimensional, tangent to $\theta^\sharp$ and $I\theta^\sharp$.

Remark 3.9 As $\theta^\sharp$ is parallel, $\mathrm{Lie}_{\theta^\sharp} g(\theta^\sharp, \theta^\sharp) = 0$, hence the flow of $\theta^\sharp$ preserves the potential ψ.

Proposition 3.10 *If the foliation $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) is quasi-regular (thus having compact leaves), then the leaf space $N/\mathcal{F}_1$ (resp. $N/\mathcal{F}_2$ is a Sasakian (resp. projective Kähler) orbifold.*

Remark 3.11 Let L be the weight line bundle associated to the Vaisman manifold *via* the subjacent l.c.K. structure. The Lee form can be interpreted as a canonical Hermitian connection in its complexification (that we also denote by L) and one can prove (see [21]) that the curvature is positive except on the Lee field. The Chern connection in L is trivial along $\mathcal{F}_2$. Therefore, if N is quasi-regular, L is a pullback of a Hermitian line bundle $\pi_* L$ on the Kähler orbifold $N/\mathcal{F}_2$. Since the projection $\pi\colon N \longrightarrow N/\mathcal{F}_2$ kills the directions on which the curvature was non-positive, the push-forward bundle $\pi_* L$ is ample (cf. [18]).

The following result from [18] will be important in the sequel:

Theorem 3.12 *[18, Proposition 4.6] A compact Vaisman manifold can be deformed into a quasi-regular Vaisman manifold, with the same Kähler covering $(\tilde{N}, h)$.*

4 Algebraic cones and CR geometry

4.1 Algebraic cones

Definition 4.1 Let X be a projective variety, and L an ample line bundle on X. The *algebraic cone* $\mathcal{C}(X, L)$ of X is the total space of non-zero vectors in L^*. A *cone structure* on $\mathcal{C}(X, L)$ is the $\mathbb{C}^*$-action arising this way (by fibrewise multiplication).

Definition 4.2 Let $\mathcal{C}(X, L)$ be an algebraic cone. Consider the associated affine variety $\overline{\mathcal{C}}(X, L) := \mathrm{Spec} \oplus_i H^0(X, L^i)$. Geometrically, $\overline{\mathcal{C}}(X, L)$ is a complex variety, obtained by adding a point at the "origin" of the cone $\mathcal{C}(X, L)$. We call $\overline{\mathcal{C}}(X, L)$ *the closure* of the algebraic cone $\mathcal{C}(X, L)$. This space is called *a closed algebraic cone*, and $\mathcal{C}(X, L)$ *an open algebraic cone*.

The definition of an algebraic cone is motivated by the following observation. Let h be a Hermitian metric on L^*, such that the curvature of the associated Hermitian connection is negative definite (such a metric exists, because L is ample). Consider a function $\mathcal{C}(X, L) \xrightarrow{\varphi} \mathbb{R}$, $\varphi(v) = h(v, v)$. Then $\partial\overline{\partial}\varphi$ is a Kähler metric on $\mathcal{C}(X, L)$ (see e.g. [21]). The associated Kähler manifold is a Riemannian cone of a unit circle bundle

$$\{v \in \mathcal{C}(X, L) \mid h(v, v) = 1\}$$

which is, therefore, Sasakian.

In the Introduction, we defined algebraic cones in a less constructive manner (see Definition 1.1). This definition is equivalent to the one given above, as follows from [18] and [19]. Starting from an algebraic cone in the sense of Definition 1.1, that is, an affine algebraic variety X with an action ρ of $\mathbb{C}^*$ contracting X to a single singular point x_0, we may embed $(X\backslash x_0)/\rho(2)$ into a diagonal Hopf manifold, as shown in [19]. This allows us to equip $(X\backslash x_0)/\rho(2)$ with a Vaisman metric. In [18], it was shown that a covering of a compact Vaisman manifold is isomorphic to the space of non-zero vectors in some anti-ample line bundle over a projective orbifold. Therefore, it is an open algebraic cone, in the sense of Definition 4.2. This implies that Definition 1.1 is equivalent to Definition 4.2.

The arguments of the present paper are built on the correspondence between the Sasakian manifolds and the algebraic cones, which is implied by the following proposition.

Proposition 4.3 *Let M be a compact Sasakian manifold, $\mathcal{C}(M)$ its cone, considered as a complex manifold. Then $\mathcal{C}(M) = \mathcal{C}(X, L)$ is an algebraic cone, associated with a projective orbifold X.*

Proof Indeed, the product $M \times S^1$ is Vaisman (see above), and is covered by the Kähler cone $\mathcal{C}(M)$. Proposition 4.6 of [18] implies that the same cone $\mathcal{C}(M)$ is a covering of a quasi-regular Vaisman manifold, that is, a total space of an elliptic fibration $E \longrightarrow X$, with X a projective orbifold. But any quasi-regular Vaisman manifold can be obtained as a quotient of a cone $\mathcal{C}(X, L)$ (see 3.11) by an equivalence $t \sim qt$, where $q \in \mathbb{C}^*$ is a fixed complex number, $|q| > 1$ (see Definition 3.8). □

The Kähler structure of the Riemannian cone $\mathcal{C}(M)$ explicitly depends on the Sasakian metric of M. However, the holomorphic structure of the cone is determined by the underlying CR-structure of M, as follows from Theorem 1.3. A weaker version of this statement is obtained immediately from standard results of complex analysis.

Proposition 4.4 *Let M be a compact Sasakian manifold, and λ a positive real number. Denote by $\mathcal{C}(M)_\lambda$ the set of all $(m, t) \in \mathcal{C}(M)$, $t \leq \lambda$. Then the holomorphic structure of $\mathcal{C}(M)_\lambda$ depends only on the CR-structure of M.*

Proof Consider the standard embedding $M \hookrightarrow \mathcal{C}(M)$, $m \longrightarrow m \times \{\lambda\}$, and let M_λ be its image, which is the boundary of a complete Stein domain $M \times [0, \lambda]/M \times \{0\}$ (see [19]), Theorem 3.1). Let V_λ be the space of CR-holomorphic functions on M_λ. Using the solution of $\overline{\partial}$-Neumann problem ([15]), we identify V_λ with the space of holomorphic functions on the Stein domain $\mathcal{C}(M)_\lambda$ which are smooth on its boundary M_λ. Then $\mathcal{C}(M)_\lambda$ is the holomorphic spectrum of the ring V_λ. □

The following problem is then natural: if one starts with an algebraic cone and fixes a pseudoconvex CR-hypersurface in it, when does the CR-structure underlie a Sasakian structure? As shown in the next section, the answer is related to the Kähler potentials on the cone.

4.2 Sasakian manifolds in algebraic cones

Theorem 4.5 *Let M be a smooth real hypersurface in a closed algebraic cone $\mathcal{C}$, considered as a CR-manifold. Assume that M is contact and pseudoconvex (this implies that M is the boundary of a Stein domain $\mathcal{C}_1$ in $\mathcal{C}$). Then M admits a Sasakian metric if and only if for some cone structure $\rho\colon \mathbb{C}^* \longrightarrow \mathrm{Aut}(\mathcal{C})$, M is S^1-invariant.*

Proof The "if" part follows from Remark 3.7, where the S^1-action is constructed. For the converse, assume that $\mathcal{C}_1$ is an open, S^1-invariant subset of a cone. To prove that its boundary M is Sasakian, we need to construct a Kähler potential which is homogeneous under the cone action and such that M is its level set. To this end, we introduce the following

Definition 4.6 *A section* of the action $\rho\colon \mathbb{R}^{>0} \longrightarrow \mathrm{Aut}(\mathcal{C})$ is a subset $V \subset \mathcal{C}$ such that $\rho(\lambda_1)V$ does not intersect $\rho(\lambda_2)V$ for $\lambda_1 \neq \lambda_2$, and $\rho(\mathbb{R}^{>0})V = \mathcal{C}$.

We now fix a Kähler potential ψ on $\mathcal{C}_1$ mapping its boundary to 1 (the existence of such a potential is assured by the strict pseudoconvexity of M; see e.g. [16, Consequence 3.2]). Let Δ_1 be a unit disk in $\mathbb{C}$. Averaging ψ with S^1-action induced by ρ, we may assume that ψ is S^1-invariant. For all $m \in M$, the discs $\rho(\Delta_1)m$, bounded by the images of S^1, belong to $\mathcal{C}_1$ (indeed, being strictly plurisubharmonic, ψ is subharmonic on all curves in $\mathcal{C}$). This implies that $M \subset \mathcal{C}$ is a section of $\rho\colon \mathbb{R}^{>0} \longrightarrow \mathrm{Aut}(\mathcal{C})$, in the sense of the above definition.

This allows us to define a map $\varphi\colon \mathcal{C} \longrightarrow \mathbb{R}^{>0}$, mapping $x \in \mathcal{C}$ to t^2, where $x \in \rho(t)M$. By construction, this map is homogeneous with respect to the cone action.

To finish the proof it remains to show that φ is a Kähler potential. Clearly, on the contact distribution of M the form $\partial\overline{\partial}\varphi$ is equal to $\varphi\omega_0$, where ω_0 is the Levi form, and it is positive because M is pseudoconvex. On the plane generated by ρ-action, $\varphi = |z|^2$, hence plurisubharmonic. Finally, these two spaces are orthogonal, because φ is $\rho(S^1)$-invariant. Then φ is plurisubharmonic. □

Remark 4.7 In the proof of Theorem 4.5, we constructed an S^1-*invariant* Sasakian metric on M. Moreover, the Reeb field of M is proportional (with constant coefficient) to the tangent field to the S^1-action. However, M can possibly admit other Sasakian metrics, not all of them necessarily S^1-invariant or having the prescribed Reeb field; see [8] or Theorem 1.11.

5 Existence and uniqueness of Sasakian structures

5.1 Sasakian structures on CR-manifolds with S^1-action

Theorem 5.1 *Let M be a compact pseudoconvex contact CR-manifold,* $\dim M \geq 5$. *Then the following assumptions are equivalent.*

(i) *M admits a transversal, CR-holomorphic action of S^1.*
(ii) *M admits a transversal, CR-holomorphic vector field.*
(iii) *M admits a Sasakian metric, compatible with the CR-structure.*

Proof The implication (i) ⇒ (ii) is clear, and (iii) ⇒ (i) follows from Remark 3.7. The implication (ii) ⇒ (i) is clear from Theorem 2.17. Indeed, since the group of CR-automorphisms of M is compact (unless M is a sphere), any diffeomorphism flow can be approximated by an S^1-action within its closure. The implication (ii) ⇒ (iii) follows from differential-geometric

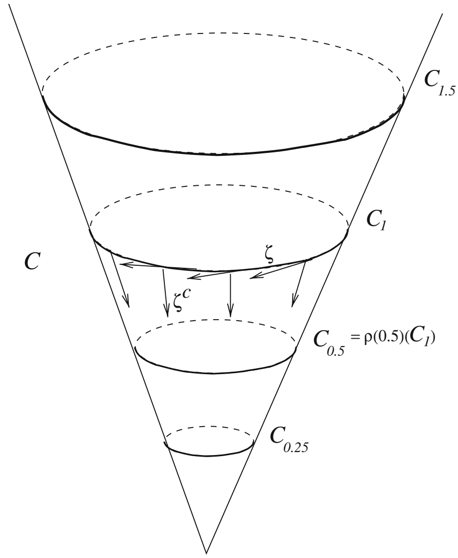

Fig. 1 Gluing $\mathcal{C}_1$ to itself

arguments (see Remark 1.5). We use another argument, which is based on complex analysis. The implication (i) ⇒ (iii), and the proof of Theorem 5.1, is given by the following proposition.

Proposition 5.2 *Let M be a compact pseudoconvex contact CR-manifold,* $\dim M \geq 5$. *Assume that M admits a proper CR-holomorphic S^1-action ρ. Then M admits an S^1-equivariant CR-embedding to an algebraic cone.*

Proof As shown in [15], M is the boundary of a Stein variety $\mathcal{C}_1$ with isolated singularities. Since $\mathcal{C}_1$ is constructed uniquely (by solving the boundary $\overline{\partial}$-Neumann problem), the S^1-action on M can be extended to a holomorphic S^1-action on $\mathcal{C}_1$. We shall explain now how to integrate ρ to a $\mathbb{C}^*$-action.

Let ζ be the tangent vector field induced by the S^1-action, and $\zeta^c = I(\zeta)$ the corresponding vector field in $T\mathcal{C}_1$. Since ζ is transversal with respect to CR-structure, ζ^c has to point inward or outward, everywhere in M (in other words, it has to point either towards the filled-in part or towards the opposite direction). Assume it is inward.

The flow of ζ^c provides a holomorphic automorphism ρ_1 of $\mathcal{C}_1$, mapping $\mathcal{C}_1$ into itself. Iterating this map, we find that we can integrate ρ from S^1 to an action of $0 < |z| \leq 1$. Inverting this construction and gluing images of $\rho(0 < |z| < \epsilon)$ together, as shown in Fig. 1, we obtain a domain $\mathcal{C}$ containing $\mathcal{C}_1$ where ρ can be integrated to a $\mathbb{C}^*$-action.

The potential φ, constructed in the proof of Theorem 4.5, can be defined in the same way now, and it gives a homogeneous, S^1-invariant plurisubharmonic function on $\mathcal{C}$ as indicated. Then $\mathcal{C}/\rho(2)$ is Vaisman, hence $\mathcal{C}$ is a cone of a projective orbifold as follows from [18, Proposition 4.6]. □

This finishes the proof of Theorem 1.2. The proof of Proposition 5.2 also leads to the following result.

Corollary 5.3 *Let M be a compact pseudoconvex contact CR-manifold, admitting a proper CR-holomorphic S^1-action ρ, and*

$$M \hookrightarrow \mathcal{C}$$

an S^1-equivariant embedding to an algebraic cone. Then $\mathcal{C}$ is uniquely determined by M and the S^1-action.

Proof By Proposition 4.4, the holomorphic structure on $\mathcal{C}_1$ is determined by the CR-structure on M. The cone $\mathcal{C}$ is reconstructed from $\mathcal{C}_1$ and the S^1-action as above. □

5.2 Uniqueness of the algebraic cone

Every Sasakian manifold is CR-embedded to an algebraic cone by Proposition 5.2. This cone is determined uniquely by an S^1-action, as follows from Corollary 5.3. On the other hand, the cone is determined by the Sasakian metric. Then, if a given Sasakian metric is invariant under two different S^1-actions, the cone associated to one S^1-action is isomorphic to the cone associated to the other S^1-action. Therefore, unless M is a sphere, Theorem 1.3 is implied by the following proposition.

Proposition 5.4 *([8, Proposition 4.4]) Let M be a CR-manifold of Sasakian type, and G the group of CR-automorphisms of M. Assume that M is not a sphere. Then M admits a G-invariant Sasakian metric g.*

6 The positive Sasakian cone of a CR-manifold

6.1 Positive Sasakian cone

The notion of positive Sasakian cone is due to [8]. We use it to classify the Sasakian metrics on a sphere.

Definition 6.1 Let (M, H) be a strictly pseudoconvex CR-manifold. The Levi form $H \otimes H \longrightarrow TM/H$ is sign-definite. This gives an orientation on TM/H. A transversal CR-holomorphic vector field is called *positive* if its projection to TM/H is everywhere positive. The *positive Sasakian cone* is the space of all transversal, positive, CR-holomorphic vector fields.

Definition 6.2 Let M be a Sasakian manifold, $\mathcal{C}(M) = M \times \mathbb{R}^{>0}$ its cone, with t a coordinate in $\mathbb{R}^{>0}$, and $\frac{d}{dt}$ the corresponding holomorphic vector field. It is clear from the definition that $\xi := I(t\frac{d}{dt})$ is tangent to the fibration $M \times \{t\}$, hence defines a vector field on M. We normalize it in such a way that $|\xi| = 1$. Then ξ is called *the Reeb field of the Sasakian manifold M*.

This definition is compatible with the one used in contact geometry (Remark 2.8). The following well-known claim is easy to prove (see, for example, [7]):

Claim 6.3 In these assumptions, ξ is transversal, positive, CR-holomorphic and Killing.

The following theorem is implied by Lemma 6.4 of [8].

Theorem 6.4 *Let M be a compact, strictly pseudoconvex CR-manifold, and $\mathcal{R}$ the Sasakian positive cone. Denote by $\mathcal{S}$ the set of Sasakian metrics on M, and let $\mathcal{S} \xrightarrow{\Psi} \mathcal{R}$ map a metric into the corresponding Reeb field. Then Ψ is a bijection.*

Proof We give an independent proof of Theorem 6.4, using the same kind of arguments as we used in the proof of Theorem 1.2. The Reeb field on a contact, pseudoconvex CR-manifold (M, H, I) determines the Sasakian metric uniquely, as can be seen from the following argument. Denote the corresponding contact form by η (see Remark 2.8). The Hermitian form $d\eta\big|_H$ is equal to the Levi form by construction; the Reeb field ξ is orthogonal to H and has length 1. Therefore, the map $\mathcal{S} \stackrel{\Psi}{\longrightarrow} \mathcal{R}$ is injective. It remains to show that Ψ is a surjection.

Let ζ be a positive, transversal CR-holomorphic vector field on a compact, strictly pseudoconvex CR-manifold M, and B the corresponding Stein domain, $\partial B = M$. Then $e^{t\zeta}$ induces an automorphism of $B = \mathrm{Spec}(\mathcal{O}_M)$, where $\mathcal{O}_M$ is the ring of CR-holomorphic functions on M. Since ζ is positive, the vector field $-\zeta^c := -I(\zeta)$ points transversally to M towards B. Therefore, the map $e^{-t\zeta^c}: B \longrightarrow B$ is well defined for small t, and maps B to a strictly smaller subset which is contained in the interior of B. Iterating this map, we obtain that $e^{-t\zeta^c}$ is well defined for all t. Inverting this procedure as in the proof of Proposition 5.2, we obtain that ζ^c induces a holomorphic action ρ of the multiplicative group $\mathbb{R}^{>0}$ on a Stein domain B_∞, which contains M as a hypersurface. Clearly, ρ is a contraction, with $\rho(\varepsilon_i)$, $\varepsilon_i \longrightarrow 0$, putting B into a sequence of smaller open balls converging to a single fixed point x_0 (see [19]). Therefore, ρ is free outside of $\{x_0\}$, and

$$B_\infty \backslash S = \rho(\mathbb{R}^{>0})M.$$

because each orbit of ρ encounters M on the way to x_0. We define a function $\varphi: (B_\infty \backslash \{x_0\}) \longrightarrow \mathbb{R}^{>0}$ by

$$\varphi(x) = \lambda^2, \quad \text{for } \rho(\lambda^{-1})x \in M.$$

Then φ is a Kähler potential on B_∞. Indeed, on the contact distribution $H \subset TM$, $\partial\overline{\partial}\varphi$ is proportional to the Levi form of M, because φ is constant on M. In particular, $\partial\overline{\partial}\varphi$ is positive on H. On the 1-dimensional complex foliation F generated by $\langle\zeta, \zeta^c\rangle$, φ is quadratic, and can be written in appropriate holomorphic coordinates as $z \longrightarrow |z - c|^2$. Finally, $\partial\overline{\partial}\varphi$ vanishes on pairs (x, y), $x \in H$, $y \in F$, because $\mathrm{Lie}_\zeta\, \varphi = 0$. Therefore, $\partial\overline{\partial}\varphi$ is positive on $F \oplus H = TB_\infty$. The function φ is by construction homogeneous with respect to the action of ρ. Therefore, the corresponding Kähler form $\partial\overline{\partial}\varphi$ is also homogeneous, and B_∞, considered as a Riemannian manifold, is identified with a Riemannian cone over M. This gives a Sasakian metric on M. It is easy to check that the corresponding Reeb field is equal to ζ. We proved that any positive transversal CR-holomorphic vector field is induced by some Sasakian metric. Theorem 6.4 is proven. □

Now we can prove Theorem 1.11. A Sasakian manifold M is *quasi-regular* if the 1-dimensional foliation F_1 induced by the Reeb field on M has compact fibers. Quasiregular Sasakian manifolds are always obtained from the construction described in Example 1.9 (see [6]). Therefore, to prove Theorem 1.11, we need to show that a given CR-manifold M admits a quasi-regular Sasakian structure, if it admits some Sasakian structure.

Denote by A_0 the 1-parameter group of CR-holomorphic isometries, generated by $e^{t\xi}$, where ξ is the Reeb field of M. Let A be its closure in the Lie group of CR-holomorphic isometries of M. Since A_0 is abelian, A is also abelian; it is compact, because the group of isometries is compact. Therefore, A is a compact torus.

Let $\mathfrak{a}$ be its Lie algebra. We consider $\mathfrak{a}$ as a subset in the space of CR-holomorphic vector fields on M. We call a vector field $\zeta \in TM$ *quasi-regular* if the corresponding 1-dimensional foliation has compact fibers. A vector field $\zeta \in \mathfrak{a}$ is quasi-regular if it is tangent to an embedding $S^1 \hookrightarrow A$. Such embeddings correspond to rational points in $\mathfrak{a}$, hence they are dense in $\mathfrak{a}$. Taking a quasiregular $\zeta \in \mathfrak{a}$ sufficiently close to ξ, we may assume that it is also transversal and positive. By Theorem 6.4, the corresponding Sasakian manifold is quasiregular. We proved Theorem 1.11.

Remark 6.5 A similar deformation-type argument was used in [13, Proposition 1.10].

6.2 Sasakian metrics on a sphere

To finish the proof of Theorem 1.3, we need to consider the case of a sphere. Let $M = S^{2n-1} \subset \mathbb{C}^n$ be an odd-dimensional sphere equipped with a standard CR-structure. We are going to classify the Sasakian metrics compatible with this CR-structure. We are interested in Sasakian metrics up to CR-automorphism.

Let G be the group of CR-automorphisms of M. It is well known that $G \cong SU(n, 1)$ (see e.g. [8]). Using the same argument as used in the proof of Proposition 4.4, we may assume that G acts as a group of holomorphic automorphisms on an open ball $B \subset \mathbb{C}^n$, $\partial B = M$. The action of $SU(n, 1)$ on B is very easy to describe explicitly. Let us identify B with a projectivization of the positive cone

$$\{\xi \in V \mid (\xi, \xi)_p > 1\},$$

where $(\cdot, \cdot)_p$ is a Hermitian form of signature $(n, 1)$ on $V = \mathbb{C}^{n+1}$. The group $U(n, 1)$ acts on V preserving the metric, hence $SU(n, 1) \subset PU(n, 1)$ acts on $B \subset \mathbb{P}V$. This action is holomorphic, therefore its restriction to $M = \partial B$ is CR-holomorphic. Clearly, $G = SU(n, 1)$ acts on the interior of B transitively. This gives

Proposition 6.6 *Let $M \subset \mathbb{C}^n$ be an odd-dimensional sphere, considered as a CR-manifold, $G = \mathrm{Aut}_{CR}(M)$ the group of CR-automorphisms, $G \cong SU(n, 1)$. Then G acts transitively on the interior part of the open ball B, $M = \partial B$.*

We are interested in classification of Sasakian structures up to CR-automorphism. A Sasakian structure on M induces a $\mathbb{C}^*$-action on a Stein domain containing B (Theorem 1.3). This way, B is identified with an open part of an algebraic cone. Since G acts on B transitively, it maps the origin of this cone into any other interior point of B.

Corollary 6.7 *In assumptions of Proposition 6.6, let g be a Sasakian metric on M, and ξ its Reeb field. As shown in Subsection 6.1, $\rho(t) := e^{-t\xi^c}$ acts on B by holomorphic contractions. Denote by x_0 the fixed point of ρ. Then, after an appropriate action of the group of CR-automorphisms of G, we may assume that x_0 is $0 \in B$.*

The group G_0 of CR-automorphisms of M fixing $0 \in B$ is identified with the stabilizer of 0 under $SU(n, 1)$-action on B, that is, with $U(n)$. Denote by $\mathcal{R}_0$ the part of positive Sasaki cone consisting of those positive transversal CR-holomorphic vector fields ξ which fix $0 \in B$. As follows from Theorem 6.4 and Corollary 6.7, every Sasakian metric on M corresponds to some $\xi \in \mathcal{R}_0$, up to a CR-automorphism. Then, the set $\mathcal{S}/G$ of isomorphism classes of Sasakian metrics is identified with $\mathcal{R}_0/G_0$

Clearly, $\mathcal{R}_0$ is the set of all $\xi \in \mathfrak{u}(n)$ which are positive and transversal, that is, have all eigenvalues α_i with $-\sqrt{-1}\,\alpha_i$ positive real numbers. The group $U(n)$ acts on $\mathcal{R}_0$ in a natural way, and each orbit is determined by the corresponding set of eigenvalues. This gives the following theorem:

Theorem 6.8 *Let $M \subset \mathbb{C}^n$ be an odd-dimensional sphere, considered as a CR-manifold, $G = \mathrm{Aut}_{CR}(M)$ the group of CR-automorphisms, $G \cong SU(n, 1)$, and $\mathcal{S}$ the set of Sasakian* *metrics on M. Then $\mathcal{S}/G$ is in natural, bijective and continuous correspondence with the set of unordered n-tuples of positive real numbers.*

From this construction, it is clear that the Riemannian cone of each Sasakian structure on a sphere is identified naturally with $\mathbb{C}^n \setminus \{0\}$. This proves Theorem 1.3 in the case when M is a sphere. We finished the proof of Theorem 1.3.

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References

1. Andreotti, A., Siu, Y.T.: Projective embeddings of pseudoconcave spaces. Ann. Sc. Norm. Sup. Pisa **24**, 231–278 (1970)
2. Arnold, V.I.: Mathematical Methods in Classical Mechanics. Springer-Verlag, New York (1978)
3. Belgun, F.A.: Normal CR structures on compact 3-manifolds. Math. Z. **238**, 441–460 (2001)
4. Belgun, F.A.: Normal CR structures on S^3. Math. Z. **244**(1), 125–151 (2003)
5. Blair, D.E.: Riemannian geometry of contact and symplectic manifolds. Progress in Math. **203**, Birkhäuser, Boston, Basel (2002)
6. Boyer, C.P., Galicki, K.: On Sasakian-Einstein Geometry. Int. J. Math. **11**, 873–909 (2000)
7. Boyer, C.P., Galicki, K.: Sasakian Geometry. Oxford Mathematical Monographs. Oxford Univ. Press (2007) to appear.
8. Boyer, C.P., Galicki, K., Simanca, S.: Canonical Sasakian metrics. math.DG/0604325.
9. Dragomir, S.: On pseudo-Hermitian immersions between strictly pseudoconvex CR manifolds. Amer. J. Math. **117**(1), 169–202 (1995)
10. Dragomir, S., Ornea, L.: Locally Conformal Kähler Geometry. Progress in Math. **155**, Birkhäuser, Boston, Basel (1998)
11. Geiges, H.: Normal contact structures on 3-manifolds. Tohoku Math. J. **49**, 415–422 (1997)
12. Grauert, H., Remmert, R.: Theory of Stein Spaces. Springer-Verlag (2004)
13. Kamishima, Y.: Standard pseudo-Hermitian structure on manifolds and Seifert fibrations. Ann. Global Anal. Geom. **12**, 261–289 (1994)
14. Lee, J.M.: Pseudo-Einstein structures on CR manifolds. Amer. J. Math. **110**, 157–178 (1988)
15. Marinescu, G., Dinh, T.-C.: On the compactification of hyperconcave ends and the theorems of Siu-Yau and Nadel. math.CV/0210485, v2. Inventiones Math. **164**, 233–248 (2006)
16. Marinescu, G., Yeganefar, N.: Embeddability of some strongly pseudoconvex CR manifolds. Trans. Amer. Math. Soc. **359**, 4757–4771 (2007)
17. Ornea L., Verbitsky, M.: Structure theorem for compact Vaisman manifolds. Math. Res. Lett. **10**, 799–805 (2003) see also math.DG/0305259
18. Ornea L., Verbitsky, M.: An immersion theorem for compact Vaisman manifolds. Math. Ann. **332**(1), 121–143 (2005) see also math.AG/0306077
19. Ornea, L., Verbitsky, M.: Locally conformal manifolds with potential. Math. Ann. (to appear), math.AG/0407231.
20. Schoen, R.: On the conformal and CR automorphism groups. Geom. Funct. Anal. **5**, 464–481 (1995)
21. Verbitsky, M.: Theorems on the vanishing of cohomology for locally conformally hyper-Kähler manifolds. (Russian). Tr. Mat. Inst. Steklova **246**, (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 64–91; translation in Proc. Steklov Inst. Math. **246**, 54–78 (2004) Also available as math.DG/0302219
22. Webster, S.M.: Pseudo-Hermitian structures on a real hypersurface. J. Differential Geom. **13**, 25–41 (1978)