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# Reduction of Vaisman structures in complex and quaternionic geometry

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**Abstract**

We consider locally conformal Kähler geometry as an equivariant (homothetic) Kähler geometry: a locally conformal Kähler manifold is, up to equivalence, a pair $(K, \Gamma)$, where $K$ is a Kähler manifold and $\Gamma$ is a discrete Lie group of biholomorphic homotheties acting freely and properly discontinuously. We define a new invariant of a locally conformal Kähler manifold $(K, \Gamma)$ as the rank of a natural quotient of $\Gamma$, and prove its invariance under reduction. This equivariant point of view leads to a proof that locally conformal Kähler reduction of compact Vaisman manifolds produces Vaisman manifolds and is equivalent to a Sasakian reduction. Moreover, we define locally conformal hyperKähler reduction as an equivariant version of hyperKähler reduction and in the compact case we show its equivalence with 3-Sasakian reduction. Finally, we show that locally conformal hyperKähler reduction induces hyperKähler with torsion (HKT) reduction of the associated HKT structure and the two reductions are compatible, even though not every HKT reduction comes from a locally conformal hyperKähler reduction.


*MSC:* 53C55; 53C25; 53D20

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## 1. Introduction

Symplectic reduction has already been extended to many other structures defined by a closed form. Among the more recent is the reduction of locally conformal Kähler structures; see [8]. One of the main results in the above paper concerned the conditions under which a particular class of compact locally conformal Kähler manifolds, namely Vaisman manifolds, is preserved by reduction.

Compact Vaisman manifolds are, in a certain way, equivalent to Sasakian manifolds, as their universal cover is a Riemannian cone over a Sasakian manifold. This was implicit in [14] and was made explicit in [8]. On the other hand, the Structure Theorem in [22] proves that any compact Vaisman manifold is a Riemannian suspension over a circle, with fibre a compact Sasakian manifold.

It is one of the purposes of the present paper to make clear the distinction between these two results in the language of *presentations*. More precisely, in [8] the authors verified that considering a locally conformal Kähler manifold as a pair $(K, \Gamma)$, where $K$ Kähler and $\Gamma$ act by biholomorphic homotheties, provides an interesting insight into locally conformal Kähler and Vaisman geometry. Such geometries are considered as equivariant versions of Kähler and Sasakian geometry, considered as *homothetic* geometries. The "distance" from the Kähler (Sasakian) geomety of $K$ is measured by a new invariant, the *rank* of the manifold $(K, \Gamma)$, which we define as the rank of the non-isometric part of the action of $\Gamma$. In this setting, a Vaisman manifold is a pair $(K, \Gamma)$, where $K = \mathcal{C}(W)$ is the Kähler cone of a Sasakian manifold and $\Gamma$ commutes elementwise with the radial flow.

In this paper we aim to delimit how much this language can be considered to be exhaustive, that is, to answer the question: can locally conformal Kähler and Vaisman geometry of a manifold be completely described by any of its presentations?

The answer is generally yes, with one proviso: among the equivalent presentations defining the same manifold, some presentations are "more equal" than others, namely, the "biggest" and the "smallest" ones.

The use of presentations leads to easy proofs of several interesting properties. In particular, we completely drive locally conformal Kähler reduction, as defined in [8], to Kähler reduction, and, in the compact Vaisman case, to Sasakian reduction, which in turn proves that compact Vaisman manifolds are closed under reduction. Indeed, here we prove the following theorem.

**Theorem.** *The locally conformal Kähler reduction of a compact Vaisman manifold is a Vaisman manifold.*

This result depends heavily on a rather striking and more general one, namely: all the isometries of the cone metric over a compact Riemannian manifold project to identities of the generator lines. Here is where the compactness of the Vaisman manifold is used: one needs it to know that the minimal presentation is a cone over a *compact* Sasakian manifold.

Vaisman manifolds also appear naturally in quaternionic geometry: every compact locally conformal hyperKähler manifold is Vaisman (see [24]). Besides their interest as a living structure in Hermitian quaternionic geometry (see, for instance, the classification in [5]), the interest in locally conformal hyperKähler manifolds is also motivated by their close relation to 3-Sasakian structures (see [19]) and to HKT structures (see [20]). It is then only natural to place Vaisman reduction in the context of locally conformal hyperKähler reduction, which we define and study in the language of presentations. This gives rise to an easy construction of locally conformal hyperKähler reduction which is completely induced by hyperKähler reduction, hence the interaction with Joyce's hypercomplex reduction and with HKT reduction is naturally managed.

The structure of the paper is as follows: each of the three main contexts (locally conformal Kähler geometry, Vaisman geometry, locally conformal hyperKähler geometry) is analysed from two points of view: first the definition and properties of presentations, then the definition and properties of maps, hence of reduction—the approach is categorical, even though the language of category is never mentioned explicitly. Hence, in Section 2 the notion of presentation of a locally conformal Kähler manifold is defined, and accordingly a new equivalent definition of locally conformal Kähler manifold is introduced, together with the notion of *rank*. In Section 3 locally conformal Kähler automorphisms and twisted Hamiltonian actions are restated in terms of presentations, and reduction is reconstructed as an equivariant process, with some detail, also leading to the fact that rank is closed under reduction. In Section 4 presentations of Vaisman manifolds are explained, and it is proven that compact Vaisman manifolds all have rank 1. In Section 5 Vaisman automorphisms (as defined in [8]) are characterized in the compact case, thus leading to the main theorem that the reduction of compact Vaisman manifolds is always Vaisman: in order to obtain this statement, several interesting results are provided, linking even more clearly compact Vaisman geometry with Sasakian geometry (Proposition 4.1, Theorems 5.1, 5.4 and 5.5). In Section 6 the language is shifted to the hypercomplex context, where some more rigidity appears, leading to a sufficient condition for a hyperKähler manifold to be a cone on a 3-Sasakian manifold (Theorem 6.5); in Section 7 locally conformal hyperKähler reduction is defined, in pure equivariant terms (Theorem 7.1); finally, in Section 8 the link with Joyce reduction and HKT geometry are explored.

## 2. Presentations of locally conformal Kähler manifolds

Usually, a locally conformal Kähler manifold is a conformal Hermitian manifold $(M, [g])$ such that $g$ is conformal to local Kähler metrics. The conformal class $[g]$ corresponds to a unique cohomology class $[\omega_g] \in H^1(M)$, whose representative $\omega_g$ is defined as the unique closed 1-form satisfying $d\Omega_g = \omega_g \wedge \Omega_g$, where $\Omega_g$ denotes the fundamental form of $g$. The 1-form $\omega_g$ is called the Lee form of $g$.

Let $M$ be a complex manifold of complex dimension of at least two. If there exists a complex covering space $\tilde{M}$ of $M$ admitting a Kähler metric $g_K$ and such that $\pi_1(M)$ acts on it by holomorphic homotheties, then one obtains a locally conformal Kähler structure $[g]$ on $M$ by pushing forward locally $g_K$ and glueing the local metrics via a partition of unity. More explicitly, in the conformal class $[g_K]$ there exists a $\pi_1(M)$-invariant metric $e^f g_K$, which induces a locally conformal Kähler metric on $M$.

Conversely, let $(M, [g])$ be a locally conformal Kähler manifold. The pull-back of any Lee form to the universal covering $\tilde{M}$ is exact, say $\tilde{\omega}_g = df$, and $e^{-f}\tilde{g}$ turns out to be a Kähler metric on $\tilde{M}$, such that $\pi_1(M)$ acts on it by holomorphic homotheties. According to the fact that the Kähler metric $e^{-f}\tilde{g}$ is defined up to homotheties, this paper is concerned with homothety classes of Kähler manifolds. In a homothetic Kähler manifold $K$, we denote $\mathrm{Hom}(K)$ the group of biholomorphic homotheties, and

$$\rho_K : \mathrm{Hom}(K) \to \mathbb{R}^+$$

the group homomorphism associating with a homothety its dilation factor, that is, defined by $f^*g = \rho_K(f)g$. We remark that both $\rho_K$ and $\mathrm{Isom}(K) = \ker \rho_K$ are well-defined for a homothetic Kähler manifold $K$.

**Remark 2.1.** Note that this puts rather severe restrictions on the groups that can be fundamental groups of locally conformal Kähler manifolds: there are groups, for instance those generated by elements of finite order, such as $SL(2, \mathbb{Z})$, which do not send any non-trivial homomorphism to $\mathbb{R}^+$.

Every locally conformal Kähler manifold can thus be obtained by a homothetic Kähler manifold $K$ and a discrete Lie group $\Gamma \subset \mathrm{Hom}(K)$ acting freely and properly discontinously on $K$. This motivates the following definition.

**Definition 2.2.** A pair $(K, \Gamma)$ is a *presentation* if $K$ is a homothetic Kähler manifold and $\Gamma$ is a discrete Lie group of biholomorphic homotheties acting freely and properly discontinously on $K$. If $M$ is a locally conformal Kähler manifold and $M = K/\Gamma$ (as locally conformal Kähler manifolds, that is, they are conformally and biholomorphically equivalent), we say that $(K, \Gamma)$ is a presentation *of* $M$.

**Example 2.3.** The most classical examples of locally conformal Kähler manifolds are the Hopf surfaces. Following Kodaira, a Hopf surface is any compact complex surface with universal covering $\mathbb{C}^2\backslash 0$. Since any discrete group of biholomorphisms of $\mathbb{C}^2\backslash 0$ is of the form $H \rtimes \mathbb{Z}$, where $H$ is a finite group (see [15–18]), any locally conformal Kähler metric on a Hopf surface can be given as a Kähler metric on $\mathbb{C}^2\backslash 0$ such that $H \rtimes \mathbb{Z}$ acts by homotheties. In this setting, $H \rtimes \mathbb{Z}$ acts by homotheties for $dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2$ if and only if $H$ is trivial (that is, if the Hopf surface is primary), the action of $\mathbb{Z}$ is diagonal (that is, if the Hopf surface has Kähler rank 1), and the diagonal action $(z_1, z_2) \mapsto (\alpha z_1, \beta z_2)$ is given by complex numbers $\alpha, \beta$ of the same length $|\alpha| = |\beta|$. All other cases have been settled in [7] and [1].

**Example 2.4.** A very recent example of a locally conformal Kähler manifold is described in [21]. For any integer $s \geq 1$, the authors construct a group $\Gamma$ acting freely and properly discontinously by biholomorphisms on $H^s \times \mathbb{C}$ (where $H$ denotes the upper-half complex plane). The Kähler form such that $\Gamma$ acts by homotheties is $i\partial\bar{\partial}F$, where the Kähler potential $F$ on $H^s \times \mathbb{C}$ is given by

$$F(z_1, \ldots, z_{s+1}) = \frac{1}{\prod_{j=1}^{s} i(z_j - \bar{z}_j)} + |z_{s+1}|^2.$$

Note that for $s = 1$ this construction recovers the Inoue surfaces $S_M$ described in [11].

**Definition 2.5.** Let $(K, \Gamma)$ be a presentation. The associated *maximal* presentation is $(\tilde{K}, \tilde{\Gamma})$, where $\tilde{K}$ is the homothetic Kähler universal covering of $K$ and $\tilde{\Gamma}$ is the lifting of $\Gamma$ to $\tilde{K}$. Then

$$(K_{\min}, \Gamma_{\min}) = \left( \frac{\tilde{K}}{\mathrm{Isom}(\tilde{K}) \cap \tilde{\Gamma}}, \frac{\tilde{\Gamma}}{\mathrm{Isom}(\tilde{K}) \cap \tilde{\Gamma}} \right)$$

is called the associated *minimal* presentation.

**Remark 2.6.** The maximal and minimal presentations associated with $(K, \Gamma)$ depend only on the locally conformal Kähler manifold $K/\Gamma$. Moreover, $\tilde{\Gamma} = \pi_1(K/\Gamma)$ and $\Gamma_{\min} = \rho_K(\Gamma)$.

The presentation of a locally conformal Kähler manifold is of course not unique, but the maximal and minimal presentations are unique as $\tilde{\Gamma}$, $\Gamma_{\min}$-spaces, respectively. This means that they can be used to distinguish between different locally conformal Kähler manifolds, as suggested by the following two definitions.

**Definition 2.7.** Let $(K, \Gamma)$, $(K', \Gamma')$ be two presentations. We say that $(K, \Gamma)$ is equivalent to $(K', \Gamma')$ if $(\tilde{K}, \tilde{\Gamma}) = (\tilde{K}', \tilde{\Gamma}')$. The $=$ sign means that there exists an equivariant map, namely, there exists a pair of maps $(f, h)$, where $f: \tilde{K} \to \tilde{K}'$ is a biholomorphic homothety, $h: \tilde{\Gamma} \to \tilde{\Gamma}'$ is an isomorphism, and for any $\gamma \in \tilde{\Gamma}$, $x \in \tilde{K}$ we have $f(\gamma x) = h(\gamma) f(x)$. Equivalently, we say that $(K, \Gamma)$ is equivalent to $(K', \Gamma')$ if $(K_{\min}, \Gamma_{\min}) = (K'_{\min}, \Gamma'_{\min})$.

**Definition 2.8.** A *locally conformal Kähler* manifold is an equivalence class $[(K, \Gamma)]$ of presentations.

It is worth stressing once again that this latter definition is equivalent to the usual one. Hence, whenever we say "[...]$M$ is a locally conformal Kähler manifold [...]" the reader can indifferently think of the usual definition or of the above new definition. Nevertheless, one should remember that, in the spirit of what is written in the Introduction, in this paper we always (henceforth) have the new definition in mind.

**Remark 2.9.** A locally conformal Kähler manifold $[(K, \Gamma)]$ is globally conformal Kähler if and only if $\Gamma \subset \mathrm{Isom}(K)$.

For any presentation $(K, \Gamma)$, since $\rho_K(\Gamma)$ is a finitely generated subgroup of $\mathbb{R}^+$, it is isomorphic to $\mathbb{Z}^r$ for a certain $r$ a priori depending on the presentation $(K, \Gamma)$. The next proposition shows that $r$ is actually an invariant of the locally conformal Kähler structure. This invariant encodes the "locally conformal" part of the structure, the Kähler part being encoded by $\ker \rho_K$.

**Proposition 2.10.** *For any presentation $(K, \Gamma)$, the rank of the free abelian group $\rho_\Gamma(\Gamma)$ depends only on the equivalence class $[(K, \Gamma)]$.*

**Proof.** The maximal presentation $(\tilde{K}, \tilde{\Gamma})$ depends only on $[(K, \Gamma)]$. We remark that $\Gamma = \tilde{\Gamma}/\pi_1(K)$ and that $\pi_1(K) \subset \ker \rho_{\tilde{K}}$ (see Remark 2.9). Moreover, the following diagram (where the vertical arrow is a surjection) commutes

$$\begin{array}{ccc} \tilde{\Gamma} & & \\ \downarrow & \searrow^{\rho_{\tilde{K}}} & \\ \Gamma & \xrightarrow[\rho_K]{} & \mathbb{R}^+ \end{array}$$

and this implies that $\mathrm{Im}\rho_K = \mathrm{Im}\rho_{\tilde{K}}$, hence the claim. □

**Definition 2.11.** The *rank* of a locally conformal Kähler manifold $M$ is the non-negative integer $r_M$ given by the previous proposition.

**Proposition 2.12.** *Let $M$ be a locally conformal Kähler manifold and $r_M$ its rank. Then*

$$0 \leq r_M \leq b_1(M)$$

*and $r_M = 0$ if and only if $M$ is globally conformal Kähler. In particular, if $b_1(M) = 1$ then $r_M = 1$.*

**Proof.** Since $\rho_{\tilde{K}}(\pi_1(M))$ is abelian, the commutator $[\pi_1(M), \pi_1(M)]$ is a subgroup of $\ker \rho_{\tilde{K}}$. Thus,

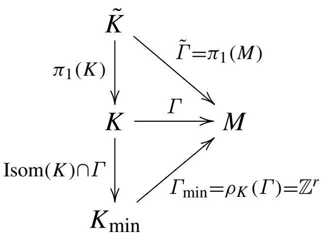
$$r_M = \operatorname{rank} \frac{\pi_1(M)}{\ker(\rho_{\tilde{K}}) \cap \pi_1(M)} \leq \operatorname{rank} \frac{\pi_1(M)}{[\pi_1(M), \pi_1(M)]} = \operatorname{rank}(H_1(M)) = b_1(M).$$

Moreover, by Remark 2.9, $r_M = 0$ if and only if $M$ is globally conformal Kähler. Finally, if $b_1(M) = 1$, the manifold cannot admit any globally conformal Kähler structure, hence $r_M = 1$. □

**Example 2.13.** The examples $M_s = (H^s \times \mathbb{C})/\Gamma$ given in 2.4 and described in [21] satisfy $\operatorname{rank}(M_s) = s$, for any integer $s \geq 1$.

**Remark 2.14.** It is worth noting that $r_M = 1$ does not imply $b_1 = 1$. We shall see in Corollary 4.7 that all compact Vaisman manifolds have rank 1, but one can obtain examples with arbitrarily large $b_1$ by taking induced Hopf bundles over curves of large genus in $\mathbb{CP}^2$.

This picture of definitions and remarks is summarized in the following diagram, which also clarifies the terminology maximal and minimal. The vertical arrows are covering maps via the action of isometries, the transversal arrows are covering maps via the action of homotheties, and $r = r_M$ is the rank of the locally conformal Kähler manifold.




(2.1)

## 3. Locally conformal Kähler automorphisms and reduction

A locally conformal Kähler automorphism is usually a biholomorphic conformal map.

**Definition 3.1.** Let $M$ be a locally conformal Kähler manifold. Denote by $\operatorname{Lck}(M)$ the Lie group of biholomorphic conformal transformations of $M$.

Coherently with our setting, we would like to describe $\operatorname{Lck}(M)$ in terms of presentations, and a natural candidate is given in the following definition.

**Definition 3.2.** Let $(K, \Gamma)$ be a presentation. Denote by $\operatorname{Hom}_\Gamma(K)$ the Lie group of $\Gamma$-equivariant biholomorphic homotheties of $K$.

The problem is that the existence of a lifting to $K$ is generally not granted, apart from the trivial case of the maximal presentation $(\tilde{K}, \tilde{\Gamma})$. It is exactly in this context that the notion of minimal presentation shows its relevance.

**Proposition 3.3.** *Let $(K, \Gamma)$ be a presentation, and let $f$ be any homothety of $K$ commuting with $\Gamma$. Then $f$ commutes with* $\operatorname{Isom}(K) \cap \Gamma$.

**Proof.** Let $\gamma \in \mathrm{Isom}(K) \cap \Gamma$. Since $f$ commutes with $\Gamma$, there exists $\gamma' \in \Gamma$ such that $f \circ \gamma = \gamma' \circ f$. Apply $\rho_\Gamma$ to both members to obtain that $\gamma'$ actually lies in $\mathrm{Isom}(K) \cap \Gamma$. □

**Corollary 3.4.** *Let $M$ be a locally conformal Kähler manifold and $(K_{\min}, \Gamma_{\min})$ its minimal presentation. Any map $f \in \mathrm{Lck}(M)$ lifts to a map $f_{\min} \in \mathrm{Hom}_{\Gamma_{\min}}(K_{\min})$.*

**Proof.** Any biholomorphic conformal transformation $f$ of $M$ lifts to a biholomorphic homothety $\tilde{f}$ of $\tilde{K}$. Then, by Proposition 3.3, the map $\tilde{f}$ commutes with $\mathrm{Isom}(\tilde{K}) \cap \tilde{\Gamma}$ and induces a biholomorphic homothety $f_{\min}$ of $K_{\min} = \tilde{K}/\mathrm{Isom}(\tilde{K}) \cap \tilde{\Gamma}$ which is $\Gamma_{\min}$-equivariant, since it is a lifting of $f$. □

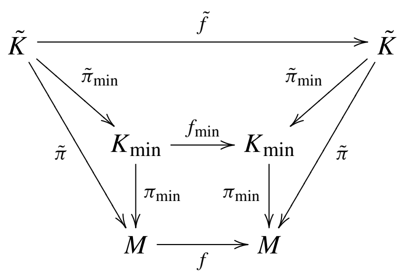
**Corollary 3.5.** *Let $M$ be a locally conformal Kähler manifold. Then any map $f \in \mathrm{Lck}(M)$ preserves the subgroup of $\pi_1(M)$ acting on $\tilde{K}$ by isometries.*

**Proof.** Denote by $p$ the covering associated with the minimal presentation. In view of Corollary 3.4, $f$ lifts to the minimal presentation, hence

$$f_*(p_*(\pi_1(K_{\min}))) = p_*(\pi_1(K_{\min})).$$

On the other hand, $p_*(\pi_1(K_{\min}))$ is the group whose action on $\tilde{K}$ produces $K_{\min}$, and by definition this is the subgroup of $\tilde{\Gamma} = \pi_1(M)$ acting on $\tilde{K}$ by isometries. □

The following diagram pictures this setting and adopts a convention that we are trying to stick to, that is, we use $\tilde{\square}$ when we refer to something related to the maximal presentation, and $\square_{\min}$ when we refer to something related to the minimal presentation.



(3.1)

In other words,

$$\mathrm{Lck}(M) = \mathrm{Hom}_{\tilde{\Gamma}}(\tilde{K})/\tilde{\Gamma} = \mathrm{Hom}_{\Gamma_{\min}}(K_{\min})/\Gamma_{\min}.$$

Moreover, any Lie subgroup $G$ of $\mathrm{Lck}(M)$ yields Lie subgroups $\tilde{G}$, $G_{\min}$ of $\mathrm{Hom}_{\tilde{\Gamma}}(\tilde{K})$, and $\mathrm{Hom}_{\Gamma_{\min}}(K_{\min})$ respectively, such that $G = \tilde{G}/\tilde{\Gamma} = G_{\min}/\Gamma_{\min}$.

**Definition 3.6.** A connected Lie subgroup $G$ of $\mathrm{Lck}(M)$ is *twisted Hamiltonian* if the identity component $\tilde{G}^\circ$ of $\tilde{G}$ is Hamiltonian on $\tilde{K}$. Equivalently, $G$ is twisted Hamiltonian if the identity component $G^\circ_{\min}$ of $G_{\min}$ is Hamiltonian on $K_{\min}$. Accordingly, we say that a map $\mu: \mathfrak{g} \to C^\infty(M)$ (or equivalently a map $\mu: M \to \mathfrak{g}^*$) is a *momentum map* for the action of $G$ on $M$ if it is the quotient of the corresponding momentum map $\tilde{\mu}$ on $\tilde{K}$, or equivalently if it is the quotient of the corresponding momentum map $\mu_{\min}$ on $K_{\min}$.

**Remark 3.7.** The connected components are necessary to ensure the existence of the momentum maps $\tilde{\mu}$ and $\mu_{\rm min}$.

**Remark 3.8.** If $G$ acts freely and properly on $\mu^{-1}(0)$, then $\tilde{G}^{\circ}$ acts freely and properly on $\tilde{\mu}^{-1}(0)$ and $G^{\circ}_{\rm min}$ acts freely and properly on ${\mu_{\rm min}}^{-1}(0)$.

**Proof.** Assume that the action of a group $G$ is proper on a manifold $X$ and lifts to a covering $X'$, and let $G'$ be the identity component of the lifted action. Let $K' \subset X' \times X'$ be compact, and let

$$L' = \{(g', x') \in G' \times X' \text{ such that } (g'(x'), x') \in K'\}$$

be its preimage. We want to show that $L'$ is compact.

Let $(g'_i, x'_i) \in L'$ be a sequence. Then $(g'_i(x'_i), x'_i) \in K'$, hence up to subsequences there exists a $(y'_0, x'_0)$ limit of $(g'_i(x'_i), x'_i)$. Then $(g_i(x_i), x_i)$ converges to $(y_0, x_0)$ and, since the action of $G$ on $X$ is proper, then

$$(g_i, x_i) \to (g_0, x_0).$$

Hence $y_0 = g_0(x_0)$, namely $(g_i(x_i), x_i) \to (g_0(x_0), x_0)$. Then there exists $g'_0 \in G'$ such that $y'_0 = g'_0(x'_0)$. The fact that $G'$ covers $G$ then implies that

$$(g'_i, x'_i) \to (g'_0, x'_0)$$

and hence that the lifted action is proper. The freedom of the lifted action is straightforward. □

This notion of twisted Hamiltonian action coincides with the one given in [8], and starts a reduction process.

In view of defining a reduction process in terms of presentations (and since, of course, we would like this process to be equivalent to the one given in [8]), here we give some technical details (given in Box I) regarding the reduction in [8], in terms of the maximal presentation $(\tilde{K}, \tilde{\Gamma})$.

Summing up, we have proven the following theorem.

**Theorem 3.9.** *Let $M$ be a locally conformal Kähler manifold. Let $G \subset \mathrm{Lck}(M)$ be twisted Hamiltonian, and suppose that $\mu^{-1}(0)$ is non-empty, 0 is a regular value for $\mu$ and the action of $G$ is free and proper on $\mu^{-1}(0)$. Then*

$$\left(\tilde{K} /\!\!/ \tilde{G}^{\circ}, \frac{\tilde{\Gamma}}{\tilde{\Gamma} \cap \tilde{G}^{\circ}}\right)$$

*is a presentation of the locally conformal Kähler reduction $M /\!\!/ G$.*

Theorem 3.9 works the same way if we substitute the maximal presentation with the minimal presentation. In this case, since

$$G^{\circ}_{\rm min} \cap \Gamma_{\rm min} = 1$$

we obtain the following corollary.

**Corollary 3.10.** *The minimal presentation for a locally conformal Kähler reduction $M /\!\!/ G$ is given by*

$$(K_{\rm min} /\!\!/ G^{\circ}_{\rm min}, \Gamma_{\rm min}).$$

First, we need some additional topological conditions on the zero set of the momentum map, that is, we suppose that $\mu^{-1}(0)$ is non-empty, 0 is a regular value for $\mu$, and the action of $G$ is free and proper on $\mu^{-1}(0)$. Then, some notations. Call $\tilde{\pi}$ the covering map of $\tilde{K}$ over $M$, whose covering maps are given by $\tilde{\Gamma}$. Points in $\tilde{K}$ are denoted by $\tilde{x}$, and this of course also means that $\tilde{\pi}(\tilde{x}) = x$, where $x \in M$. Elements of $\tilde{G}^\circ$ lifting $g \in G$ are denoted by $\tilde{g}$. Then, $\tilde{G}^\circ$ acts freely and properly on the non-empty $\tilde{\mu}^{-1}(0)$, and the Kähler reduction $\tilde{K}/\!\!/\tilde{G}^\circ$ is defined. Here we show that $\tilde{\Gamma}$ is $\tilde{G}^\circ$-equivariant, that is,

$$\tilde{\gamma}\tilde{G}^\circ = \tilde{G}^\circ\tilde{\gamma} \qquad \text{for any } \tilde{\gamma} \in \tilde{\Gamma}. \tag{3.2}$$

Indeed, take a point $\tilde{x} \in \tilde{K}$, and consider the intersection

$$\tilde{\gamma}\tilde{G}^\circ(\tilde{x}) \cap \tilde{G}^\circ\tilde{\gamma}(\tilde{x}).$$

This is clearly closed, and it can be easily shown to be also open in both $\tilde{\gamma}\tilde{G}^\circ(\tilde{x})$ and $\tilde{G}^\circ\tilde{\gamma}(\tilde{x})$. Thus (remembering that $\tilde{G}^\circ$ is connected), we obtain Formula (3.2). The fact that $\tilde{\Gamma}$ is $\tilde{G}^\circ$-equivariant means that $\tilde{\Gamma}$ acts on the Kähler reduction $\tilde{K}/\!\!/\tilde{G}^\circ$ by biholomorphic homotheties. In particular, we have a diffeomorphism

$$\frac{\tilde{K}/\!\!/\tilde{G}^\circ}{\tilde{\Gamma}} = M/\!\!/G.$$

We claim that $\tilde{\Gamma}$ acts properly discontinously on $\tilde{K}/\!\!/\tilde{G}^\circ$. In fact, the quotient could not be Hausdorff otherwise. Now suppose that there exists $\gamma \in \tilde{\Gamma}$ with a fixed point $\tilde{G}^\circ\tilde{x}$ in $\tilde{K}/\!\!/\tilde{G}^\circ$. Then we can find $\tilde{g}$ such that $\gamma\tilde{x} = \tilde{g}\tilde{x}$, and this implies $g(x) = x$ on $M$. Since the action of $G$ on $M$ is free by hypothesis, we have $g = \mathrm{Id}_M$. This means that $\tilde{g}$ is actually in $\tilde{\Gamma}$ and, since it coincides with $\gamma$ on $\tilde{x}$, it must be exactly $\gamma$. We have proved that

$$\frac{\tilde{\Gamma}}{\tilde{\Gamma} \cap \tilde{G}^\circ}$$

acts freely on $\tilde{K}/\!\!/\tilde{G}^\circ$.

Box I.

In particular, this implies that the rank is preserved under reduction.

**Corollary 3.11.** *Let $M$ be a locally conformal Kähler manifold, and let $G$ be a twisted Hamiltonian subgroup of* $\mathrm{Lck}(M)$. *Then*

$$r_{M/\!\!/G} = r_M.$$

This discussion motivates the hypothesis in the following definition.

**Definition 3.12.** Let $[(K, \Gamma)]$ be a locally conformal Kähler manifold. Let $G$ be a connected Hamiltonian subgroup of $\mathrm{Hom}_\Gamma(K)$ (this implies that $\Gamma$ is $G$-equivariant) with Kähler momentum map $\mu \colon K \to \mathfrak{g}^*$. Suppose that $\mu^{-1}(0)$ is non-empty, 0 is a regular value for $\mu$, and the action of $G$ is free and proper on $\mu^{-1}(0)$, so that the Kähler reduction $K/\!\!/G$ is defined. Suppose, moreover, that the action of $\Gamma$ on $K/\!\!/G$ is properly discontinous, and that the action of $\Gamma/(\Gamma \cap G)$ on $K/\!\!/G$ is free. Then

$$\left[\left(K/\!\!/G, \frac{\Gamma}{\Gamma \cap G}\right)\right]$$

is the *reduction* of the locally conformal Kähler manifold $[(K, \Gamma)]$.

## 4. Vaisman presentations

In the category of locally conformal Kähler manifolds, a remarkable subclass is given by the so-called Vaisman manifolds, after the name of the author who first recognized their importance [26,27]. They are usually defined as locally conformal Kähler manifolds having parallel Lee form, and this definition belongs to the Hermitian setting.

However, it is possible to define Vaisman manifolds using the point of view of presentations. This is done in Theorem 4.2, with the aid of the following proposition which can be proved by direct computation.

**Proposition 4.1.** *Let $W$ be a Riemannian manifold and let $f$ be a homothety of its Riemannian cone $\mathcal{C}(W)$ with dilation factor $\rho_f$. Suppose, moreover, that $f$ commutes with the radial flow $\phi_s$ given by $\phi_s(w,t) = (w, s\cdot t)$. Then $f(w,t) = (\psi_f(w), \rho_f \cdot t)$, where $\psi_f$ is an isometry of $W$. In particular, the isometries of $W$ are the isometries of the cone $\mathcal{C}(W)$ commuting with the radial flow $\phi_s$.*

**Theorem 4.2.** *A locally conformal Kähler manifold $[(K,\Gamma)]$ is Vaisman if and only if $K$ is the Kähler cone $\mathcal{C}(W)$ of a Sasakian manifold $W$ and $\Gamma$ commutes elementwise with the radial flow $\phi_s(w,t) = (w, s\cdot t)$ of the cone.*

**Proof.** The universal covering of a Vaiman manifold is a cone over a Sasakian manifold, as was implicit in the work of Vaisman [26] who, using the de Rham Theorem, observed the decomposition of the universal cover as the product of the real line with a Sasakian manifold that covers the leaves of the foliation $\omega^{\sharp^\perp}$, these having an induced Sasakian structure. That the lifted globally conformal Kähler metric is conformal with a cone metric is then immediate. This fact, together with Proposition 4.1, implies that all the presentations of a Vaisman manifold are cones.

Vice versa, it has been proved in [8] that any manifold admitting such a presentation is Vaisman. □

**Definition 4.3.** A pair $(\mathcal{C}(W),\Gamma)$ is a *Vaisman presentation* if $W$ is a Sasakian manifold, $\mathcal{C}(W)$ its Kähler cone, and $\Gamma$ a discrete Lie group of biholomorphic homotheties commuting elementwise with the radial flow $\phi_s$ and acting freely and properly discontinously on $\mathcal{C}(W)$. Correspondingly, the equivalence class $[(\mathcal{C}(W),\Gamma)]$ is called a Vaisman manifold.

Thus, whenever $M$ is a Vaisman manifold, $\tilde{K}$ is replaced by $\mathcal{C}(\tilde{W})$ and $K_{\min}$ is replaced by $\mathcal{C}(W_{\min})$, where $\tilde{W}$ is a simply connected Sasakian manifold and $W_{\min}$ is the "smallest" Sasakian manifold such that its Kähler cone $\mathcal{C}(W_{\min})$ covers $M$. Moreover, the additional property of $\tilde{\Gamma}$, $\Gamma_{\min}$ commuting with the radial flow, holds.

**Example 4.4.** Consider primary Hopf surfaces of Kähler rank 0, that is, of the form $(\mathbb{C}^2\backslash 0)/\Gamma$, where $\Gamma$ is generated by $(z,w) \mapsto (\alpha z + \lambda w^m, \alpha^m w)$ for a non-zero complex number $\lambda$. We know from [1] that, on this kind of surface, there are no Vaisman structures. Accordingly, it is easy to check that $\Gamma$ does not commute with the radial flow of the standard cone structure on $\mathbb{C}^2\backslash 0$.

**Example 4.5.** The example given in 2.3 of Hopf surfaces can be generalized to any dimension. Consider the compact complex manifold

$$H_{\alpha_1,\dots,\alpha_n} = \frac{\mathbb{C}^n\backslash 0}{\Gamma}$$

with $\Gamma$ generated by $(z_1, \ldots, z_n) \mapsto (\alpha_1 z_1, \ldots, \alpha_n z_n)$, where $\alpha_i$ are complex numbers of equal length $\neq 1$. Then $\Gamma$ acts by homotheties for the standard metric $dz_1 \otimes d\bar{z}_1 + \cdots + dz_n \otimes d\bar{z}_n$, and the resulting $H_{\alpha_1,\ldots,\alpha_n}$ is locally conformal Kähler. Moreover, since $\mathbb{C}^n \backslash 0$ with the standard metric is the Kähler cone of the standard Sasakian structure on the sphere $S^{2n-1}$, and $\Gamma$ commutes with the radial flow, the Hopf manifold $H_{\alpha_1,\ldots,\alpha_n}$ is Vaisman. The case of $\alpha_i$ generic is settled in [14] deforming the standard Sasakian structure on the sphere, thus obtaining a Vaisman structure on any $H_{\alpha_1,\ldots,\alpha_n}$.

For a compact Vaisman manifold $M$, the picture can be made more precise: one can associate a compact Sasakian manifold $W$ as any of the fibers of a canonically defined Riemannian submersion $M \to S^1$ and a biholomorphic homothety $\phi$ of the Kähler cone of $W$ such that $M = [(\mathcal{C}(W), \langle\phi\rangle)]$ [22, Structure Theorem]. Note that if $(\mathcal{C}(W'), \mathbb{Z})$ is any other presentation of $M$, then (up to homotheties) $W' = W$. Since any presentation of the form $(K, \mathbb{Z})$ is necessarily minimal, we have the following corollary.

**Corollary 4.6.** *The minimal presentation of any compact Vaisman manifold is* $(\mathcal{C}(W_{\min}), \mathbb{Z})$*, for a compact Sasakian manifold* $W_{\min}$.

**Corollary 4.7.** *If* $M$ *is a compact Vaisman manifold, then* $r_M = 1$.

## 5. Vaisman automorphisms and reduction

At this point, we are left with the description of automorphisms for Vaisman presentations. Given the fact that the additional structure in the Vaisman case is the compatibility of $\Gamma$ with the radial flow, we could define $\mathrm{Sas}(M)$ as those maps $f \in \mathrm{Lck}(M)$ such that $\tilde{f}$ (or equivalently, as we will see, $f_{\min}$) commutes with the radial flow: in fact, commuting with the radial flow implies reducing to the Sasakian structure of the base of the cone, due to Proposition 4.1.

Hence, $\mathrm{Sas}(M)$ can be equivalently defined as the group of locally conformal Kähler automorphism induced by the Sasaki automorphisms of the base of the maximal cone via

$$f(w, t) = (\psi_f(w), \rho_f \cdot t).$$

The first result of this section, in Theorem 5.4, is that, in the compact case, the additional hypothesis on a $\Gamma$-invariant automorphism of commuting with the radial flow is actually redundant, hence $\mathrm{Sas}(M) = \mathrm{Lck}(M)$.

The main tool to prove the equivalence is the following theorem, together with the Structure Theorem in [22].

**Theorem 5.1.** *For any compact Riemannian manifold* $W$*, all the homotheties of the cone* $\mathcal{C}(W)$ *are given by* $(w, t) \mapsto (\psi(w), \rho \cdot t)$*, where* $\rho$ *is the dilation factor and* $\psi$ *is an isometry of* $W$*. In particular, all the isometries of the cone* $\mathcal{C}(W)$ *come from those of* $W$.

**Proof.** Let $d$ and $\tilde{d}$ be the distances on $W$ and on the cone $W \times \mathbb{R}^+$, respectively. The metric completion $(W \times \mathbb{R}^+)^*$ of $\tilde{d}$ is obtained by adding just one point, which we call the origin 0. This can be seen as follows.

Since the cone metric is given by $\tilde{g} = dr^2 + r^2 g$, the length of germs of curves $[\gamma(t)] = [(w(t), r(t))]$ satisfies

$$\tilde{g}(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2} = (\dot{r}^2(t) + r^2(t) g(\dot{w}(t), \dot{w}(t)))^{1/2} \geq |\dot{r}(t)|,$$

and this implies the following inequality for $\tilde{d}$:

$$\tilde{d}((w, r), (v, s)) \geq |s - r|.$$

Thus, if $\{(w_n, r_n)\}$ is a Cauchy sequence not converging in $W \times \mathbb{R}^+$, then $r_n \to 0$. If $(w, r)$, $(v, s)$ are points of $W \times \mathbb{R}^+$, then the triangular inequality implies that

$$\tilde{d}((w, r), (v, s)) \leq \min\{r, s\} d(w, v) + |s - r|.$$

It follows that if $\{(w_n, r_n)\}$, $\{(v_n, s_n)\}$ are Cauchy sequences not converging in $W \times \mathbb{R}^+$, then $\tilde{d}((w_n, r_n), (v_n, s_n)) \to 0$.

Next, observe that every isometry of the Riemannian cone $W \times \mathbb{R}^+$ can be extended by mapping $0 \mapsto 0$ to a transformation of $(W \times \mathbb{R}^+)^*$ preserving $\tilde{d}^*$ that is preserving rays and level submanifolds of the cone. This means that every isometry of the cone $W \times \mathbb{R}^+$ is of the form $(\psi, \mathrm{Id}_{\mathbb{R}^+})$, where $\psi$ is an isometry of $W$. The claim for a general homothety follows. □

**Remark 5.2.** Note that, to prove that $r_n \to 0$, we only need the completeness of $W$, whereas to prove that $\tilde{d}((w_n, r_n), (v_n, s_n)) \to 0$, we need its compactness. Moreover, the compactness hypothesis is essential to complete the cone metric with only one point: take, for example, $W = \mathbb{R}$, and the Cauchy sequences $\{(-n, 1/n)\}$, $\{(n, 1/n)\}$.

The following statement is a small *detour*.

**Corollary 5.3.** *Let $\Gamma$ be a discrete Lie group of homotheties acting freely and properly discontinously on a Riemannian cone $\mathcal{C}(W)$, where $W$ is a compact Riemannian manifold. Then $\Gamma \simeq I \rtimes \mathbb{Z}$, where $I$ is a finite subgroup of isometries of $W$.*

**Proof.** Let $\rho : \Gamma \to \mathbb{R}^+$ be the map defined by the dilation factor of elements of $\Gamma$. Note that if $\rho(\Gamma)$ is not cyclic, then it contains two elements $\alpha, \beta$ such that $\log \alpha / \log \beta$ is irrational. This in turn implies that $\rho(\Gamma)$ is dense in $\mathbb{R}^+$, a fact which, together with the compactness of $W$, contradicts the proper discontinuity of $\Gamma$. Thus $\rho(\Gamma) \simeq \mathbb{Z}$. Moreover, since isometries on a Riemannian cone $\mathcal{C}(W)$ are actually isometries of the base space $W$ by Theorem 5.1, the proper discontinuity of $\Gamma$ and the compactness of $W$ imply that the normal subgroup $I = \rho^{-1}(1)$ of isometries in $\Gamma$ is finite, and $\Gamma$ is a finite extension of $\mathbb{Z}$:

$$0 \to I \to \Gamma \to \mathbb{Z} \to 0.$$

This implies that $\Gamma \simeq I \rtimes \mathbb{Z}$. □

We can now prove the first result of this section.

**Theorem 5.4.** *Let $M$ be a compact Vaisman manifold and let $f \in \mathrm{Lck}(M)$. Let $(\mathcal{C}(W), \Gamma)$ be a Vaisman presentation of $M$ and suppose there exists a lifting $f' \in \mathrm{Hom}_\Gamma(\mathcal{C}(W))$ of $f$. Then*

$$f'(w, t) = (\psi_{f'}(w), \rho_{f'} \cdot t)$$

*where $\psi_{f'}$ is a Sasakian automorphism of $W$. Hence*

$$\mathrm{Sas}(M) = \mathrm{Lck}(M).$$

*In particular, if $f'$ is an isometry, then $f' = \psi_{f'} \times \mathrm{Id}$.*

**Proof.** Consider the following diagram.

$$\begin{CD} \mathcal{C}(W) @>{f'}>> \mathcal{C}(W) \\ @V{\pi}VV @VV{\pi}V \\ \mathcal{C}(W_{\min}) @>{f_{\min}}>> \mathcal{C}(W_{\min}) \end{CD} \tag{5.1}$$

Since $\mathcal{C}(W_{\min})$ is Kähler, it follows that the covering maps of $\pi$ are isometries. According to Theorem 4.2 and Proposition 4.1, $\pi$ is given by a projection of Sasakian manifolds. Moreover, $f_{\min} = \psi \times \rho$ by Theorem 5.1, and the diagram becomes the following:

$$\begin{CD} \mathcal{C}(W) @>{f'}>> \mathcal{C}(W) \\ @V{\pi \times \mathrm{Id}}VV @VV{\pi \times \mathrm{Id}}V \\ \mathcal{C}(W_{\min}) @>{\psi \times \rho}>> \mathcal{C}(W_{\min}) \end{CD} \tag{5.2}$$

This implies that

$$(\pi \times \mathrm{Id}) \circ f' = (\psi \times \rho) \circ (\pi \times \mathrm{Id}) = (\psi \circ \pi) \times \rho$$

and thus $f'(w, t) = (f_1'(w, t), \rho \cdot t)$, where $f_1'$ satisfies

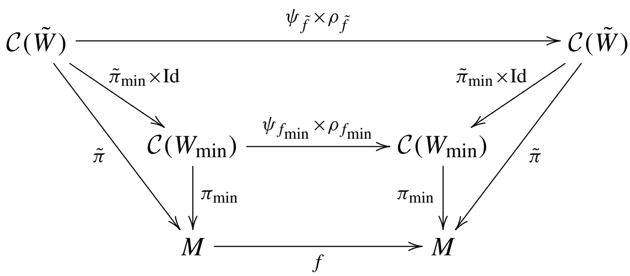
$$\pi(f_1'(w, t)) = \psi(\pi(w)).$$

This last equation implies that $f_1'(w, \mathbb{R})$ is discrete in $W_{\min}$, and

$$f'(w, t) = (f_1'(w, t), \rho \cdot t) = (f_1'(w), \rho \cdot t).$$

To end the proof, note that $f'$, being a Kähler automorphism, forces $f_1'$ to be a Sasakian automorphism. □

The following is diagram (3.1), adapted to the compact Vaisman case.



(5.3)

In [8], the notion of twisted Hamiltonian action for compact Vaisman manifolds was partially linked to the cone structure of the presentation. Those locally conformal Kähler automorphisms induced by isometries of a covering cone of the form

$$f = \psi_f \times \mathrm{Id}$$

where $\psi_f$ is a Sasakian isometry of the base, were called *Vaisman automorphisms* of the compact Vaisman manifold. It was shown that if all of the elements of a group $G \subset \mathrm{Lck}(M)$ are Vaisman automorphisms, then its action is twisted Hamiltonian. It was moreover proven that twisted

Hamiltonian actions of Vaisman automorphisms induced a Vaisman structure on the reduction. We can now prove that there are no twisted Hamiltonian actions but those ones.

**Theorem 5.5.** *The action of a connected group G on a compact Vaisman manifold M is twisted Hamiltonian if and only if it is by Vaisman automorphisms; if, moreover, the topological hypothesis of the Reduction Theorem is satisfied, then*

$$M /\!\!/ G = [(\mathcal{C}(W_{\min} /\!\!/ G^{\circ}_{\min}), \Gamma_{\min})] = \left[\left(\mathcal{C}(\tilde{W} /\!\!/ \tilde{G}^{\circ}), \frac{\tilde{\Gamma}}{\tilde{\Gamma} \cap \tilde{G}^{\circ}}\right)\right].$$

**Proof.** That the action of a group of Vaisman automorphisms admits a momentum map, hence is twisted Hamiltonian, is proved in Theorem 5.13 of [8].

Vice versa, let $G$ be any twisted Hamiltonian connected Lie subgroup of $\mathrm{Lck}(M)$. Since $\mathrm{Lck}(M) = \mathrm{Sas}(M)$, the elements of $\tilde{G}^{\circ}$ are of the form

$$f(w, t) = (\psi_f(w), \rho_f \cdot t),$$

and, since $\tilde{G}^{\circ}$ is connected, $\rho|\tilde{G}^{\circ}$ is constantly equal to 1, that is $\rho_f = 1$ for all $f \in \tilde{G}^{\circ}$, and hence $G$ is of Vaisman automorphisms.

The final claim is Theorem 5.15 of [8] in the present notation. □

This proves the theorem in the Introduction.

**Remark 5.6.** The theorem in the Introduction holds true for any Vaisman presentation $[(\mathcal{C}(W), \Gamma)]$ and $G$ acting on $W$.

**Corollary 5.7.** *The Structure Theorem of* [22] *and Vaisman reduction are compatible.*

**Proof.** Note that, in the equivalence

$$[(\mathcal{C}(W_{\min}), \Gamma_{\min})] /\!\!/ G = [(\mathcal{C}(W_{\min} /\!\!/ G^{\circ}_{\min}), \Gamma_{\min})]$$

the group $\Gamma_{\min}$ is preserved. This implies that the reduced Sasakian manifold $W_{\min} /\!\!/ G^{\circ}_{\min}$ is still given by the Structure Theorem. □

**Remark 5.8.** Whereas the minimal presentation is preserved, as shown above in Corollary 5.7, the maximal presentation is not, because $\tilde{W} /\!\!/ \tilde{G}^{\circ}$ needs not be simply connected.

**Remark 5.9.** If $\mathcal{C}(W_{\min}) \stackrel{\mathbb{Z}}{\to} M$ is the covering given by the minimal presentation of a compact Vaisman manifold and $\pi$ is the projection of the cone onto its radius, it is easy to check that $\pi$ is equivariant with respect to the action of the covering maps on $\mathcal{C}(W_{\min})$ and the action of $n \in \mathbb{Z}$ on $t \in \mathbb{R}$ given by $n + t$. This describes in an alternative way the projection over $S^1$ of the Structure Theorem in [22], and moreover provides a structure theorem for other types of locally conformal structures, where the compactness of the base of the cone is given by other methods (see [12] for an application to $G_2$, Spin(7) and Spin(9) structures).

## 6. Locally conformal hyperKähler presentations

Usually, a locally conformal hyperKähler manifold is a conformal hyperhermitian manifold $(M, [g])$ such that $g$ is conformal to local hyperKähler metrics.

In the same way as for locally conformal Kähler manifolds, we observe that a hyperKähler manifold $H$ and a discrete Lie group $\Gamma$ of hypercomplex homotheties acting freely and properly discontinously on $H$ give rise to a locally conformal hyperKähler manifold $H/\Gamma$, and vice versa that every locally conformal hyperKähler manifold can be viewed this way.

Since the hard part of the work is already done in the locally conformal Kähler setting, here we sketch the definitions and properties shared in the hyperKähler case. For the sake of simplicity, we do not use a different notation/terminology.

**Definition 6.1.** Given a homothetic hyperKähler manifold $H$ and a discrete Lie group $\Gamma$ of hypercomplex homotheties acting freely and properly discontinously on $H$, the pair $(H, \Gamma)$ is called a *presentation*. If a locally conformal hyperKähler manifold $M$ is given, and $M = H/\Gamma$ are locally conformal hyperKähler manifolds, then we say that $(H, \Gamma)$ is a presentation *of* $M$.

As before, the maximal and the minimal presentations are defined, and give a way to distinguish between equivalent presentations.

**Definition 6.2.** A *locally conformal hyperKähler* manifold is an equivalence class $[(H, \Gamma)]$ of presentations.

**Example 6.3.** Let $q \in \mathbb{H}\backslash 0$ be a non-zero quaternion of length different to 1. Then the right multiplication by $q$ generates a cocompact, free and properly discontinous action of homotheties on $\mathbb{H}^n\backslash 0$ equipped with the standard metric. Denote by $\Gamma$ the infinite cyclic group $\langle h \mapsto h \cdot q\rangle$. Then $(\mathbb{H}^n\backslash 0, \Gamma)$ is a compact locally conformal hyperKähler manifold, called a quaternionic Hopf manifold.

By definition, any locally conformal hyperKähler manifold is locally conformal Kähler, but much more is true in the compact case.

**Theorem 6.4** ([4,6]). *A compact locally conformal hyperKähler manifold is Vaisman for each one of its three underlying locally conformal Kähler structures.*

This shows that, in the compact case, locally conformal hyperKähler manifolds are the hypercomplex analogs of Vaisman manifolds and not only of locally conformal Kähler manifolds. This leads to the following theorem. Before stating it, recall that a Riemannian manifold $(W, g)$ is 3-Sasakian if, by definition, the Riemannian cone metric on $W \times \mathbb{R}^+$ has holonomy contained in $\mathrm{Sp}(n)$; see, for instance, [2].

**Theorem 6.5.** *Let* $[(H, \Gamma)]$ *be a compact locally conformal hyperKähler manifold. Then* $H$ *is the hyperKähler cone of a compact 3-Sasakian manifold.*

**Proof.** By Theorems 6.4 and 4.2, $H$ is the Kähler cone of three different compact Sasakian manifold. Using Theorem 5.1, one sees that they are isometric, say $W$ the common underlying Riemannian manifold. Then $W$ is tautologically 3-Sasakian, because its Riemannian cone $H$ is hyperKähler. □

**Corollary 6.6.** *Let* $H$ *be a hyperKähler manifold. Let* $\Gamma$ *be a cocompact discrete Lie group of hypercomplex homotheties of* $H$ *acting freely and properly discontinously. Then* $H$ *is the hyperKähler cone of a compact 3-Sasakian manifold, and* $\Gamma$ *commutes with the radial flow.*

**Example 6.7.** For the quaternionic Hopf manifold given in Example 6.3, the 3-Sasakian manifold $W$ given by Theorem 6.5 is the standard sphere $S^{4n-1}$.

## 7. Locally conformal hyperKähler reduction

Let $[(H, \Gamma)]$ be a locally conformal hyperKähler manifold. Denote by $\mathrm{Hom}_\Gamma(H)$ the Lie group of $\Gamma$-equivariant hypercomplex homotheties of $H$, and remark that $\mathrm{Hom}_\Gamma(H)/\Gamma$ coincides with the Lie group $\mathrm{Lchk}(M)$ of hypercomplex conformal transformations of $M = H/\Gamma$ whenever $(H, \Gamma)$ is the maximal or the minimal presentation. Accordingly, a connected Lie subgroup $G$ of $\mathrm{Lchk}(M)$ is *twisted Hamiltonian* if the identity component $\tilde{G}^\circ$ of $\tilde{G}$ or $G^\circ_{\mathrm{min}}$ of $G_{\mathrm{min}}$ is Hamiltonian in the hyperKähler sense. We choose the same term Hamiltonian for both the complex and the quaternionic contexts: the name Hamilton is of course related both to Hamiltonian mechanics and to quaternions, and it does not seem appropriate to look for a different term for Hamiltonian maps in quaternionic geometry.

If $G$ is a twisted Hamiltonian subgroup of $\mathrm{Lchk}(M)$, the quotient of the hyperKähler momentum map $\tilde{\mu}$ on $\tilde{H}$ is the *locally conformal hyperKähler momentum map*. Nothing is different if we use the minimal presentation instead.

**Theorem 7.1.** *Let $[(H, \Gamma)]$ be a locally conformal hyperKähler manifold, say $M = H/\Gamma$ in the usual sense. Let G be a twisted Hamiltonian Lie subgroup of* $\mathrm{Lchk}(M)$, *with momentum map $\mu$. Suppose that the usual topological conditions are satisfied, that is, $\mu^{-1}(0)$ is non-empty, 0 is a regular value for $\mu$, and G acts freely and properly on $\mu^{-1}(0)$. Then*

$$\left[\left(\tilde{H} /\!/\!/ \tilde{G}^\circ, \frac{\tilde{\Gamma}}{\tilde{\Gamma} \cap \tilde{G}^\circ}\right)\right] = [(H_{\mathrm{min}} /\!/\!/ G^\circ_{\mathrm{min}}, \Gamma_{\mathrm{min}})]$$

*is a locally conformal hyperKähler manifold, where $/\!/\!/$ denotes the hyperKähler reduction.*

**Proof.** The action of $\tilde{G}^\circ$ commutes with $\tilde{\Gamma}$ by hypothesis, $\tilde{\Gamma}$ acts properly discontinously, and

$$\frac{\tilde{\Gamma}}{\tilde{\Gamma} \cap \tilde{G}^\circ}$$

acts freely by hypercomplex homotheties on the hyperKähler reduction $\tilde{H} /\!/\!/ \tilde{G}^\circ$. Likewise, if we use $(H_{\mathrm{min}} /\!/\!/ G^\circ_{\mathrm{min}}, \Gamma_{\mathrm{min}})$ instead. □

In practice, to reduce a locally conformal hyperKähler manifold $M$ presented as $(H, \Gamma)$, we lift $G$ to the hyperKähler universal covering $\tilde{H}$ of $H$, take its identity component $\tilde{G}^\circ$, and reduce $\tilde{H}$. We then take the hyperKähler reduction $\tilde{H} /\!/\!/ \tilde{G}^\circ$. The reduction we were looking for is presented by

$$\left(\tilde{H} /\!/\!/ \tilde{G}^\circ, \frac{\pi_1(M)}{\tilde{G}^\circ \cap \pi_1(M)}\right).$$

If we want to get rid of the remaining isometries, because they do not give any contribution to the locally conformal part of the structure, then we switch to the minimal presentation.

**Remark 7.2.** If we start with a subgroup $G$ of $\mathrm{Hom}_\Gamma(H)$, and we do not know that it appears as a lifting of a free action on $H/\Gamma$, then we must add the hypothesis that $\Gamma/(G \cap \Gamma)$ is free and properly discontinous on $H$.

**Definition 7.3.** Let $[(H, \Gamma)]$ be a locally conformal hyperKähler manifold. Let $G$ be a connected Hamiltonian subgroup of $\mathrm{Hom}_\Gamma(H)$ (this implies that $\Gamma$ is $G$-equivariant) with hyperKähler momentum map $\mu: H \to \mathfrak{g}^* \otimes \mathbb{R}^3$. Suppose that $\mu^{-1}(0)$ is non-empty, 0 is a regular value for

$\mu$, and the action of $G$ is free and proper on $\mu^{-1}(0)$, so that the hyperKähler reduction $H /\!/\!/ G$ is defined. Suppose, moreover, that the action of $\Gamma$ on $H /\!/\!/ G$ is properly discontinous, and that the action of $\Gamma/(\Gamma \cap G)$ on $H /\!/\!/ G$ is free. Then

$$\left[\left(H /\!/\!/ G, \frac{\Gamma}{\Gamma \cap G}\right)\right]$$

is the *reduction* of the locally conformal hyperKähler manifold $[(H, \Gamma)]$.

Using Theorem 6.5, we relate compact locally conformal hyperKähler reduction to compact 3-Sasakian reduction.

**Corollary 7.4.** *If* $[(\mathcal{C}(W), \Gamma)]$ *is a compact locally conformal hyperKähler manifold, and G is a twisted Hamiltonian Lie subgroup of* $\mathrm{Lchk}(M)$ *providing a locally conformal hyperKähler reduction, then* $\tilde{G}^{\circ}$, $G^{\circ}_{\min}$ *also provides a 3-Sasakian reduction of* $\tilde{W}$, $W_{\min}$, *respectively.*

As in the compact Vaisman case, with any compact locally conformal hyperKähler manifold $M$ one associates a compact 3-Sasakian manifold $W$ as any of the fibers of a canonically defined Riemannian submersion $M \to S^1$ and a hypercomplex homothety $\phi$ of the hyperKähler cone of $W$ such that $M = (\mathcal{C}(W), \langle\phi\rangle)$ [29, Structure Theorem]. Accordingly, we have the following analogs of Corollary 5.7.

**Corollary 7.5.** *The Structure Theorem of* [29] *and locally conformal hyperKähler reduction are compatible.*

**Remark 7.6.** Due to Corollary 7.4, finding examples of compact locally conformal hyperKähler reduction involves finding discrete Lie groups $\Gamma$ of hypercomplex homotheties acting freely and properly discontinously on the hyperKähler cone $\mathcal{C}(W /\!/\!/ G)$ over a reduced 3-Sasakian manifold $W /\!/\!/ G$ and commuting with the radial flow of this cone. Here, $W$ is a simply connected 3-Sasakian manifold. According to Proposition 4.1, $\Gamma$ splits as a semidirect product of an isometry part $I$ and a "true" homothety part $\mathbb{Z} = \langle f \times \rho\rangle$, where $f$ is an isometry of $(W /\!/\!/ G)/I$ and $\rho$ denotes translations by $\rho \in \mathbb{R}$ along the radius of $\mathcal{C}((W /\!/\!/ G)/I)$.

If we prefer the compact setting, we switch to the minimal presentation. Hence, we look for $\Gamma$ acting on $\mathcal{C}(W /\!/\!/ G)$, where $W$ is a compact 3-Sasakian manifold. In this case, $\Gamma$ splits as $I \rtimes \mathbb{Z}$ for a finite isometry part $I$.

The isometry type of $\mathcal{C}(W /\!/\!/ G)/\Gamma$ is then related to the isotopy class of $f$ by

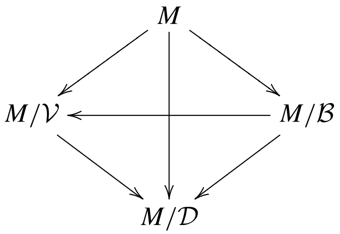
$$\frac{\mathcal{C}(W /\!/\!/ G)}{\Gamma} = \frac{\mathcal{C}((W /\!/\!/ G)/I)}{\langle f \times \rho\rangle} = \frac{((W /\!/\!/ G)/I) \times [0, \rho]}{(x, 0) \sim (f(x), \rho)}.$$

In particular, $\mathcal{C}(W /\!/\!/ G)/\Gamma$ is a product if and only if $f$ is isotopic to the identity.

**Remark 7.7.** The well-known Diamond Diagram (DD) pictured in [2], which relates complex, positive quaternion-Kähler, 3-Sasakian and hyperKähler geometries to each other, has been ported to the compact locally conformal hyperKähler setting in [19] in the following way. Now let $(M, [g], I, J, K)$ be a compact locally conformal hyperKähler manifold, where $g$ is the Gauduchon metric, and denote by $\omega$ the corresponding length 1 Lee form. Then define three foliations $\mathcal{B}$, $\mathcal{V}$, $\mathcal{D}$ on $M$, respectively, as $\langle\omega\rangle$, $\langle\omega, I\omega\rangle$, $\langle\omega, I\omega, J\omega, K\omega\rangle$, and suppose the leaves of these foliations are compact. Then

- $M/\mathcal{B}$ is a 3-Sasakian orbifold;
- $M/\mathcal{V}$ is a positive Kähler–Einstein orbifold;
- $M/\mathcal{D}$ is a positive quaternion-Kähler orbifold.

Moreover, the following diagram holds:




(7.1)

Using the argument of Remark 7.6, one obtains diagram (7.1) from the corresponding DD. For instance, if $M$ is presented as $(\mathcal{C}(W), I \rtimes \mathbb{Z})$, then $M/\mathcal{B}$ is given by $W/I$. It follows that to any reduction of a locally conformal hyperKähler manifold corresponds a reduction of any of the spaces in (7.1).

**Example 7.8.** Let $M = (\mathbb{H}^n \backslash 0, \Gamma)$ be a quaternionic Hopf manifold, as described in Examples 6.3 and 6.7. Consider the left action of $S^1$ on $M$ described by

$$t \cdot (h_1, \ldots, h_n) = (e^{it} h_1, \ldots, e^{it} h_n) \qquad t \in S^1, (h_1, \ldots h_n) \in \mathbb{H}^n \backslash 0. \tag{7.2}$$

This action is $\Gamma$-equivariant and Hamiltonian for the standard metric on $\mathbb{H}^n \backslash 0$, so it is, by definition, a twisted Hamiltonian action on $M$. The corresponding 3-Sasakian action (see Corollary 7.4) on $S^{4n-1}$ is described by the same formula (7.2), where in this case $(h_1, \ldots h_n) \in S^{4n-1}$. The 3-Sasakian quotient of the sphere is the space (see [3])

$$\frac{\mathrm{U}(n)}{\mathrm{U}(n-2) \times \mathrm{U}(1)}.$$

Hence, Remark 7.6 implies that the locally conformal hyperKähler quotient is

$$\frac{\mathrm{U}(n)}{\mathrm{U}(n-2) \times \mathrm{U}(1)} \times \mathrm{U}(1).$$

**Remark 7.9.** The usual definition of locally conformal hyperKähler structures can be resumed in the requirement for a hyperhermitian structure with fundamental forms $\Omega_i$ to satisfy $d\Omega_i = \omega \wedge \Omega_i$ for the same closed Lee form $\omega$. A locally conformal hyperKähler reduction can then alternatively be defined by means of the Lee form $\omega$ and its associated twisted differential $d^\omega = d - \omega \wedge \cdot$, in the same way as for the locally conformal Kähler case. This construction is equivalent to the one given above by means of the presentations.

**Remark 7.10.** In the compact case, we can mimic the proof in [10] to provide a direct construction of the locally conformal hyperKähler quotient. We briefly indicate the argument.

Let $(M, [g], I, J, K)$ be a *compact* locally conformal hyperKähler manifold, and denote by $\Omega_i$ the corresponding fundamental forms. Let $G$ be a twisted Hamiltonian subgroup of $\mathrm{Lchk}(M)$, and $\mu = (\mu_1, \mu_2, \mu_3)$ the corresponding momentum map. Suppose that 0 is a regular value of $\mu$, and that $G$ acts freely and properly on $\mu^{-1}(0)$. Let $N = \mu_2^{-1}(0) \cap \mu_3^{-1}(0)$. We show that this is a complex submanifold of $(M, I)$. Indeed, for any $x \in N$, we have $T_x^\perp N = J\mathfrak{g}(x) \oplus K\mathfrak{g}(x)$, where $\mathfrak{g}(x)$ is the vector space spanned by the values in $x$ of the fundamental fields associated with the elements of $\mathfrak{g}$. On the other hand, a vector $X \in T_x M$ belongs to $T_x \mu_i^{-1}(0)$ if and only if

$\Omega_{ix}(X,\cdot) = d^{\omega}\mu^X_{ix} = 0$. Now let $X \in \mathfrak{X}(M)$ and $Y$ be any fundamental field. Then, on $N$:

$$g(IX, KY) = -g(JX, Y) = \Omega_2(X, Y) = 0,$$
$$g(IX, JY) = g(KX, Y) = \Omega_3(X, Y) = 0.$$

Hence $N$ is a complex submanifold of the Vaisman manifold $(M, [g], I)$. By construction, it is a closed submanifold, thus, by [25] (see also [29, Proposition 2.1]), the Lee field is tangential to $N$ and $(N, [g_{|_N}], I)$ is actually a Vaisman manifold.[1] Now $G$ acts on $(N, [g_{|_N}], I)$ as a twisted Hamiltonian subgroup of $\mathrm{Lck}(M)$. The associated momentum map is the restriction to $N$ of $\mu_1$, therefore $\mu^{-1}(0) = N \cap \mu_1^{-1}(0)$. Then, according to [8], the quotient $\mu^{-1}(0)/G$ is a locally conformal Kähler manifold with respect to the projected conformal class and complex structure. The same argument for $J$ and $K$ ends the proof.

## 8. Joyce hypercomplex reduction and HKT reduction

Locally conformal hyperKähler manifolds are first of all hypercomplex manifolds, and it seems natural to ask whether the Joyce hypercomplex reduction given in [13] is compatible with the locally conformal hyperKähler reduction defined in Section 7. Recall that, in the hypercomplex reduction, a momentum map for the action of a Lie group $G$ on a hypercomplex manifold $(M, I, J, K)$ is defined as any triple of maps $\mu_i : M \to \mathfrak{g}^*$ satisfying

$$I d\mu_1 = J d\mu_2 = K d\mu_3 \tag{8.1}$$

and $I d\mu_1(X) \neq 0$ for every non-zero fundamental vector field. If such a momentum map exists, then $(\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0))/G$ is a hypercomplex manifold that we denote by $M /\!\!/\!\!/_{\mathrm{Joyce}} G$.

**Theorem 8.1.** *Let $G$ be a twisted Hamiltonian Lie subgroup of* $\mathrm{Lchk}(M)$. *Then a locally conformal hyperKähler momentum map for the action of $G$ is also a hypercomplex momentum map. Moreover,*

$$M /\!\!/\!\!/ G = M /\!\!/\!\!/_{\mathrm{Joyce}} G$$

*as hypercomplex manifolds.*

**Proof.** Let $[(H, \Gamma)]$ be the locally conformal hyperKähler manifold. Denote by $(g, I, J, K)$ the hyperKähler structure on $H$, by $\Omega_i$ the fundamental forms, and by

$$\begin{array}{llll} \mu_i : & \mathfrak{g} & \longrightarrow & C^\infty(H) \\ & X & \longmapsto & \mu_i^X \quad i = 1, 2, 3 \end{array}$$

the Kähler momentum maps. Then, by definition,

$$d\mu_i^X = i_X \Omega_i$$

and this implies that

$$\left.\begin{array}{l} I d\mu_1^X = I i_X \Omega_1 \\ J d\mu_2^X = J i_X \Omega_2 \\ K d\mu_3^X = K i_X \Omega_3 \end{array}\right\} = g(X, \cdot).$$

Hence, the locally conformal hyperKähler momentum map satisfies (8.1) and the claim follows. □

[1] The compactness of $M$ is essential to show that $N$ is compact and to apply the cited result.

**Remark 8.2.** A Hermitian manifold equipped with a locally conformal hyperKähler structure might admit Joyce reductions that are not locally conformal hyperKähler reductions.

As an example, let $M = (\mathbb{H}^3\backslash 0, \Gamma)$ be the quaternionic Hopf manifold given in Example 6.3, where, for simplicity, $\Gamma$ is spanned by a real number in (0, 1). Taking into account that $M$ is diffeomorphic to $S^{11} \times S^1$, we define an action of $S^1 \subset \mathbb{C}$ on $M$ by left multiplication, that is, if $\theta \in S^1$ and $(z_1, \ldots, z_7)$ are the complex coordinates describing $S^{11} \times S^1 \subset \mathbb{C}^6 \times \mathbb{C}$, then $\theta$ acts by

$$\theta \cdot (z_1, \ldots, z_7) = (\mathrm{e}^{2\pi \mathrm{i}\theta} z_1, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta} z_7). \tag{8.2}$$

According to [23, Example 6.3], this action admits a Joyce momentum map with respect to the standard hypercomplex structure (right multiplication by quaternions). The hypercomplex quotient is SU(3) which, being simply connected, does not admit any locally conformal hyperKähler structure. This means that this Joyce reduction is *not* a locally conformal hyperKähler reduction. In fact, this action on $S^{11} \times S^1$ rotates the seventh complex coordinate, that is, it is effective along the $S^1$ factor of the Hopf manifold, and hence lifts to an action by non-trivial homotheties on $H^3\backslash 0$ which, a fortiori, cannot be a Hamiltonian action.

In passing, we remark on the following somehow curious fact (compare also with Example 7.8). Denote by $\mu_{\mathrm{Joyce}}$ the lifting to $\mathbb{H}^3\backslash 0$ of the Joyce momentum map for the action (8.2) chosen in [23]. Consider the following action of $S^1$ on $S^{11} \times S^1$:

$$\theta \cdot (z_1, \ldots, z_7) = (\mathrm{e}^{2\pi \mathrm{i}\theta} z_1, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta} z_6, z_7). \tag{8.3}$$

This action lifts to the left complex multiplication on $\mathbb{H}^3\backslash 0$, which is Hamiltonian with respect to the flat metric. Thus, by definition, the action (8.3) is twisted Hamiltonian on the quaternionic Hopf manifold and admits a locally conformal hyperKähler momentum map. Denote by $\mu$ the lifting to $\mathbb{H}^3\backslash 0$ of this locally conformal hyperKähler momentum map. Then $\mu_{\mathrm{Joyce}} = \mu$.

Nevertheless, since it is originated by different actions, this momentum map produces different quotients. With respect to (8.2), the quotient is SU(3), whereas with respect to (8.3) the quotient is

$$\frac{\mathrm{SU}(3)}{S^1} \times S^1.$$

We now discuss the relation between locally conformal hyperKähler reduction and HKT reduction. Recall first that, on a hyperhermitian manifold $(M, g)$, a connection $D$ is called hyperKähler with torsion (HKT) if it is hyperhermitian (namely, $DI = DJ = DK = 0$, $Dg = 0$) and its torsion is totally skew-symmetric (that is, $g(\mathrm{Tor}^D(X, Y), Z)$ is a 3-form).

The following result shows that compact locally conformal hyperKähler geometry (in which the Lee form can always be assumed parallel and unitary) is related to HKT geometry.

**Theorem 8.3** (*[20]*). *Let $(M, [g], I, J, K)$ be a locally conformal hyperKähler manifold with unitary parallel Lee form $\omega$ associated with the metric $g$. Then the metric*

$$\hat{g} = g - \frac{1}{2}\{\omega \otimes \omega + I\omega \otimes I\omega + J\omega \otimes J\omega + K\omega \otimes K\omega\} \tag{8.4}$$

*is HKT.*

As a reduction scheme was recently provided for HKT structures (see [9]), it is natural to discuss the interplay between the two reductions.

Recall that, based on Joyce's hypercomplex reduction (see [13]), the HKT reduction has the peculiarity that a "good" action of a group of hypercomplex automorphisms does not automatically produce a momentum map. What is proved in [9] is that if a group acts by hypercomplex isometries with respect to the HKT metric and *if Joyce's hypercomplex momentum map exists*, then the hypercomplex quotient inherits a natural HKT structure.

**Theorem 8.4.** *Let M be a compact locally conformal hyperKähler manifold, G a compact twisted Hamiltonian Lie group, and $\mu$ the corresponding momentum map, with 0 as a regular value. Suppose that G acts freely on $\mu^{-1}(0)$, so that the locally conformal hyperKähler reduction of Theorem 7.1 is defined. Then $\mu$ is a hypercomplex momentum map, and G acts by isometries for the associated HKT metric. Moreover, the HKT structure of the quotient is induced by the locally conformal hyperKähler reduced structure by the relation (8.4).*

**Proof.** Let $g$ be the Gauduchon metric of $M$. Then $G$ acts by isometries for $g$, thus preserving the corresponding Lee form and, since $\hat{g}$ is defined using only $G$-invariant tensors, $G$ acts also by $\hat{g}$-isometries. In this situation, $\mu$ satisfies Joyce's conditions, hence $\mu$ is also a HKT momentum map and the HKT quotient exists. Moreover, the reduced HKT structure lives on the reduced locally conformal hyperKähler manifold. As the projection $\mu^{-1}(0) \to \mu^{-1}(0)/G$ is a Riemannian submersion with respect to both the locally conformal hyperKähler and HKT metric, and since the Lee form is projectable, we see that the reduced HKT metric is the one induced by the reduced locally conformal hyperKähler metric. □

Compact HKT manifolds do not have a potential, but if the HKT structure is induced by a locally conformal hyperKähler structure, then the Lee form is a *potential 1-form* as defined in [20]. Thus we have the following corollary.

**Corollary 8.5.** *All manifolds obtained by the HKT reduction associated with a locally conformal hyperKähler reduction admit a potential 1-form.*

**Remark 8.6.** It is not known if generic HKT reduction preserves potential 1-forms.

**Remark 8.7.** Not all HKT structures are induced by a locally conformal hyperKähler structure (see [28]), so one may ask if the HKT reduction can be induced by a locally conformal hyperKähler structure in such cases.

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