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LOCALLY CONFORMALLY KÄHLER METRICS ON HOPF SURFACES

by P. GAUDUCHON & L. ORNEA

To the memory of Franco Tricerri

1. Introduction.

Let M be an even-dimensional, oriented, smooth manifold. A Hermitian structure on M is a pair (J, g) consisting of an *integrable* almost-complex structure J, and a Riemannian metric g such that $g(JX, JY) = g(X, Y)$ for any vector fields X, Y. The Hermitian structure is *Kähler* if J is parallel with respect to the Levi-Civita connection D^g of g; equivalently, as J is integrable, (J, g) is *Kähler* if the *Kähler form* ω, defined by $\omega(X, Y) = g(JX, Y)$, is closed. More generally, (J, g) is called *locally conformally Kähler*, *l.c.K* for short, if for each point x of M there exist an open neighbourhood $\mathcal{U}$ of x and a positive function f on $\mathcal{U}$ so that the pair $(J, f^{-2}g)$ is Kähler, see [7] for a general overview.

When J is understood, we say that g is Kähler, l.c.K. etc. whenever the corresponding Hermitian structure (J, g) is Kähler, l.c.K. etc.

When M is four-dimensional, the only case considered in this paper, the defect for a Hermitian structure (J, g) to be Kähler is measured by the *Lee form*, the real 1-form θ determined by

$$d\omega = -2\theta \wedge \omega, \tag{1}$$

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see [18], [23], [8]. (*Warning*: The definition of the Lee form in the literature may differ from the above definition by a factor $\pm\frac{1}{2}$.)

Then, (J, g) is l.c.K if and only if the Lee form θ is closed, i.e. the defect for (J, g) to be l.c.K. is measured by the 2-form $d\theta$.

A special occurence of l.c.K. metrics is when the Lee form θ is *parallel* with respect to the Levi-Civita connection of g. This case reduces to the Kähler case (θ identically zero) if M is compact, with even first Betti number b_1 [25]; if b_1 is odd, θ is (everywhere) non-zero and (M, J, g) is usually called a *generalized Hopf manifold*. The reason for this name is a God-given series of examples on a sub-class of Hopf surfaces (see below); however, in the present context, when the complex surfaces of interest are Hopf surfaces and not all are *generalized Hopf manifolds* (see Remark 1), we may prefer to call them *Vaisman surfaces*, as *e.g.* in [7].

Notice that not all complex surfaces admitting l.c.K. metrics admit l.c.K. metric with parallel Lee form, see [22], [3]. However, if the surface is compact, the Lee form of any l.c.K. structure can always be made *harmonic* by a conformal change of the metric, see [8].

The l.c.K. condition is conformally invariant i.e. concerns the *conformal Hermitian structure* $(J, [g])$, where $[g]$ denotes the conformal class of g. In particular, $d\theta$ is conformally invariant, but θ and ω are not: the Lee form $\tilde{\theta}$ and the Kähler form $\tilde{\omega}$ of the Hermitian structure $(\tilde{g} = f^{-2}g, J)$ are given by $\tilde{\theta} = \theta + \frac{df}{f}$ and $\tilde{\omega} = f^{-2}\omega$. These rules of transformation can be interpreted as follows. On any n-dimensional smooth manifold M, let L be the *bundle of real scalars of weight 1*, the oriented real line determined (via the $G\ell(n, \mathbb{R})$-principal bundle of (all) frames on M) by the representation $A \in G\ell(n, \mathbb{R}) \mapsto |\det A|^{\frac{1}{n}}$. Then, each Riemannian metric g in the conformal class $[g]$ determines a positive section, hence a trivialization, ℓ, of L and the Kähler form ω appears as the expression of a conformally invariant L^2-valued 2-form, say $\omega_{\rm conf}$, with respect to ℓ. In the same way, the Lee form θ can be viewed as the connection 1-form with respect to ℓ of a conformally invariant linear connection ∇^L on L, the so-called *Weyl derivative* determined by the conformal Hermitian structure $(J, [g])$. Finally, the identity (1) means that the *conformal Kähler form* $\omega_{\rm conf}$ *is closed* as a L^2-valued 2-form, when L^2 comes equipped with the linear connection ∇^{L^2} induced by ∇^L. Then, the 2-form $d\theta$ is equal (up to the sign) to the curvature 2-form of ∇^L, so that the l.c.K. condition just means that the connection ∇^L is *flat*.

In the case when M is compact and b_1 is even it is well-known that any l.c.K. Hermitian structure is actually *globally* conformal to a Kähler Hermitian structure [25]. Conversely, it is also well-known (but much more difficult to prove, see for example [20], [10], [2]) that any compact complex surface (M, J) with even first Betti number admits a Kähler metric; moreover, many explicit examples of Kähler Hermitian structures are known, many of them provided by the complex algebraic geometry.

The situation when M is compact with *odd* first Betti number is quite different: it is still unknown whether there exist compact complex surfaces with b_1 odd *not* admitting a l.c.K. metric, but, on the other hand, only few examples have been constructed by now in an explicit way: apart from the case of Hopf surfaces considered in the present paper, these are essentially the l.c.K. metrics constructed by F. Tricerri on some classes of *Inoue surfaces* [22] and examples appearing in [5], [6], [26].

A *Hopf surface* is a compact complex surface whose universal covering is $W = \mathbb{C}^2 - \{(0,0)\}$. More precisely, *primary* Hopf surfaces have their fundamental group isomorphic to $\mathbb{Z}$, generated by the transformation γ defined by

$$x = (u, v) \mapsto (\alpha u + \lambda v^m, \beta v), \tag{2}$$

for any $x = (u, v) \in \mathbb{C}^2 - \{(0,0)\}$; here, m is some integer and α, β, λ are complex numbers such that

$$|\alpha| \geq |\beta| > 1, \tag{3}$$

and

$$(\alpha - \beta^m)\lambda = 0. \tag{4}$$

All primary Hopf surfaces are diffeomorphic to the product $S^3 \times S^1$ and any Hopf surface is finitely covered by a primary one, [13], [14].

Following [11], the primary Hopf surfaces fall into two disjoint classes, according to their *Kähler rank*:

– The *Hopf surfaces of class* 1, whose fundamental group is generated by a transformation γ as above for which $\lambda = 0$ and α, β are any two complex numbers satisfying (3); the corresponding Hopf surface will be denoted by $M_{\alpha,\beta}$.

– The *Hopf surfaces of class* 0, corresponding to $\lambda \neq 0$ and $\alpha = \beta^m$ for some positive integer m; the corresponding Hopf surface will be denoted by $\tilde{M}_{\beta,m,\lambda}$; as observed in [11], for any β, m and any two λ, μ in $\mathbb{C}^*$, $\tilde{M}_{\beta,m,\lambda}$ and $\tilde{M}_{\beta,m,\mu}$ are isomorphic as complex surfaces [11, Proposition 60].

For all Hopf surfaces, the complex structure induced by the natural complex structure of $\mathbb{C}^2 - \{(0,0)\}$ is denoted by J; a l.c.K. metric is always understood with respect to J.

The class 1 contains the subclass of $M_{\alpha,\beta}$'s such that $|\alpha| = |\beta|$. For these special Hopf manifolds, a l.c.K. metric is easily constructed as follows. Let ρ be the *distance function to the origin*, i.e. the (smooth) positive real function on W defined by $\rho(x) = \sqrt{|u|^2 + |v|^2}$ for any $x = (u,v)$ in $W = \mathbb{C}^2 - \{(0,0)\}$. Then, $\frac{1}{4} dd^c \rho^2$ is the Kähler form of the natural flat Hermitian metric (here and henceforth, the operator d^c, acting on functions, is defined by $d^c f(X) = -df(JX)$ so that $dd^c f = 2i\partial\bar{\partial} f$). The 2-form $\frac{1}{4\rho^2} dd^c \rho^2$ clearly descends as the Kähler form of a well-defined, obviously l.c.K., Hermitian metric $g_{\alpha,\beta}$ on the Hopf surface $M_{\alpha,\beta}$. It is then easy to check that the Lee form is parallel.

The main goal of this paper is to prove

THEOREM 1. — *Each primary Hopf surface admits a l.c.K. metric. Moreover, each primary Hopf surface of class* 1 *admits a l.c.K. metric with parallel (non-zero) Lee form.*

An explicit construction of a l.c.K. metric with parallel Lee form on each (primary) Hopf surfaces of class 1, as well as an explicit description of the corresponding Sasakian geometry, are given in Sections 2 and 3 (see, in particular, Proposition 1, Corollary 1, Proposition 3 and Remark 6).

Previous attempts to write (non globally defined) l.c.K. metrics on Hopf surfaces appear in [19].

The *existence* of l.c.K. Hermitian metrics on $M_{\alpha,\beta}$ for $|\alpha|$ and $|\beta|$ different, but close to each other, has been proved by C. LeBrun [17]. The argument goes as follows. First, notice that the line bundle L of any Hopf surface $M_{\alpha,\beta}$ is naturally identified to the real line bundle $(W \times \mathbb{R})/\mathbb{Z}$, where the action of $\mathbb{Z}$ on $W \times \mathbb{R}$ is described by $1.((u,v),a) = ((\alpha u, \beta v), |\alpha|^{\frac{1}{2}} |\beta|^{\frac{1}{2}} a)$ (observe that $|\alpha|^{\frac{1}{2}} |\beta|^{\frac{1}{2}}$ is equal to $|\det \gamma_*|^{\frac{1}{4}}$, where γ_* is the differential of γ). The line bundle so defined admits a natural flat connection which coincides with the Weyl derivative ∇^L; in particular the pullback of (L, ∇^L) on W coincides with the trivial line bundle $W \times \mathbb{R}$ equipped with the trivial connection. Finally, the 2-form $\frac{1}{4} dd^c \rho^2$ on W descends on M as a L^2-valued 2-form on $M_{\alpha,\beta}$, which is obviously closed with respect to ∇^L. Then, a deformation argument using the identification $L = (W \times \mathbb{R})/\mathbb{Z}$

(see Section 4) shows that $M_{\alpha,\beta}$ still admits a l.c.K. Hermitian metric for $|\alpha| \neq |\beta|$, provided that $|\alpha|$ and $|\beta|$ are close enough) to each other [17].

In Section 2, following a suggestion in [17], we give an explicit formulation for these l.c.K. metrics and show that the same formulation actually provides a l.c.K. Hermitian metric on *any* (primary) Hopf surface of class 1, see Proposition 1 and Corollary 1.

It then appears that all the l.c.K. Hermitian metrics obtained in this way have a parallel, non-zero, Lee form, and are related to a very simple class of Sasakian structures on the sphere S^3, of which a precise description is given in Section 3 (Proposition 3 and Remark 6).

Finally, LeBrun's argument still applies to prove the second statement of Theorem 1, see Section 4.

Remark 1. — Theorem 1 gives no information as to the existence or the non-existence of (l.c.K.) Hermitian metric with parallel Lee form on Hopf surfaces of class 0. However, this question has been solved recently by F. Belgun [3], so that Theorem 1 can actually be completed by the following statement: *Hopf surfaces of class 0 admit no (l.c.K.) Hermitian metrics with parallel Lee form.*

2. Construction of l.c.K. metrics on $M_{\alpha,\beta}$.

Fix any two complex numbers α, β satisfying (3) and let $\phi_{\alpha,\beta}$ the (smooth) function determined on W by

$$|u|^2|\alpha|^{-2\phi_{\alpha,\beta}(x)} + |v|^2|\beta|^{-2\phi_{\alpha,\beta}(x)} = 1, \tag{5}$$

for any $x = (u, v)$ in W. Notice that $\phi_{\alpha,\beta}$ is well-defined since, for x fixed, the function $t \mapsto |u|^2|\alpha|^t + |v|^2|\beta|^t$ is strictly increasing from 0 to $+\infty$.

Then, $\phi_{\alpha,\beta}$ satisfies the following equivariance property:

$$\phi_{\alpha,\beta}(\gamma \cdot x) = \phi_{\alpha,\beta}(x) + 1 \tag{6}$$

for any x in $\tilde{M}$.

Indeed, we have

$$\begin{aligned} 1 &= |\alpha u|^2|\alpha|^{-2\phi_{\alpha,\beta}(\gamma\cdot x)} + |\beta v|^2|\beta|^{-2\phi_{\alpha,\beta}(\gamma\cdot x)} \\ &= |u|^2|\alpha|^{2-2\phi_{\alpha,\beta}(\gamma\cdot x)} + |v|^2|\beta|^{2-2\phi_{\alpha,\beta}(\gamma\cdot x)} \\ &= |u|^2|\alpha|^{-2\phi_{\alpha,\beta}(x)} + |v|^2|\beta|^{-2\phi_{\alpha,\beta}(x)} \end{aligned}$$

for any x in W.

We then get a diffeomorphism from the corresponding Hopf surface (of class 1) $M_{\alpha,\beta}$ onto the product $S^3 \times S^1$ as follows (here S^3 denotes the 3-dimensional sphere, viewed as the unit sphere in $\mathbb{C}^2$, and S^1 denotes the circle, identified with the quotient $\mathbb{R}/\mathbb{Z}$). Let $\tilde{\psi}$ be the map from W to $S^3 \times S^1$ defined by

$$x = (u,v) \mapsto ((u\,\alpha^{-\phi_{\alpha,\beta}(x)}, v\,\beta^{-\phi_{\alpha,\beta}(x)}), \phi_{\alpha,\beta}(x) \bmod \mathbb{Z}). \tag{7}$$

Then, due to (6), $\tilde{\psi}$ is γ -invariant hence determines a map, ψ, from $M_{\alpha,\beta}$ to $S^3 \times S^1$, which is clearly a diffeomorphism; the inverse ψ^{-1} is given by

$$(z, t \bmod \mathbb{Z}) \mapsto [u = \alpha^t z_1, v = \beta^t z_2], \tag{8}$$

where $z = (z_1, z_2)$, $|z_1|^2 + |z_2|^2 = 1$, is a point of $S^3 \subset \mathbb{C}^2$ and $t \bmod \mathbb{Z}$ is an element of $S^1 = \mathbb{R}/\mathbb{Z}$; here, $[u,v]$ denotes the class of (u,v) mod. $\Gamma_{\alpha,\beta}$. Observe that the diffeomorphism ψ depends on the choice of an argument for α and for β, say $\mathfrak{Arg}\,\alpha$ and $\mathfrak{Arg}\,\beta$.

Remark 2. — For any choice of $\mathfrak{Arg}\alpha$ and $\mathfrak{Arg}\beta$, the above action of $\mathbb{Z}$ on $\tilde{M}$ is the restriction of an action of the (additive) group $\mathbb{R}$ defined by

$$t \cdot (u,v) = (\alpha^t u, \beta^t v), \tag{9}$$

for any t in $\mathbb{R}$. Then, $\phi_{\alpha,\beta}$ can be described as follows: for any $x = (u,v)$ in $\tilde{M}$, $\phi_{\alpha,\beta}(x)$ is the unique element of $\mathbb{R}$ such that $(-\phi_{\alpha,\beta}(x)) \cdot x$ belongs to the unit sphere of $\mathbb{C}^2$.

We denote by $\Phi_{\alpha,\beta}$ the real positive function on W defined by $\Phi_{\alpha,\beta} = e^{(k_1+k_2)\phi_{\alpha,\beta}}$; alternatively, $\Phi_{\alpha,\beta}$ is determined by

$$\rho_1^2\,\Phi_{\alpha,\beta}^{-\frac{2k_1}{k_1+k_2}} + \rho_2^2\,\Phi_{\alpha,\beta}^{-\frac{2k_2}{k_1+k_2}} = 1, \tag{10}$$

where k_1, k_2 are the (positive) real numbers given by

$$k_1 = \ln|\alpha|,\ k_2 = \ln|\beta|, \tag{11}$$

and ρ_1, ρ_2 are the functions on $\tilde{M}$ defined by

$$\rho_1(x) = |u|,\ \rho_2(x) = |v|. \tag{12}$$

In this notations (3) translates to

$$k_1 \geq k_2 > 0. \tag{13}$$

Then, by (6), $\Phi_{\alpha,\beta}$ satisfies the following equivariance property with respect to the action of γ:

$$\Phi_{\alpha,\beta}(\gamma \cdot x) = |\alpha||\beta| \cdot \Phi_{\alpha,\beta}(x). \tag{14}$$

In other words, $\Phi_{\alpha,\beta}$ descends on $M_{\alpha,\beta}$ as a (positive) section of L^2.

PROPOSITION 1. — *For any pair of complex numbers α, β satisfying (3), the real 2-form $\frac{1}{4}dd^c\Phi_{\alpha,\beta}$ is the Kähler form of a Hermitian metric on $\tilde{M}$.*

Proof. — For simplicity, $\Phi_{\alpha,\beta}$ will be denoted by Φ. Then, it follows readily from (10) that the differential of Φ is given by

$$d\Phi = \frac{1}{\Delta}\big(\Phi^{\frac{k_2-k_1}{k_1+k_2}}(ud\bar{u} + \bar{u}du) + \Phi^{\frac{k_1-k_2}{k_1+k_2}}(vd\bar{v} + \bar{v}dv)\big), \tag{15}$$

where Δ is the positive function defined by

$$\Delta = \frac{2k_1\,\rho_1^2\,\Phi^{\frac{-2k_1}{k_1+k_2}} + 2k_2\,\rho_2^2\,\Phi^{\frac{-2k_2}{k_1+k_2}}}{k_1+k_2}. \tag{16}$$

From (15), we infer:

$$\begin{aligned}\partial^2_{u,\bar{u}}\Phi = \frac{2\Phi^{\frac{k_2-k_1}{k_1+k_2}}}{\Delta^3(k_1+k_2)^2}\big(&k_1(k_1+k_2)\rho_1^4\Phi^{\frac{-4k_1}{k_1+k_2}} + 2k_2^2\rho_2^4\Phi^{\frac{-4k_2}{k_1+k_2}}\\ &+ k_2(k_1+3k_2)\rho_1^2\rho_2^2\Phi^{-2}\big);\end{aligned} \tag{17}$$

$$\begin{aligned}\partial^2_{v,\bar{v}}\Phi = \frac{2\Phi^{\frac{k_1-k_2}{k_1+k_2}}}{\Delta^3(k_1+k_2)^2}\big(&k_2(k_1+k_2)\rho_2^4\Phi^{\frac{-4k_2}{k_1+k_2}} + 2k_1^2\rho_1^4\Phi^{\frac{-4k_1}{k_1+k_2}}\\ &+ k_1(k_2+3k_1)\rho_1^2\rho_2^2\Phi^{-2}\big);\end{aligned} \tag{18}$$

$$\partial^2_{u,\bar{v}}\Phi = \frac{2\bar{u}v\Phi^{-1}}{\Delta^3(k_1+k_2)^2}(k_1-k_2)\big(k_1\rho_1^2\Phi^{\frac{-2k_1}{k_1+k_2}} - k_2\rho_2^2\Phi^{\frac{-2k_2}{k_1+k_2}}\big). \tag{19}$$

The claim is that the Hermitian matrix

$$A = \begin{pmatrix}\partial^2_{u,\bar{u}}\Phi & \partial^2_{u,\bar{v}}\Phi\\ \partial^2_{v,\bar{u}}\Phi & \partial^2_{v,\bar{v}}\Phi\end{pmatrix}$$

is *positive*; this in turn is equivalent to the fact that the trace and the determinant are positive.

By (17) and (18), $\partial^2_{u,\bar{u}}\Phi$ and $\partial^2_{v,\bar{v}}\Phi$ are both positive; it then remains to check that the determinant of A is positive. By a straightforward calculation, this determinant is equal to

$$\begin{aligned}\det A = \frac{8}{\Delta^6(k_1+k_2)^3}\big(&k_1^3\rho_1^8\Phi^{\frac{-8k_1}{k_1+k_2}} + k_2^3\rho_2^8\Phi^{\frac{-8k_2}{k_1+k_2}}\\ &+ 3k_1k_2(k_1+k_2)\rho_1^4\rho_2^4\Phi^{-4}\\ &+ k_1^2(k_1+3k_2)\rho_1^6\rho_2^2\Phi^{\frac{-6k_1-2k_2}{k_1+k_2}}\\ &+ k_2^2(k_2+3k_1)\rho_1^2\rho_2^6\Phi^{\frac{-2k_1-6k_2}{k_1+k_2}}\big)\\ = \frac{1}{\Delta^3},&\end{aligned} \tag{21}$$

which is obviously positive for any k_1, k_2 positive.

COROLLARY 1. — *For any pair of complex numbers α, β satisfying (3), the 2-form $\omega_{\alpha,\beta} = \dfrac{1}{4\Phi_{\alpha,\beta}} dd^c \Phi_{\alpha,\beta}$ is well-defined on $M_{\alpha,\beta}$ and is the Kähler form of a locally conformally Kähler structure, $(g_{\alpha,\beta}, J)$.*

Proof. — The fact that $\omega_{\alpha,\beta}$ is well-defined on $M_{\alpha,\beta}$ follows readily from (14). By the above proposition, $\omega_{\alpha,\beta}$ is the Kähler form of a, clearly l.c.K., Hermitian structure. □

Remark 3. — The 2-form $\frac{1}{4} dd^c \Phi_{\alpha,\beta}$ descends on $M_{\alpha,\beta}$ as a L^2-valued 2-form, equal to the conformal Kähler form of the l.c.K. Hermitian structure $(g_{\alpha,\beta}, J)$. In the special case that $|\alpha| = |\beta|$ or, equivalently, $k_1 = k_2$, we recover $\Phi_{\alpha,\beta} = \rho^2$.

In the general situation, we actually get a 1-parameter family of l.c.K. Hermitian structures obtained by choosing any positive real number ℓ and by considering, instead of $\dfrac{1}{4\Phi_{\alpha,\beta}} dd^c \Phi_{\alpha,\beta}$, the new Kähler form $\dfrac{1}{4\Phi^{\ell}_{\alpha,\beta}} dd^c \Phi^{\ell}_{\alpha,\beta}$. This amounts to replacing k_1, k_2 by ℓk_1, ℓk_2 in the above formulae.

This can be done in particular in the case $|\alpha| = |\beta|$; then, the Kähler form on W is equal to $\frac{1}{4} dd^c \rho^{2\ell}$ and the corresponding Riemannian metric g_ℓ can be described as follows:

$$g_\ell(X, X) = \ell\, \rho^{(2\ell-2)} (\ell |X_{\mathrm{rad}}|^2 + |X^{\perp}_{\mathrm{rad}}|^2), \tag{22}$$

with the following notation: For any vector X at the point x of W, X_{rad} denotes the *radial component* of X, i.e. the orthogonal projection of X on the complex line $\mathbb{C} \cdot x$ (viewed as a real 2-plane), and $X^{\perp}_{\mathrm{rad}}$ denotes the *transversal component* of X, i.e. the orthogonal projection of X on the orthogonal complex line $(\mathbb{C} \cdot x)^{\perp}$ (here, *orthogonal* means orthogonal with respect to the natural flat metric of $\mathbb{C}^2$).

The Lee form $\theta_{\alpha,\beta}$ of the Hermitian structure $(g_{\alpha,\beta}, J)$ is clearly equal to $\frac{1}{2} \Phi^{-1}_{\alpha,\beta}\, d\Phi_{\alpha,\beta}$.

Let $V_{\alpha,\beta}$ denote the dual vector field of $\theta_{\alpha,\beta}$ with respect to $g_{\alpha,\beta}$, the so-called *Lee vector field.*

A direct computation using (17), (18), (19) shows that the pull-back vector field of $V_{\alpha,\beta}$ on W, still denoted by $V_{\alpha,\beta}$, is expressed by

$$V_{\alpha,\beta}(x) = \left(\frac{2k_1}{k_1+k_2}u, \frac{2k_2}{k_1+k_2}v\right). \tag{23}$$

Again, a direct, but lengthy, computation shows that $V_{\alpha,\beta}$ is of norm 1 with respect to $g_{\alpha,\beta}$ and is parallel with respect to the Levi-Civita connection of $g_{\alpha,\beta}$. These facts will however become more easily apparent in the framework of the next section.

3. Associated Sasakian structures.

3.1. Three-dimensional Sasakian structures.

We begin this section with some general considerations concerning three-dimensional Sasakian structures (see e.g. [4]) for more information).

A *Sasakian structure* on some oriented, three-dimensional smooth manifold N is a pair (g, Z), where g is a Riemannian metric and Z a unit Killing vector field with respect to g, such that

$$D^g Z = *Z \; ; \tag{24}$$

here, D^g is the Levi-Civita connection of g, $*$ is the Hodge operator determined by the metric and the chosen orientation and $*Z$ is viewed as a skew-symmetric operator, also denoted by I; we thus have $I(Z) = 0$ and the restriction of I to $Q := Z^{\perp}$ coincides with the uniquely defined complex structure compatible with the metric and the induced orientation.

The distribution Q constitutes a *contact structure* and the Riemannian dual 1-form of Z with respect to g, η, is a *contact 1-form* for Q.

Notice that (3) implies

$$g(X,Y) = \frac{1}{2} d\eta(X, IY), \tag{25}$$

for any sections X, Y of Q.

In general, for any contact structure Q and any choice of a contact 1-form η, the corresponding *Reeb vector field* is the vector field V determined by the two conditions: $\eta(V) = 1$, $i_V d\eta = 0$. In the case of a Sasakian structure as above, the Reeb vector field of the contact structure Q with respect to the contact 1-form η is clearly equal to Z.

We denote by R^g the curvature tensor of D^g. Since Z is a Killing vector field, it satisfies the *Kostant identity*: $D^g_X(D^g Z) = R^g_{X,Z}$ [16]. We thus get

$$R^g_{Z,Y} = Z \wedge Y, \tag{26}$$

for any vector field Y. In particular, the sectional curvature of g is equal to 1 for any 2-plane containing Z.

Since N is three-dimensional, R^g is entirely determined by the Ricci tensor Ric^g and it is easy to deduce from (26) that Z is an eigenvector field for Ric^g (viewed as a symmetric operator) with respect to the constant eigenvalue 2, whereas Q is an eigen-subbundle of Ric^g with respect to the (in general non-constant) eigenvalue $\frac{\mathrm{Scal}^g}{2} - 1$, where Scal^g denotes the scalar curvature of g; Ric^g can thus be written as follows:

$$\mathrm{Ric}^g = \Big(\frac{\mathrm{Scal}^g}{2} - 1\Big)\, g + \Big(3 - \frac{\mathrm{Scal}^g}{2}\Big)\, \eta \otimes \eta. \tag{27}$$

The Levi-Civita connection D^g can be computed by using the well-known 6-terms formula, see e.g. [12]; it is given by the following table, where X denotes any *unit* section of Q, and Y any vector field on N:

$$D^g_Y Z = IY, \tag{28}$$

$$\begin{aligned} D^g_Y X = &\big((g([Z,X],IX) - 1)\, \eta(Y) - g([X,IX],Y)\big)\, IX \\ &+ g(Y,IX)\, Z. \end{aligned} \tag{29}$$

It follows that the sectional curvature restricted to Q, $K^g(Q)$, is given by

$$\begin{aligned} K^g(Q) = &1 - 2g([Z,X],IX) - g([X,IX],[X,IX]) \\ &+ X \cdot g([X,IX],IX) - IX \cdot g([X,IX],X), \end{aligned} \tag{30}$$

for any *unit* section, X, of Q. Then, $\mathrm{Scal}^g = 2(2 + K^g(Q))$ is immediately deduced from (30).

Remark 4. — The Levi-Civita connection D^g does not preserve the sub-bundle Q, but induces a linear connection ∇ on Q by orthogonal projection

$$\nabla_Y X = \big((g([Z,X],IX) - 1)\, \eta(Y) - g([X,IX],Y)\big)\, IX, \tag{31}$$

for any unit section X of Q and for any vector field Y on N. This connection is clearly I-linear and preserves the metric g, i.e. is a Hermitian connection when Q is viewed as a Hermitian complex line bundle over N. The (real) connection 1-form of ∇ with respect to X, viewed as a (unit) *gauge* of the Hermitian line bundle Q, is then the real 1-form ζ defined by

$$\zeta = (g([Z, X], IX) - 1)\,\eta - [X, IX]^\flat, \tag{32}$$

where $[X, IX]^\flat$ denotes the dual 1-form of the vector field $[X, IX]$. Then, (30) can be written as follows:

$$\begin{aligned} K(Q) &= -d\zeta(X, IX) - 1 \\ &= \Omega^\nabla(X, IX) - 1, \end{aligned} \tag{33}$$

where $\Omega^\nabla = -d\zeta$ is the (real) curvature form of ∇.

3.2. Sasakian versus Hermitian geometry.

We here describe the well-known correspondence between three-dimensional Sasakian manifolds and l.c.K. Hermitian complex surfaces with parallel unit Lee form, [25].

First, start from a three-dimensional Sasakian manifold (N, g, Z) as above and consider the product manifold $M = N \times \mathbb{R}$; let M be equipped with the product Riemannian metric, still denoted by g, of g and the standard metric of the factor $\mathbb{R}$, and with the almost complex structure J defined as follows:

$$J_{|Q} = I_{|Q}, \; JZ = T, \tag{34}$$

where $T := \partial/\partial t$ is the vector field determined by the natural parameter, t, of the factor $\mathbb{R}$. Let again D^g denote the Levi-Civita connection of g on M. Then, it follows from (24) that

$$D^g_U J = dt \wedge JU - \eta \wedge U, \tag{35}$$

for any vector field U on M. This implies that J is integrable and that the Lee form θ of the Hermitian structure (J, g) is the 1-form dt, see e.g. [23] or [1]. In particular, θ is D^g-parallel, of norm 1.

This construction can be compactified in the following manner. Let σ be any Sasakian transformation of (N, g, Z), i.e. any (direct) diffeomorphism of N preserving g and Z, and choose a positive real number ℓ; then the (Riemannian) *suspension* $M_{\sigma,\ell}$ of σ over the circle of length ℓ is obtained by identifying $N \times \{0\}$ and $N \times \{1\}$ via σ in the product $N \times [0, \ell]$ of N by the closed segment $[0, \ell]$.

The natural projection π from $M_{\sigma,\ell}$ to the circle $S^1_\ell = \mathbb{R}/\mathbb{Z} \cdot \ell$ is thus a Riemannian submersion and the natural vector field d/dt on S^1_ℓ admits a natural unit lift, $\tilde{T}$ on $M_{\sigma,\ell}$, orthogonal to the fibers of π, whose flow at time 1 coincides with σ. Applying this construction to the Sasakian three-manifold (N, g, Z) for any σ and any ℓ we eventually get a Hermitian structure with parallel (unit) Lee form on $M_{\sigma,\ell}$ by putting $JZ = \tilde{T}$.

Conversely, let (M, J, g) be a Hermitian complex surface. Let θ and V denote the Lee form and the Lee vector field. Since J is integrable, we have

$$D^g_U J = \theta \wedge JU + J\theta \wedge U, \tag{36}$$

for any vector field U on M (as usual, the RHS of (36) has to be considered as a skew-symmetric operator). If, moreover, θ, hence also V, are D^g-parallel, then the metric g splits locally as a Riemannian product $N \times \mathbb{R}$, where (N, g_N) is a three-dimensional Riemannian manifold and $V = d/dt$. By rescaling g if necessary, we can assume that V is of norm 1, as well as the vector field $Z := JV$; now, Z can be viewed as a vector field on N and (36) directly implies (24), showing that (g_N, Z) is a Sasakian structure on N.

3.3. Deformation of Sasakian structures.

Start from any Sasakian structure (g, Z) on N, fix any real positive function f on N and consider the new contact 1-form $\eta_f = \frac{1}{f}\eta$. Denote by Z_f the Reeb vector field corresponding to the same contact structure Q and the contact 1-form η_f; then,

$$Z_f = f\, Z + Z_f^Q, \tag{37}$$

where Z_f^Q is a section of Q, uniquely determined by the identity

$$df + i_{Z_f^Q} d\eta = 0. \tag{38}$$

It follows that

$$Z_f^Q = \frac{1}{2} I(df_{|Q})^\sharp, \tag{39}$$

where $df_{|Q}$ denotes the restriction of df to Q and $df^\sharp_{|Q}$ the section of Q dual to $df_{|Q}$ with respect to the the restriction of g to Q.

Let g_f be the Riemannian metric on N defined as follows:

1. Z_f is of norm 1 w.r.t g_f;

2. Q and Z_f are orthogonal with respect to g_f;

3. $g_f(X,Y) = \frac{1}{2}d\eta_f(X,IY) = \frac{1}{f}\,g(X,Y)$, for any sections X,Y of Q.

We shall refer to the metric g_f as *the metric obtained by deforming the Sasakian structure* (g,Z) *by the function* f, see [21].

Notice that η_f is the dual 1-form of Z_f with respect to g_f.

Now we have

PROPOSITION 2. — *The pair* (g_f, Z_f) *is a Sasakian structure on* N *if and only if the following condition is satisfied:*

$$\mathrm{Hess}^{g} f(X,Y) = \mathrm{Hess}^{g} f(IX,IY), \tag{40}$$

for any sections X,Y *of* Q*, where* $\mathrm{Hess}^{g} f = D^g df$ *denotes the Hessian of* f *with respect to* g*; equivalently, the restriction of* $\mathrm{Hess}^{g} f$ *to* Q *is a multiple of the restriction of* g.

Proof. — Let D^{g_f} be the Levi-Civita connection of g_f. We first show that $D^{g_f}_{Z_f}Z_f = 0$. Indeed, $g_f(D^{g_f}_{Z_f}Z_f, Z_f) = 0$, since Z_f is of norm 1 with respect to g_f, and, for any section X of Q, we have $g_f(D^{g_f}_{Z_f}Z_f, X) = g_f([X,Z_f],Z_f) = \eta_f([X,Z_f]) = -d\eta_f(X,Z_f) = 0$. Then, for any sections X,Y of Q, we have

$$\begin{aligned} g_f(D^{g_f}_X Z_f, Y) = &-\frac{1}{2}\eta_f([X,Y]) \\ &+\frac{1}{2}\big(Z_f\cdot g_f(X,Y) - g_f([Z_f,X],Y) - g_f([Z_f,Y],X)\big). \end{aligned}$$

This shows that the pair (g_f, Z_f) is Sasakian if and only if Z_f is Killing with respect to the induced metric g_f; moreover, in the present case that Z is already a Killing vector field with respect to g, Z_f is a Killing vector field with respect to g_f if and only if

$$g(D^g_X Z^Q_f, Y) + g(X, D^g_Y Z^Q_f) = df(Z)\,g(X,Y), \tag{41}$$

holds for any sections X,Y of Q. We then have

$$\begin{aligned} g(D^g_X Z^Q_f, Y) &= \frac{1}{2}X\cdot g(I(df_{|Q})^\sharp, Y) - \frac{1}{2}g(I(df_{|Q})^\sharp, D^g_X Y) \\ &= -\frac{1}{2}X\cdot df(IY) + \frac{1}{2}df(ID^g_X Y) \\ &= -\frac{1}{2}\mathrm{Hess}^{g} f(X,IY) - \frac{1}{2}df((D^g_X I)Y) \\ &= -\frac{1}{2}\mathrm{Hess}^{g} f(X,IY) + \frac{1}{2}df(Z)\,g(X,Y), \end{aligned}$$

which shows that (41) is true if and only (40) is satisfied. The last statement comes from Q being of rank 2 (notice that, except for this last statement, the argument holds in any dimension). □

By using (30), we get the following formulation for the scalar curvature of g_f:

$$\widetilde{\mathrm{Scal}^{g_f}} = 2\Big(3 + 4(f-1) + 4\mathrm{Hess}^g f(X,X) - 3\,\frac{((df(X))^2 + (df(JX))^2)}{f}\Big), \tag{42}$$

for any unit section X of Q.

3.4. Sasakian structures attached to $M_{\alpha,\beta}$.

Recall that the Hopf surface $M_{\alpha,\beta}$ as a manifold has been identified to the product $S^3 \times S^1$ by $\psi : M_{\alpha,\beta} \mapsto S^3 \times S^1$, defined by (7) and its inverse $\psi^{-1} : S^3 \times S^1 \mapsto M_{\alpha,\beta}$ described by (8).

We adopt the following notations. The sphere S^3 is realized as the set of elements of $\mathbb{C}^2$ of norm 1: a generic element of S^3 is denoted by $z = (z_1, z_2)$, where z_1, z_2 are complex numbers such that $|z_1|^2 + |z_2|^2 = 1$. Accordingly, a generic vector X of S^3 at z is identified to a pair of complex numbers $(X_1 = \dot{z_1}, X_2 = \dot{z_2})$ satisfying $\mathfrak{Re}(X_1\overline{z_1} + X_2\overline{z_2}) = 0$ ($\mathfrak{Re}$ and $\mathfrak{Im}$ denote respectively the real and imaginary part of a complex number). We denote by Z the vector field on S^3 generated by the natural action of S^1, so that

$$Z = (iz_1, iz_2).$$

We denote by $Q := Z^\perp$ the rank 2 vector sub-bundle of TS^3 orthogonal to Z with respect to the standard metric, g, of S^3 (of constant sectional curvature $+1$). The natural complex structure of $Q = Z^\perp$ is denoted by i. We denote by E, iE the generators of Q defined by

$$E = (\overline{z_2}, -\overline{z_1}),\ iE = (i\overline{z_2}, -i\overline{z_1}).$$

For any complex number μ, μE stands for the *real* vector field $\mathfrak{Re}\mu E + \mathfrak{Im}\mu\, iE$. The three (real) vector fields Z, E, iE are (unit) Killing vector fields with respect to g and generate the (real) Lie algebra of left-invariant vector fields of S^3, when S^3 is identified to the Lie group $Sp(1)$ of unit quaternions, via the usual identification $\mathbb{H} = \mathbb{C} \oplus j\,\mathbb{C}$. Their brackets are given by

$$[Z, E] = -2iE,\ [E, iE] = -2Z,\ [iE, Z] = -2E. \tag{43}$$

Also recall that, if D^g denotes the Levi-Civita connection of g, we have

$$D^g_Z Z = 0,\ D^g_X Z = iX, \tag{44}$$

for any X in Q, i.e. the pair (g, Z) is a *Sasakian structure*, called the *canonical* Sasakian structure of S^3. The corresponding contact 1-form is denoted by η: $\eta(X) = g(Z, X)$, for any vector field X on S^3.

The vector fields Z, E, iE will be also considered as vector fields on $S^3 \times S^1$ (with the same notation). As for the factor $S^1 = \mathbb{R}/\mathbb{Z}$, we denote by t the natural parameter of $\mathbb{R}$ and by T the vector field $\partial/\partial t$, also considered as a vector field on $S^3 \times S^1$.

For later convenience, we consider the complex function, F, on $S^3 \times S^1$ defined by

$$\begin{aligned} F(z) &= \ln\alpha|z_1|^2 + \ln\beta|z_2|^2 \\ &= k_1|z_1|^2 + k_2|z_2|^2 + i\left(\mathfrak{Arg}\,\alpha\,|z_1|^2 + \mathfrak{Arg}\,\beta\,|z_2|^2\right). \end{aligned} \tag{45}$$

Viewed as functions on $M_{\alpha,\beta}$, $|z_1|^2$, $|z_2|^2$ and $\mathfrak{Re}F$ are respectively equal to $\rho_1^2 \Phi^{\frac{-2k_1}{k_1+k_2}}$, $\rho_2^2 \Phi^{\frac{-2k_2}{k_1+k_2}}$ and $\dfrac{(k_1+k_2)}{2}\Delta$.

The image of any vector field $U = (U_1, U_2)$ of $M_{\alpha,\beta}$ by the differential ψ_* of ψ is of the form (X, aT), where $X = (X_1, X_2)$ is tangent to S^3 and a is a real function on $S^3 \times S^1$. It is easily checked that

$$a = \frac{\mathfrak{Re}(U_1(x)\,\overline{u}\,|\alpha|^{-2t} + U_2(x)\,\overline{v}\,|\beta|^{-2t})}{\mathfrak{Re}F}, \tag{46}$$

$$X_1 = \alpha^{-t}U_1 - a\,\ln\alpha\, z_1,\ X_2 = \beta^{-t}U_2 - a\,\ln\beta\, z_2. \tag{47}$$

Finally, the vector field X can be written uniquely as $bZ + \mu E$, where b is a real function and μ a complex function on $S^3 \times S^1$, given by

$$b = -i\,(X_1\overline{z_1} + X_2\overline{z_2}),\ \mu = X_1\, z_2 - X_2\, z_1. \tag{48}$$

We conclude that the complex structure J of $M_{\alpha,\beta}$, transported on $S^3 \times S^1$ by ψ, is described by the following table:

$$\begin{aligned} JT &= \frac{1}{\mathfrak{Re}F}\left(-\mathfrak{Im}F\,T + |F|^2\,Z + i\overline{F}(\ln\alpha - \ln\beta)\,z_1 z_2\,E\right), \\ JZ &= \frac{1}{\mathfrak{Re}F}\left(-T + \mathfrak{Im}F\,Z + (\ln\alpha - \ln\beta)\,z_1 z_2\,E\right), \\ JE &= iE. \end{aligned} \tag{49}$$

The Kähler form $\omega_{\alpha,\beta}$, transported on $S^3 \times S^1$ by ψ, is given by

$$\begin{aligned} \omega_{\alpha,\beta} &= \frac{1}{4}e^{-(k_1+k_2)t}\,dd^c e^{(k_1+k_2)t} \\ &= \frac{(k_1+k_2)}{4}dd^c t + \frac{(k_1+k_2)^2}{4}dt \wedge d^c t, \end{aligned} \tag{50}$$

where d^c refers to the operator J defined by (49).

From (49), we obtain

$$dt(T) = 1,\ dt(Z) = dt(E) = dt(iE) = 0, \tag{51}$$

$$d^ct(T) = \frac{\mathfrak{Im}F}{\mathfrak{Re}F},\ d^ct(Z) = \frac{1}{\mathfrak{Re}F},\ d^ct(E) = d^ct(iE) = 0, \tag{52}$$

hence also, the following table for $\omega_{\alpha,\beta}$

$$\begin{aligned}
\omega_{\alpha,\beta}(T,Z) &= \frac{(k_1+k_2)^2}{4\mathfrak{Re}F},\\
\omega_{\alpha,\beta}(T,\lambda E) &= \frac{(k_1+k_2)}{2(\mathfrak{Re}F)^2}\,\mathfrak{Re}(\overline{\lambda}z_1z_2)\,(k_1\mathfrak{Arg}\beta - k_2\mathfrak{Arg}\alpha),\\
\omega_{\alpha,\beta}(Z,\lambda E) &= \frac{(k_1+k_2)}{2(\mathfrak{Re}F)^2}\,(k_1-k_2)\,\mathfrak{Re}(\overline{\lambda}z_1z_2),\\
\omega_{\alpha,\beta}(\lambda E,\mu E) &= -\frac{(k_1+k_2)}{2\mathfrak{Re}F}\,\mathfrak{Im}(\lambda\overline{\mu}),
\end{aligned} \tag{53}$$

for any complex numbers λ, μ. We finally derive the following table for the metric $g_{\alpha,\beta} = \omega_{\alpha,\beta}(\cdot, J\cdot)$:

(54)

$$\begin{aligned}
g_{\alpha,\beta}(T,T) &= \frac{(k_1+k_2)^2|F|^2}{4(\mathfrak{Re}F)^2} + \frac{(k_1+k_2)(k_1\mathfrak{Arg}\beta - k_2\mathfrak{Arg}\alpha)^2}{2(\mathfrak{Re}F)^3}|z_1|^2|z_2|^2,\\
g_{\alpha,\beta}(T,Z) &= \frac{(k_1+k_2)^2\mathfrak{Im}H}{4(\mathfrak{Re}F)^2} + \frac{(k_1^2-k_2^2)(k_1\mathfrak{Arg}\beta - k_2\mathfrak{Arg}\alpha)}{2(\mathfrak{Re}F)^2}|z_1|^2|z_2|^2,\\
g_{\alpha,\beta}(T,\lambda E) &= \frac{(k_1+k_2)(k_1\mathfrak{Arg}\beta - k_2\mathfrak{Arg}\alpha)}{2(\mathfrak{Re}F)^2}\mathfrak{Im}(\overline{\lambda}z_1z_2),
\end{aligned}$$

$$\begin{aligned}
g_{\alpha,\beta}(Z,Z) &= \frac{(k_1+k_2)^2}{4(\mathfrak{Re}F)^2} + \frac{(k_1+k_2)(k_1-k_2)^2}{2(\mathfrak{Re}F)^3}\,|z_1|^2|z_2|^2,\\
g_{\alpha,\beta}(E,E) &= g_{\alpha,\beta}(iE,iE) = \frac{(k_1+k_2)}{2\mathfrak{Re}F}\\
g_{\alpha,\beta}(Z,\lambda E) &= \frac{(k_1^2-k_2^2)}{2(\mathfrak{Re}F)^2}\,\mathfrak{Im}(\overline{\lambda}z_1z_2),\\
g_{\alpha,\beta}(E,iE) &= 0.
\end{aligned} \tag{55}$$

From (23), we infer that the Lee vector field $V_{\alpha,\beta}$, viewed as a vector field on $S^3 \times S^1$ via ψ, is written as:

$$V_{\alpha,\beta} = \frac{2}{(k_1+k_2)}\big(T - \mathfrak{Im}F\,Z - (\mathfrak{Arg}\,\alpha - \mathfrak{Arg}\,\beta)\,i\,z_1z_2\,E\big). \tag{56}$$

The vector field $JV_{\alpha,\beta}$ is thus equal to

$$JV_{\alpha,\beta} = \frac{2}{(k_1+k_2)} \mathfrak{Re} F\, Z + (k_1 - k_2) i z_1 z_2\, E). \tag{57}$$

In particular, $JV_{\alpha,\beta}$ is independent of t and is tangent to the factor S^3, hence can ve viewed as a vector field on S^3; as such, it will be denoted by Z_Δ.

We then have

$$\begin{aligned} Z_\Delta &= (\frac{2k_1}{k_1+k_2} i z_1, \frac{2k_2}{k_1+k_2} i z_2) \\ &= \frac{2}{(k_1+k_2)} (\mathfrak{Re} F\, Z + (k_1 - k_2)\, i z_1 z_2\, E) \\ &= Z + \frac{2(k_1-k_2)}{(k_1+k_2)} Z_R, \end{aligned} \tag{58}$$

where Z_R is the vector field on S^3 defined by $Z_R = (iz_1, -iz_2)$. It can be seen that Z_R is a *right-invariant* (unit) Killing vector field for the standard metric g of S^3. In particular, Z_Δ is itself a Killing vector field with respect to g.

We observe that the restriction of $g_{\alpha,\beta}$ on each fiber of the natural fibration $\pi : S^3 \times S^1 \to S^1$ is independent of t, hence can be considered as a Riemannian metric on the sphere S^3; this metric is denoted by g_Δ.

PROPOSITION 3. — *The pair (g_Δ, Z_Δ) is a Sasakian structure on S^3, actually coincides with the Sasakian structure obtained by deforming the canonical Sasakian structure (g, Z) of S^3 by the function Δ defined by*

$$\Delta(z) = \frac{2}{(k_1+k_2)} \mathfrak{Re} F = \frac{2k_1|z_1|^2 + 2k_2|z_2|^2}{k_1+k_2}. \tag{59}$$

Proof. — By using (55) we check that Z_Δ is of norm 1 and is orthogonal to Q with respect to the metric g_Δ. Then, with respect to the triple Z_Δ, E, iE, g_Δ is described by the following table:

$$\begin{aligned} &g_\Delta(Z_\Delta, Z_\Delta) = 1, \\ &g_\Delta(Z_\Delta, E) = g_\Delta(Z_\Delta, F) = g_\Delta(E, F) = 0, \\ &g_\Delta(E, E) = g_\Delta(iE, iE) = \frac{1}{\Delta}. \end{aligned} \tag{60}$$

Now, the vector field Z_Δ can be written

$$Z_\Delta = \Delta\, Z - \frac{2(k_1-k_2)}{(k_1+k_2)} \mathfrak{Re}(z_1 z_2)\, E + \frac{2(k_1-k_2)}{(k_1+k_2)} \mathfrak{Im}(z_1 z_2)\, F. \tag{61}$$

On the other hand, we clearly have

$$d\Delta(E) = \frac{4(k_1 - k_2)}{(k_1 + k_2)}\mathfrak{Re}(z_1 z_2),\ d\Delta(F) = \frac{4(k_1 - k_2)}{(k_1 + k_2)}\mathfrak{Im}(z_1 z_2). \tag{62}$$

These prove that Z_Δ is the Reeb vector field of the contact structure Q with respect to the contact 1-form $\frac{\eta}{\Delta}$. By (60), the metric g_Δ coincides with the Riemannian metric determined by the Reeb vector field Z_Δ. It remains to check that Δ satisfies the condition of Proposition 2, which is clear. □

COROLLARY 2. — *For any complex numbers α, β satisfying (3), the Lee form of the l.c.K. Hermitian structure $(g_{\alpha,\beta}, J)$ is parallel with respect to $D^{g_{\alpha,\beta}}$.*

Proof. — As already observed in Section 3.2, (24) together with (36) imply that the Lee vector field $V_{\alpha,\beta}$, hence also the Lee form $\theta_{\alpha,\beta}$, is parallel with respect to $g_{\alpha,\beta}$. □

Remark 5. — By (42) and the above proposition, we infer that the scalar curvature $\mathrm{Scal}^{g_{\alpha,\beta}}$, which is also equal to the scalar curvature Scal^{g_Δ} of g_Δ, is given by

$$\mathrm{Scal}^{g_{\alpha,\beta}} = 6\left(1 - 4\frac{(k_1 - k_2)}{(k_1 + k_2)}\frac{(k_1|z_1|^2 - k_2|z_2|^2)}{(k_1|z_1|^2 + k_2|z_2|^2)}\right). \tag{63}$$

In particular, $\mathrm{Scal}^{g_{\alpha,\beta}}$ is not constant, except in the case that $k_1 = k_2$, *i.e.* $|\alpha| = |\beta|$.

Remark 6. — It follows readily from (23) that the flow $\Psi^{V_{\alpha,\beta}}$ of $V_{\alpha,\beta}$ on $S^3 \times S^1$ is given by

$$\begin{aligned}\Psi_s^{V_{\alpha,\beta}}((z,t)) = \Big(&\big(e^{-i\frac{2s}{(k_1+k_2)}\mathfrak{Arg}\,\alpha}\cdot z_1, e^{-i\frac{2s}{(k_1+k_2)}\mathfrak{Arg}\,\beta}\cdot z_2\big),\\ &t + \frac{2s}{(k_1+k_2)} \bmod \mathbb{Z}\Big).\end{aligned} \tag{64}$$

This flow preserves the fibration π and induces an isometry with respect to g_Δ from each fiber to the corresponding target fiber. In particular, after one rotation over S^1, this isometry is the isometry $\sigma_{\alpha,\beta}$ from S^3 to itself defined by

$$\sigma_{\alpha,\beta}((z,t)) = \Big(\big(e^{-i\mathfrak{Arg}\,\alpha}\cdot z_1, e^{-i\mathfrak{Arg}\,\beta}\cdot z_2\big), t\Big). \tag{65}$$

Finally, the l.c.K. metric $g_{\alpha,\beta}$ on the Hopf surface $M_{\alpha,\beta}$ is obtained by the following procedure (see [9] for the case $k_1 = k_2$):

1. Equip the sphere S^3 with the Riemannian metric g_Δ obtained by deforming the canonical Sasakian structure (g, Z) by the function Δ defined by (59) (see Section 3.3).

2. Realize $(M_{\alpha,\beta}, g_{\alpha,\beta})$ as the suspension of the isometry $\sigma_{\alpha,\beta}$ defined by (65) over the circle of length $\frac{(k_1+k_2)}{2}$ (see Section 3.2).

Proof of Theorem 1.

The first statement has been proved in the preceding sections, see in particular Proposition 1 and Proposition 3.

In order to prove the second statement, *i.e.* the *existence* of l.c.K. metrics on all Hopf surfaces of class 0, we use a specific deformation argument due to C. LeBrun [17]. Here are details. Fix any complex number β such that $|\beta| > 1$ and any positive integer m. Consider the three-dimensional complex manifold $\mathcal{M}$ defined as the quotient of $\mathbb{C} \times (\mathbb{C}^2 - \{(0,0)\})$ by the group $\tilde{\Gamma}_{\beta,m} \equiv \mathbb{Z}$ generated by the transformation $\tilde{\gamma}_{\beta,m} : (\lambda, (u,v)) \mapsto (\lambda, (\beta^m u + \lambda v^m, \beta v))$. Let p be the natural projection from $\mathcal{M}$ onto $\mathbb{C}$ which assigns λ to the class of $(\lambda, (u,v))$. Then, p is a holomorphic fibration whose fiber at $\lambda = 0$ is the Hopf surface of class 1 $M_{\beta^m,\beta}$ whereas fibers at $\lambda \neq 0$ are Hopf surfaces of class 0, all isomorphic to each other as recalled in the first section.

The bundle of scalars of weight 1 on $\mathcal{M}$ (see Section 1) is naturally identified to the quotient of the product bundle $\mathbb{C} \times (\mathbb{C}^2 - \{(0,0)\}) \times \mathbb{R}$ by $\tilde{\gamma}_{\beta,m} \equiv \mathbb{Z}$ acting by $1.(\lambda, (u,v), a) = (\lambda, (\beta^m u + \lambda v^m, \beta v), |\beta|^{\frac{m+1}{2}} a)$. (Notice that $|\beta|^{\frac{m+1}{2}}$ is equal to $|\det(\tilde{\gamma}_{\beta,m}) * |^{\frac{1}{4}}$.) Let $\mathcal{L}$ denote this bundle.

As already observed, the function $\Phi_{\beta^m,\beta}$ introduced in Section 2 can be considered as a section of $\mathcal{L}^2$ over $p^{-1}(0) = M_{\beta^m,\beta}$. It extends to a smooth section, $\tilde{\Phi}$ of $\mathcal{L}^2$ on $\mathcal{U}$ for some neighbourhood $\mathcal{U}$ of 0 in $\mathbb{C}$. For any λ in $\mathcal{U}$, let $\tilde{\Phi}_\lambda$ be the restriction of $\tilde{\Phi}$ to the fiber $p^{-1}(\lambda)$, also viewed as a function on $W = \mathbb{C}^2 - \{(0,0)\}$. Then, $\tilde{\Phi}_0$ is equal to $\Phi_{\beta^m,\beta}$. By Proposition 1, $\frac{1}{4} dd^c \Phi_{\beta^m,\beta}$ is a Kähler form on $\tilde{M}$; by continuity, the same is true for $\frac{1}{4} dd^c \tilde{\Phi}_\lambda$, so that $\frac{1}{4\tilde{\Phi}_\lambda} dd^c \tilde{\Phi}_\lambda$ is the Kähler form of a l.c.K. metric on $p^{-1}(\lambda)$. We thus get a l.c.K. metric on $\tilde{M}_{\beta,m,\lambda}$ for any λ in $\mathcal{U}$, hence for for any λ in $\mathbb{C}$. By varying β and m, we eventually get a l.c.K. metric for each primary Hopf surface of class 0.

Remark 7. — Note that the above deformation argument is specific to the Hopf surface. A general stability theorem for l.c.K. structures, as in the Kähler case, [15], is still lacking in the literature.

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