FINITENESS CONDITIONS FOR HOPF SUPERALGEBRAS

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Abstract. We show that as in the Hopf algebra case, the space of left integrals on a Hopf superalgebra A is nonzero if and only if A is co-Frobenius as a coalgebra. Our proof uses bosonization and a general result which is of independent interest: if C is a comodule coalgebra over a finite dimensional Hopf algebra H, then the smash coproduct $C \rtimes H$ is (quasi)co-Frobenius if and only if so is C.
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1. Introduction and preliminaries

The theory of integrals for Hopf algebras was initiated by Larson and Sweedler in the 1960's, and continued with Sullivan's proof of the uniqueness of integrals in early 1970's. Integrals have been of great use in understanding the structure of Hopf algebras. Their study was in close relationship to finiteness properties of the underlying coalgebra structure of the Hopf algebra. More precisely, a Hopf algebra has non-zero integrals if and only if it is co-Frobenius as a coalgebra. In the case of Hopf superalgebras, a systematic study of integrals was initiated in [6] and continued in [7], where it has been related to integration on Lie supergroups. It is natural to ask whether the existence of integrals is also related to finiteness coalgebra properties in the Hopf superalgebra case. In this paper we answer this in

Theorem A. *Let A be a Hopf superalgebra. Then the following are equivalent.*
(i) A has non-zero left integrals.
(ii) A is left co-Frobenius as a coalgebra.
(iii) A is left quasi-co-Frobenius as a coalgebra.
(iv) A is a left semiperfect coalgebra.
Moreover, these conditions are also equivalent to their right hand side versions.

We give the definitions of (quasi)-co-Frobenius coalgebras and semiperfect coalgebras in Section 2, and of integrals for Hopf superalgebras in Section 3.

A standard method in the study of Hopf superalgebras is the reduction to usual Hopf algebras by applying a process of bosonization. More precisely if

A is a Hopf superalgebra, then there is an action and a coaction of the group Hopf algebra $k\mathbb{Z}_2$ of the cyclic group of order 2 on A, and then the tensor product $A \otimes k\mathbb{Z}_2$ has an algebra structure (a smash product) and a coalgebra structure (a smash coproduct), which are compatible and make $A \otimes k\mathbb{Z}_2$ a Hopf algebra. In this way, integrals on A are investigated in connection to integrals on its bosonization in [6], leading to a uniqueness theorem for integrals on Hopf superalgebras. Our approach for proving Theorem A is similar, so we need to understand how finiteness properties transfer between A and its bosonization. In fact we prove a much more general transfer result, which is of interest by itself.

Theorem B. *Let C be a right comodule coalgebra over a finite dimensional Hopf algebra H, and let $C \rtimes H$ be the associated smash coproduct. The following assertions hold.*
(1) *$C \rtimes H$ is left quasi-co-Frobenius if and only if so is C.*
(2) *$C \rtimes H$ is left co-Frobenius if and only if so is C.*

We give the necessary definitions and the proof of Theorem B in Section 2.

2. Co-Frobenius smash coproducts

Let H be a finite dimensional Hopf algebra over a field k. A right H-comodule coalgebra is a coalgebra C with comultiplication denoted by $c \mapsto \sum c_1 \otimes c_2$ for $c \in C$, and counit ε, such that C is also a right H-comodule with coaction denoted by $c \mapsto \sum c_{(0)} \otimes c_{(1)} \in C \otimes H$, and the following compatibility conditions hold

$$\sum c_{(0)1} \otimes c_{(0)2} \otimes c_{(1)} = \sum c_{(0)} \otimes c_{2(0)} \otimes c_{1(1)}c_{2(1)}$$

$$\sum \varepsilon(c_{(0)})c_{(1)} = \varepsilon(c)1_H$$

for any $c \in C$.

If C is a right H-comodule coalgebra, the smash coproduct $C \rtimes H$ associated with a right H-comodule coalgebra C is the space $C \otimes H$, with $c \otimes h$ denoted by $c \rtimes h$ for any $c \in C, h \in H$, endowed with a coalgebra structure whose comultiplication is

$$\Delta(c \rtimes h) = \sum (c_1 \rtimes c_{2(1)}h_2) \otimes (c_{2(0)} \rtimes h_1)$$

and counit is given by $\varepsilon(c \rtimes h) = \varepsilon(c)\varepsilon(h)$.

The map $\pi : C \rtimes H \to C$, $\pi(c \rtimes h) = \varepsilon(h)c$ is a coalgebra morphism, thus $C \rtimes H$ is a left C-comodule and a right C-comodule via π. Moreover, we have that

- $C \rtimes H \simeq C^{\dim(H)}$ as left C-comodules; $\varphi : C \rtimes H \to C \otimes H$, $\varphi(c \rtimes h) = c \otimes h$ is such an isomorphism.
- $C \rtimes H \simeq C^{\dim(H)}$ as right C-comodules; $\psi : C \rtimes H \to H \otimes C$, $\psi(c \rtimes h) = \sum c_{(1)}h \otimes c_{(0)}$ is such an isomorphism, with inverse $\psi^{-1}(h \otimes c) =$

$\sum c_{(0)} \rtimes S(c_{(1)})h$, where S is the antipode of H.

Now we recall from [2, Chapter 3] the definitions of several coalgebra properties.

• A coalgebra C is called left co-Frobenius if C embeds into its dual C^* as a left C^*-module; if C is finite dimensional, C is left co-Frobenius if and only if its dual algebra C^* is Frobenius.

• C is called left quasi-co-Frobenius if C embeds into a free left C^*-module, or equivalently, C is a projective right C-comodule.

• C is called left semiperfect if any finite dimensional left C-comodule has a projective cover, or equivalently, the injective envelope of any simple right C-comodule is finite dimensional, and furthermore equivalent to the rational part $Rat(C^*_{C^*})$ part of the right C^*-module C^* is a dense subset of C^* in the finite topology.

It is known that C left co-Frobenius $\Rightarrow$ C is left quasi-co-Frobenius $\Rightarrow$ C is left semiperfect. There are also versions to the right of these coalgebra properties, and note that neither of them is left-right symmetric.

Proof of Theorem B. Denote $n = \dim(H)$.

(1) Assume that $C \rtimes H$ is left quasi-co-Frobenius. Then $C \rtimes H$ is a projective right $C \rtimes H$-comodule. Consider the pair of adjoint functors

$$\mathcal{M}^{C\rtimes H} \underset{-\square_C(C\rtimes H)}{\overset{F}{\rightleftarrows}} \mathcal{M}^C$$

where F is induced by $\pi : C \rtimes H \to C$; here $\square_C$ denotes the cotensor product over C, see [2, Section 2.3]. Since $C \rtimes H$ is a free left C-comodule, the functor $-\square_C(C \rtimes H)$ is exact, and then its left adjoint F takes projectives to projectives. In particular, $C \rtimes H$ is a projective right C-comodule, and then so is C since $C \rtimes H \simeq C^n$ in $\mathcal{M}^C$. This shows that C is left quasi-co-Frobenius.

Conversely, assume that C is left quasi-co-Frobenius. Then so is the $n \times n$-matrix coalgebra $M^c(n, C) = M^c(n, k) \otimes C$. By the duality theorem for coactions [3], we have $(C \rtimes H) \rtimes H^* \simeq M^c(n, C)$, so $C \rtimes H$ is left quasi-co-Frobenius by the first implication that we proved above, applied to the right H^*-comodule coalgebra $C \rtimes H$.

(2) Assume that $C \rtimes H$ is left co-Frobenius, so $C \rtimes H$ embeds into $(C \rtimes H)^*$ as a left $(C \rtimes H)^*$-module. Regard this as an embedding of left C^*-modules.

Now $C \rtimes H \simeq C^n$ as right C-comodules, or equivalently, as left C^*-modules. On the other hand, since $C \rtimes H \simeq C^n$ as left C-comodules, we get that $(C \rtimes H)^* \simeq (C^*)^n$ as left C^*-modules. Hence the embedding of $C \rtimes H$ into $(C \rtimes H)^*$ produces an embedding $C^n \hookrightarrow (C^*)^n$ as left C^*-modules; in fact $C^n \hookrightarrow Rat({}_{C^*}C^*)^n$.

The coradical C_0 of C is the sum of all simple subcoalgebras of C, so it can be written as a sum of minimal right coideals, as well as a sum of minimal left coideals. Write $C_0 = \oplus_{j\in J} T_j = \oplus_{i\in I} S_i$, where the T_j's are simple right C-comodules, and the S_i's are simple left C-comodules. Each isomorphism type occurs finitely many times among the T_j's (since isomorphic minimal right coideals occur only from the same simple subcoalgebra in a representation of C_0 as a direct sum of simple subcoalgebras). We have that

$$C = \oplus_{j\in J} E(T_j) = \oplus_{i\in I} E(S_i),$$

where $E(X)$ denotes an injective envelope of the right (left) coideal X inside C. By (1), C is left quasi-co-Frobenius, so it is also left semiperfect, showing that each $E(T_j)$ is finite dimensional. Moreover, by [1, Proposition 1.3]

$$Rat({}_{C^*}C^*) \subset \oplus_{j\in J} E(T_j)^* \subset Rat(C^*_{C^*}) \subset \oplus_{\in I} E(S_i)^* \subset C^*.$$

Since $C^n \hookrightarrow Rat({}_{C^*}C^*)^n$ as left C^*-modules, and we obtain that

$$\oplus_{j\in J} E(T_j)^n = C^n \hookrightarrow \oplus_{\in I} (E(S_i)^*)^n$$

as left C^*-modules. Note that since $E(T_j)$ is finite dimensional and injective as a right C-comodule, it is also injective as a left C^*-module.

Let $T_{j_1}, \dots, T_{j_h}$ be all the T_j's isomorphic to a certain simple right C-comodule. Then $E(T_{j_1})^n \oplus \dots \oplus E(T_{j_h})^n$ is an injective left C^*-module, and it embeds into $\oplus_{i\in I}(E(S_i)^*)^n$, so

$$E(T_{j_1})^n \oplus \dots \oplus E(T_{j_h})^n \oplus X = \oplus_{i\in I}(E(S_i)^*)^n$$

for some left C^*-module X. By [4, Lemma 1.4] (see also [1, Proposition 1.5]), the endomorphism ring of the left C^*-module $E(S_i)^*$ is local (thus $E(S_i)^*$ is indecomposable). By a lemma used in the proof of Azumaya theorem, we find distinct $i_1, \dots, i_h \in I$ such that $E(S_{i_v})^* \simeq E(T_{j_v})$ for any $1 \le v \le h$. Then each $E(S_{i_v})$ has finite dimension and $S_{i_1}, \dots, S_{i_h}$ are isomorphic, since $E(S_{i_v}) \simeq E(T_{j_v})^*$, so S_{i_v} is the socle of $E(T_{j_v})^*$. We obtain that

$$E(T_{j_1}) \oplus \dots \oplus E(T_{j_1}) \simeq E(S_{i_1})^* \oplus \dots \oplus E(S_{i_h})^* \tag{2.1}$$

and either side lies inside $Rat({}_{C^*}C^*)$. If we write the relation (2.1) for each isomorphism type of simple right C-comodule, and note that if T_j is not isomorphic to $T_{j'}$, $E(T_j) \simeq E(S_i)^*$ and $E(T_{j'}) \simeq E(S_{i'})^*$, we cannot have $S_i \simeq S_{i'}$, we obtain by taking the direct sum of all these relations that $C = \oplus_{j\in J} E(T_j) \hookrightarrow Rat({}_{C^*}C^*)$, showing that C is left co-Frobenius.

Conversely, if C is left co-Frobenius, then so is $M^c(n, C) = M^c(n, k) \otimes C$. Using as in (1) the duality theorem and the implication proved above, we get that $C \rtimes H$ is left co-Frobenius.

Remark 2.1. *The transfer of semiperfectness between a comodule coalgebra and the smash coproduct was discussed in* [1, Theorem 4.8 and Proposition

4.10], *where it was showed that if C is a left comodule coalgebra over the finite dimensional Hopf algebra H, then $C \rtimes H$ is left semiperfect if and only if so is C.*

3. Hopf superalgebras with integrals

Consider the category $\mathcal{M}^{k\mathbb{Z}_2}$ of right comodules over the group Hopf algebra $k\mathbb{Z}_2$, which is exactly the category of $\mathbb{Z}_2$-graded vector spaces. This is a monoidal category with the usual tensor product of $\mathbb{Z}_2$-graded vector spaces, and moreover, it is braided with the braiding $c_{V,W} : V \otimes W \to W \otimes V$ defined by $c_{V,W}(v \otimes w) = (-1)^{|v|\cdot|w|} w \otimes v$ for any $\mathbb{Z}_2$-graded vector spaces V and W, and any homogeneous elements $v \in V$, $w \in W$; here $|v|$ denotes the degree of the homogeneous element v.

A Hopf superalgebra is a Hopf algebra in the braided category $\mathcal{M}^{k\mathbb{Z}_2}$, i.e., it is a $\mathbb{Z}_2$-graded vector space A which is the same time a $\mathbb{Z}_2$-graded algebra and a $\mathbb{Z}_2$-graded coalgebra, such that the comultiplication Δ and the counit ε satisfy the relations

$$\begin{aligned} \Delta(ab) &= \sum (-1)^{|a_2|\cdot|b_1|} a_1 b_1 \otimes a_2 b_2 \\ \varepsilon(ab) &= \varepsilon(a)\varepsilon(b) \end{aligned}$$

for any homogeneous $a, b \in A$; we consider representations of $\Delta(a) = \sum a_1 \otimes a_2$ with homogeneous a_1's and a_2's.

The following hold for the antipode S of A

$$S(ba) = (-1)^{|a|\cdot|b|} S(a)S(b) \tag{3.1}$$

$$\Delta(S(a)) = \sum (-1)^{|a_1|\cdot|a_2|} S(a_2) \otimes S(a_1) \tag{3.2}$$

for any homogeneous elements $a, b \in A$.

A left integral on a Hopf superalgebra A is an element $t \in A^*$ such that $\sum t(a_2)a_1 = T(a)1$ for any $a \in A$. It is proved in [6, Theorem 1] that the dimension of the space of left integrals on A is at most 1, and if it is 1, the space of left integrals is spanned by a homogeneous element of A^*.

If A is a Hopf superalgebra, then the right $k\mathbb{Z}_2$-coaction on A induces a left action of the dual Hopf algebra $(k\mathbb{Z}_2)^*$. Let $p_{\hat{0}}, p_{\hat{1}}$ be the basis of $(k\mathbb{Z}_2)^*$ dual to the basis $\hat{0}, \hat{1}$ of $k\mathbb{Z}_2$. As k has characteristic $\neq 2$, it contains a primitive root of unity of order 2, so there is a Hopf algebra isomorphism $(k\mathbb{Z}_2)^* \simeq k\mathbb{Z}_2$, which takes $p_{\hat{0}}$ to $\frac{1}{2}(\hat{0} + \hat{1})$ (regarded as a sum of grouplike elements, not as a sum in $\mathbb{Z}_2$), and $p_{\hat{1}}$ to $\frac{1}{2}(\hat{0} - \hat{1})$. Hence $k\mathbb{Z}_2$ acts from the left, as well from the right as it is commutative, on A. This right action works as follows: if $a = a_{\hat{0}} + a_{\hat{1}} \in A$, then $a \cdot \hat{0} = a$ and $a \cdot \hat{1} = a_{\hat{0}} - a_{\hat{1}}$.

With these structures, A is a Hopf algebra in the category of (right) Yetter-Drinfeld modules over $k\mathbb{Z}_2$, and then we can consider its bosonization $A \rtimes k\mathbb{Z}_2$, which is a regular Hopf algebra with comultiplication given by the smash coproduct and multiplication given by the smash product.

At this point we explain that a fundamental theorem of Hopf modules holds in the braided category $\mathcal{M}^{k\mathbb{Z}_2}$. If A is a Hopf superalgebra, an A-Hopf supermodule is a $\mathbb{Z}_2$-graded vector space which is a right A-module and a right A-comodule such that

$$\sum (ma)_{(0)} \otimes (ma)_{(1)} = \sum (-1)^{|m_{(1)}|\cdot|a_1|} m_{(0)}a_1 \otimes m_{(1)}a_2$$

for any homogeneous $m \in M$, $a \in A$; here $m \mapsto \sum m_{(0)} \otimes m_{(1)}$ denotes the A-coaction on M. We denote $M^{\mathrm{co}\,A} = \{m \in M | \sum m_{(0)} \otimes m_{(1)} = m \otimes 1\}$, the subspace of A-coinvariants of M. By an adaptation of the proof of the fundamental theorem of Hopf modules in the usual Hopf algebra case, using equations (3.1), (3.2), we obtain the following.

Proposition 3.1. *Let M be an A-Hopf supermodule. Then $\varphi : M^{\mathrm{co}\,A} \otimes A \to M$ defined by $\varphi(m \otimes a) = ma$, is an isomorphism of A-Hopf supermodules.*

Proof. Let $g : M \to M$ be defined by $g(m) = \sum m_{(0)}S(m_{(1)})$. Then

$$\begin{aligned}
\sum g(m)_{(0)} \otimes g(m)_{(1)} &= \sum (-1)^{|m_{(0)(1)}|\cdot|S(m_{(1)})_1|} m_{(0)(0)}S(m_{(1)})_1 \\
&\qquad \otimes m_{(0)(1)}S(m_{(1)})_2 \\
&= \sum (-1)^{|m_{(1)}|\cdot|S(m_{(2)2})|}(-1)^{|m_{(2)1}|\cdot|m_{(2)2}|} m_{(0)}S(m_{(2)2}) \\
&\qquad \otimes m_{(1)}S(m_{(2)1}) \\
&= \sum (-1)^{|m_{(1)}|\cdot|m_{(3)}|}(-1)^{|m_{(2)}|\cdot|m_{(3)}|} m_{(0)}S(m_{(3)}) \\
&\qquad \otimes m_{(1)}S(m_{(2)}) \\
&= \sum (-1)^{(|m_{(1)}|+|m_{(2)}|)\cdot|m_{(3)}|} m_{(0)}S(m_{(3)}) \\
&\qquad \otimes m_{(1)}S(m_{(2)}) \\
&= (-1)^{|m_{(1)}|\cdot|m_{(2)}|} m_{(0)}S(m_{(2)}) \otimes \varepsilon(m_{(1)})1 \\
&= \sum m_{(0)}S(m_{(2)})\varepsilon(m_{(1)}) \otimes 1 \\
&= m_{(0)}S(m_{(1)}) \otimes 1 \\
&= g(m) \otimes 1
\end{aligned}$$

which shows that $g(m) \in M^{\mathrm{co}\,A}$. Then we can define $\psi : M \to M^{\mathrm{co}\,A} \otimes A$ by $\psi(m) = \sum g(m_{(0)} \otimes m_{(1)}$, and as for usual Hopf algebras one shows that ϕ and ψ are inverse each other. □

Following the same approach as for Hopf algebras, we have the following after direct, but tedious computations.

Proposition 3.2. *Let A be a Hopf superalgebra. The following assertions hold.*

(i) The rational part $Rat({}_{A^*}A^*)$ of the left A^*-module A^* is a right A-comodule. Moreover, $Rat({}_{A^*}A^*)^{\mathrm{co}\,A}$ is the space of left integrals on A.
(ii) A^* is a right A-module with action given by

$$a^* \leftharpoonup a = (-1)^{|a^*|\cdot|a|}(S(a) \rightharpoonup a^*),$$

for any homogeneous elements $a \in A$, $a^* \in A^*$, where $\rightharpoonup$ denotes the usual left A-action on A^*.
(iii) $Rat({}_{A^*}A^*)$ is a submodule of the right A-module A^* in (ii).
(iv) $Rat({}_{A^*}A^*)$ is an A-Hopf supermodule with the coaction of (i) and action of (iii).

Corollary 3.3. *Let A be a Hopf superalgebra. Then there exist non-zero left integrals on A if and only if $Rat({}_{A^*}A^*) \neq 0$.*

Proof. It follows from Proposition 3.2 and Proposition 3.1. □

Now we are in the position to prove Theorem A. For reasons which will be clear when we develop the proof, it is useful to rewrite the statement in a more complete way as follows.

Theorem 3.4. *Let A be a Hopf superalgebra. Then the following are equivalent.*
(i) A has non-zero left integrals.
(ii) A is left co-Frobenius as a coalgebra.
(iii) A is left quasi-co-Frobenius as a coalgebra.
(iv) A is a left semiperfect coalgebra.
(i') A has non-zero right integrals.
(ii') A is right co-Frobenius as a coalgebra.
(iii') A is right quasi-co-Frobenius as a coalgebra.
(iv') A is a right semiperfect coalgebra.

Proof. We first recall that if A is a usual Hopf algebra, the assertions in the theorem are equivalent by the Lin-Larson-Sweedler-Sullivan Theorem, see [2, Theorem 5.3.2].
(i)⇒(ii) If there is a non-zero left integral on A, it is showed in the proof of [6, Theorem 1] that the bosonization $A \rtimes k\mathbb{Z}_2$ of A has a non-zero left integral, so then it is left co-Frobenius. Then A is left co-Frobenius by Theorem B.
(ii)⇒(iii) and (iii)⇒(iv) are clear.
(iv)⇒(i') If A is left semiperfect, then $A_r^{*\mathrm{rat}} \neq 0$, so by the right hand side of Corollary 3.3 there are non-zero right integrals on A.
(i')⇒(ii')⇒(iii')⇒(iv') similar to (i)⇒(ii)⇒(iii)⇒(iv).
(iv')⇒(i) similar to (iv)⇒(i'). □

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References

[1] S. Dăscălescu, C. Năstăsescu, B. Toader, Doi-Hopf modules associated to comodule coalgebras, Comm. Algebra **41** (2013), 1854-1864.

[2] S. Dăscălescu, C. Năstăsescu, Ş. Raianu, *Hopf Algebras. An Introduction*, Pure and Applied Mathematics **235**, Marcel Dekker, Inc., New York-Basel, 2001.

[3] S. Dăscălescu, Ş. Raianu, Y. H. Zhang, Finite Hopf-Galois coextensions, crossed coproducts, and duality, J. Algebra **178** (1995), 400-413.

[4] M. C. Iovanov, Co-Frobenius coalgebras, J. Algebra **303** (2006), 146-153.

[5] S. Montgomery, *Hopf algebras and their actions on rings.* CBMS Regional Conf. Series in Math. No. 82, A. M. S., Providence, RI, 1993.

[6] M. Scheunert, R.B. Zhang, Invariant integration on classical and quantum Lie supergroups, J. Math. Phys. **42** (2001), 3871-3897.

[7] M. Scheunert, R.B. Zhang, Integration on Lie supergroups: a Hopf superalgebra approach, J. Algebra **292** (2005), 324-342.

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