PSEUDO V-HARMONIC MORPHISMS

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ABSTRACT. We introduce the notion of V-minimality, for V a smooth vector field on a Riemannian manifold. This is a natural extension of the classical notion of minimality. To emphasis the utility of this notion we present generalizations of some results from [1]. Specifically, we prove that a PHH submersion is V-harmonic if and only if it has minimal fibres and a PHH V-harmonic submersion pulls back complex submanifolds to V minimal submanifolds.

1. INTRODUCTION

Let (M, g) and (N, h) be compact Riemannian manifolds, V a smooth vector field on M and $\varphi : M \to N$ a smooth map. In [9],[5] the authors introduced the notion of V-harmonic map, see Definition 2.1 that naturally generalizes the classical notion of harmonic map. Unlike the latter, V-harmonicity is not defined via a variational problem, but rather by imposing the vanishing of a modified tension field, called the *V-tension field of φ*. If $V = 0$, the two notions coincide, more generally, the same is true if V is vertical.

In this generalized context, V-harmonic morphisms appear naturally, [9], and are directly connected to the minimality of the fibres, exactly as in the classical case, Theorem 2.6.

In the case of Kähler target manifolds, harmonic morphisms generalize to pseudo-harmonic morphisms [8]. Furthermore, if a natural extra-condition that the naturally almost complex structure on the horizontal distribution satisfies a Kähler-type condition, then the harmonicity implies the minimality of the fibres. These maps, called pseudo-horizontally homothetic (PHH) enjoy other geometric properties, [1].

The main goal of this paper is to provide a natural notion of V-minimality. As for V-harmonicity, minimality corresponds to the case $V = 0$. We connect this notion to PHH maps and their V-harmonicity.

The outline of the paper is as follows. Firstly, we recall the notions needed for our aim, V-harmonic maps, PHH and PHWC maps, and we briefly review some of their properties. We then define pseudo V-harmonic morphisms, Definition 3.1 and V-minimal submanifolds, Definition 3.4. The main results of the paper are Theorem 3.2, Theorem 3.5 and Theorem 3.6. Theorem 3.5 shows that for a PHH submersion, V-harmonicity

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is equivalent to having V-minimal fibres. In Theorem 3.6, we prove that a PHH V-harmonic submersion pulls back complex submanifolds to V-minimal submanifolds.

2. Preliminaries

2.1. V - Harmonic Maps and Morphisms.

We remind the reader of some basic facts on V-Harmonic Maps and Morphisms that we shall use in the sequel [9],[5].

Definition 2.1. ([5], [9]) Let (M^m, g) and (N^n, h) be two Riemannian manifolds, V a smooth vector field on M, and $\varphi : M \to N$ a smooth map. The map φ is called *V-harmonic* if satisfies:

$$\tau_V(\varphi) := \tau(\varphi) + d\varphi(V) = 0, \tag{1}$$

where $\tau(\varphi)$ is the tension field of the map φ. Since the differential $d\varphi$ of φ can be viewed as a section of the bundle $T^*M \otimes \varphi^{-1}TN$, $d\varphi(V)$ is a section of the bundle $\varphi^{-1}TN$. The tension $\tau_V(\varphi)$ is called the *V-tension field of φ*.

Remark 2.2. ([5], [9]) In particular,

1. For $V = 0$, the map φ is harmonic.
2. A smooth function $f : M \to \mathbf{R}$ is said to be *V-harmonic* if:

$$\Delta_V(f) := \Delta(f) + < V, \nabla f > = 0.$$

By taking the trace of the second fundamental form, we obtain the tension field of the map φ, $\tau(\varphi) = trace \nabla d\varphi$, which is a section of $\varphi^{-1}TN$. In local coordinates $(x_i)_{i=\overline{1,m}}$ on M and $(y_\alpha)_{\alpha=\overline{1,n}}$ on N, respectively, it has the following expression (see for example [4], [10]):

$$\tau(\varphi) = \sum_{\alpha=1}^{n} \tau(\varphi)^\alpha \frac{\partial}{\partial y_\alpha}, \tag{2}$$

where, denoting by ${}^M\Gamma_{ij}^k$ and ${}^N L_{\beta\gamma}^\alpha$ the Christoffel symbols of M and N, and $\varphi^\alpha = \varphi \circ y_\alpha$,

$$\tau(\varphi)^\alpha = \sum_{i,j=1}^{m} g^{ij} \left(\frac{\partial^2 \varphi^\alpha}{\partial x_i \partial x_j} - \sum_{k=1}^{m} {}^M\Gamma_{ij}^k \frac{\partial \varphi^\alpha}{\partial x_k} + \sum_{\beta,\gamma=1}^{n} {}^N L_{\beta\gamma}^\alpha \frac{\partial \varphi^\beta}{\partial x_i} \frac{\partial \varphi^\gamma}{\partial x_j} \right). \tag{3}$$

For V a smooth vector field on M, given in local coordinates on M by $V = \sum_{i=1}^{m} V_i \frac{\partial}{\partial x_i}$, the V-tension field of the map φ has the following expression in local coordinates:

$$\tau_V(\varphi) = \sum_{\alpha=1}^{n} \tau(\varphi)^\alpha \frac{\partial}{\partial y_\alpha} + \sum_{\alpha=1}^{n} \sum_{i=1}^{m} V_i \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial}{\partial y_\alpha}. \tag{4}$$

Using the properties of the second fundamental form and of the tension field of the composition of two maps, (see [4]), we have the following lemma:

Lemma 2.3. ([9]) *The V-tension field of the composition of two maps $\varphi : M \to N$ and $\psi : N \to P$ is given by:*

$$\tau_V(\psi \circ \varphi) = d\psi(\tau_V(\varphi)) + trace \nabla d\psi(d\varphi, d\varphi). \tag{5}$$

As in the case of harmonic maps, Zhao (Definition 1.2 in [9]) defined *V-harmonic morphisms* as maps between Riemannian manifolds, $\varphi : M \to N$, which pulls back local harmonic functions on N to local V-harmonic functions on M.

The following results characterise the harmonic morphisms.

Theorem 2.4. ([9]) *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. Then the following conditions are equivalent:*

1) *φ is a V-harmonic morphism;*
2) *φ is a horizontally weakly conformal V-harmonic map;*
3) *$\forall \psi : W \to P$ a smooth map from an open subset $W \subset N$ with $\varphi^{-1}(W) \neq \emptyset$, to a Riemann manifold P, we have: $\tau_V(\psi \circ \varphi) = \lambda^2 \tau(\psi)$, for some smooth function $\lambda^2 : M \to [0, \infty)$;*
4) *$\forall \psi : W \to P$ a harmonic map from an open subset $W \subset N$ with $\varphi^{-1}(W) \neq \emptyset$, to a Riemann manifold P, the map $\psi \circ \varphi$ is a V-harmonic map;*
5) *$\exists \lambda^2 : M \to [0, \infty)$ a smooth function such that: $\Delta_V(f \circ \varphi) = \lambda^2 \Delta f$, for any function f defined on an open subset W of N with $\varphi^{-1}(W) \neq \emptyset$.*

Corollary 2.5. ([9]) *For $\varphi : M \to N$ a V-harmonic morphism with dilation λ and $\psi : N \to P$ a harmonic morphism with dilation θ, the composition $\psi \circ \varphi$ is a V-harmonic morphism with dilation $\lambda(\theta \circ \varphi)$.*

Theorem 2.6. ([9]) *For a horizontally weakly conformal map $\varphi : M \to N$ with dilation λ, any two of the following conditions imply the third:*

1) *φ is a V-harmonic map (and so a V-harmonic morphisms);*
2) *$V + \nabla log \lambda^{2-n}$ is vertical at regular points;*
3) *the fibres of φ are minimal at regular points.*

2.2. **Pseudo Harmonic Morphisms. Pseudo-Horizontally Homothetic Maps.** The notion of harmonic morphisms can be generalized when the target manifold is endowed with a Kähler structure (see [8], [6]).

Let us consider a smooth map $\varphi : (M^m, g) \to (N^{2n}, J, h)$ from a Riemannian manifold to a Kähler manifold. The map φ is said to be a *pseudo-harmonic morphism* (PHM) if and only if it pulls back local holomorphic functions on N to local harmonic maps from M to $\mathbf{C}$.

For any $x \in M$, denote by $d\varphi_x^* : T_{\varphi(x)}N \to T_xM$ the adjoint map of the tangent linear map $d\varphi_x : T_xM \to T_{\varphi(x)}N$.

If X is a local section on the pull-back bundle $\varphi^{-1}TN$, then $d\varphi^*(X)$ is a local horizontal vector field on M.

Definition 2.7. ([8]) The map φ is called *pseudo-horizontally (weakly) conformal (shortening PHWC) at $x \in M$* if $[d\varphi_x \circ d\varphi_x^*, J] = 0$.

The map φ is called *pseudo-horizontally (weakly) conformal* if it is pseudo-horizontally (weakly) conformal at every point of M.

Then, *pseudo-harmonic morphism* can also be characterised as harmonic, pseudo-horizontally (weakly) conformal maps (see [8], [6]).

The local description of PHWC condition is given by the following (see [8]): let $(x_i)_{i=\overline{1,m}}$ be real local coordinates on M, $(z_\alpha)_{\alpha=\overline{1,n}}$ be complex local coordinates on N, and $\varphi^\alpha = z_\alpha \circ \varphi$, $\forall \alpha = 1, ..., n$. Then the PHWC condition for φ reads:

$$\sum_{i,j=1}^{m} g_M^{ij} \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_j} = 0 \tag{6}$$

for all $\alpha, \beta = 1, ..., n$.

A special class of pseudo-horizontally weakly conformal maps, called *pseudo-horizontally homothetic maps*.

Definition 2.8. ([1]) A map $\varphi : (M^m, g) \to (N^{2n}, J, h)$ is called *pseudo-horizontally homothetic at x* (PHH) if is PHWC at a point $x \in M$ and satisfy:

$$d\varphi_x \left((\nabla_v^M d\varphi^*(JY))_x \right) = J_{\varphi(x)} d\varphi_x \left((\nabla_v^M d\varphi^*(Y))_x \right), \tag{7}$$

for any horizontal tangent vector $v \in T_x M$ and any vector field Y locally defined on a neighbourhood of $\varphi(x)$.

If the map φ is PHWC, then φ is called *pseudo-horizontally homothetic* if and only if it is pseudo-horizontally homothetic at any point of M i.e. if and only if

$$d\varphi(\nabla_X^M d\varphi^*(JY)) = J d\varphi(\nabla_X^M d\varphi^*(Y)), \tag{8}$$

for any horizontal vector field X on M and any vector field Y on N.

The condition (8) is true for every horizontal vector field X on M and any section Y of $\varphi^{-1}TN$. This remark allows us to work with the larger space of sections in the pull back bundle $\varphi^{-1}TN$ instead of vector fields on N.

One of the basic properties of pseudo-horizontally homothetic maps (Proposition 3.3, [1]) shows that a PHH submersion is a harmonic map if and only if it has minimal fibres. Also, pseudo-horizontally homothetic maps are good tools to construct minimal submanifolds (Theorem 4.1, [1]).

3. Pseudo V-Harmonic Morphisms. V-minimal submanifolds

3.1. Pseudo V-Harmonic Morphisms.

Generalizing the class of harmonic maps and morphisms, respectively to V-harmonic maps and pseudo harmonic morphisms we obtain pseudo V-harmonic morphisms with a description similar to pseudo harmonic morphisms.

Definition 3.1. Let (M^m, g) be a Riemannian manifold of real dimension m, (N^{2n}, J, h) a Hermitian manifold of complex dimension n, $\varphi : M \to N$ a smooth map and V a smooth vector field on M. The map φ is called *pseudo V-harmonic morphism* (shortening PVHM) if φ is *V-harmonic* and *pseudo horizontally weakly conformal.*

The characterization of *pseudo harmonic morphism* given in [8] remain true in the general case of *V-harmonic* maps.

Theorem 3.2. *Let $\varphi : M \to N$ be a smooth map from a Riemannian manifold (M^m, g) to a Kähler one (N^{2n}, J, h) and V a smooth vector field on M. Then φ is pseudo V-harmonic morphism (PVHM) if and only if it pulls back local complex valued holomorphic functions on N to local V-harmonic functions on M.*

Proof. Let us consider $p \in M$ a given point, $\{x_i\}_{i=\overline{1,m}}$ local coordinates at p, $\{z_\alpha\}_{\alpha=\overline{1,n}}$ local complex coordinates at $\varphi(p) \in N$, and $\varphi^\alpha = z_\alpha \circ \varphi$. By ${}^M\Gamma^k_{ij}$ and ${}^N L^\alpha_{\beta\gamma}$ are denoted the Christoffel symbols on M and N, respectively. The vector field V, in local coordinates in M, reads: $V = \sum\limits_{i=1}^{m} V_i \frac{\partial}{\partial x_i}$.

Suppose that φ is $PVHM$ and $f : N \to \mathbf{C}$ is a local holomorphic function on N.

Using the V-tension field of composition of two maps (Lemma 2.3) and the V-harmonicity of the map φ (Definition 2.1):

$$\tau_V(f \circ \varphi) = df(\tau_V(\varphi)) + trace\nabla df(d\varphi, d\varphi) = trace\nabla df(d\varphi, d\varphi).$$

In order to prove that $\tau_V(f \circ \varphi) = 0$, since $d\varphi_p(\frac{\partial}{\partial x_i}(p)) = \sum\limits_{\alpha=1}^{n} \frac{\partial \varphi^\alpha}{\partial x_i}(p)\frac{\partial}{\partial z_\alpha}(\varphi(p))$, we compute at p:

$$\begin{aligned}
\nabla df\left(d\varphi\left(\tfrac{\partial}{\partial x_i}\right), d\varphi\left(\tfrac{\partial}{\partial x_j}\right)\right) &= \nabla df\left(\sum_{\alpha=1}^{n} \frac{\partial \varphi^\alpha}{\partial x_i}\frac{\partial}{\partial z_\alpha}, \sum_{\beta=1}^{n} \frac{\partial \varphi^\beta}{\partial x_j}\frac{\partial}{\partial z_\beta}\right) = \\
&= \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \nabla df\left(\frac{\partial \varphi^\alpha}{\partial x_i}\frac{\partial}{\partial z_\alpha}, \frac{\partial \varphi^\beta}{\partial x_j}\frac{\partial}{\partial z_\beta}\right) = \\
&= \sum_{\alpha,\beta=1}^{n} \frac{\partial \varphi^\alpha}{\partial x_i}\frac{\partial \varphi^\beta}{\partial x_j}\nabla df\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}\right)
\end{aligned} \tag{9}$$

To compute $\nabla df\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}\right)$, first let us remark that $f^{-1}T\mathbf{C} = N \times \mathbf{C}$ is the trivial vector bundle and the fibre $(f^{-1}T\mathbf{C})_{\varphi(p)} = \mathbf{C}$. The induced connection on the pull-back bundle $f^{-1}T\mathbf{C}$ is defined by:

$$\nabla^f_X \sigma := X(\sigma), \forall \sigma \in \Gamma(f^{-1}T\mathbf{C}), (\sigma : N \to \mathbf{C} \text{ is a map})$$

So,

(10)

$$\begin{aligned}
\nabla df\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}\right) &= \nabla^f_{\frac{\partial}{\partial z_\alpha}} df\left(\frac{\partial}{\partial z_\beta}\right) - df\left(\nabla^N_{\frac{\partial}{\partial z_\alpha}}\frac{\partial}{\partial z_\beta}\right) = \frac{\partial}{\partial z_\alpha}\left(\frac{\partial f}{\partial z_\beta}\right) - df\left(\nabla^N_{\frac{\partial}{\partial z_\alpha}}\frac{\partial}{\partial z_\beta}\right) = \\
&= \frac{\partial^2 f}{\partial z_\alpha \partial z_\beta} - df\left(\sum_{\gamma=1}^{n} {}^N L^\gamma_{\alpha\beta}\frac{\partial}{\partial z_\gamma}\right) = \frac{\partial^2 f}{\partial z_\alpha \partial z_\beta} - \sum_{\gamma=1}^{n} {}^N L^\gamma_{\alpha\beta} df\left(\frac{\partial}{\partial z_\gamma}\right) = \\
&= \frac{\partial^2 f}{\partial z_\alpha \partial z_\beta} - \sum_{\gamma=1}^{n} {}^N L^\gamma_{\alpha\beta}\frac{\partial f}{\partial z_\gamma}
\end{aligned}$$

From the equations (9) and (10) we obtain:

$$\nabla df\left(d\varphi\left(\frac{\partial}{\partial x_i}\right), d\varphi\left(\frac{\partial}{\partial x_j}\right)\right) = \sum_{\alpha,\beta=1}^{n} \frac{\partial \varphi^\alpha}{\partial x_i}\frac{\partial \varphi^\beta}{\partial x_j}\frac{\partial^2 f}{\partial z_\alpha \partial z_\beta} - \sum_{\alpha,\beta,\gamma=1}^{n} \frac{\partial \varphi^\alpha}{\partial x_i}\frac{\partial \varphi^\beta}{\partial x_j}{}^N L^\gamma_{\alpha\beta}\frac{\partial f}{\partial z_\gamma}$$

and

$$trace\nabla df\left(d\varphi\left(\frac{\partial}{\partial x_i}\right), d\varphi\left(\frac{\partial}{\partial x_j}\right)\right) =$$

(11) $$= \sum_{i,j=1}^{m} g^{ij}\left(\sum_{\alpha,\beta=1}^{n}\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j}\frac{\partial^2 f}{\partial z_\alpha\partial z_\beta} - \sum_{\alpha,\beta,\gamma=1}^{n}\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j}{}^N L^\gamma_{\alpha\beta}\frac{\partial f}{\partial z_\gamma}\right) =$$

$$= \sum_{\alpha,\beta=1}^{n}\left(\sum_{i,j=1}^{m} g^{ij}\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j}\right)\frac{\partial^2 f}{\partial z_\alpha\partial z_\beta} - \sum_{\alpha,\beta,\gamma=1}^{n}\left(\sum_{i,j=1}^{m} g^{ij}\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j}\right){}^N L^\gamma_{\alpha\beta}\frac{\partial f}{\partial z_\gamma}$$

As φ is PHWC (see (6)), the term $\sum_{i,j=1}^{m} g^{ij}\frac{\partial\varphi^s}{\partial x_i}\frac{\partial\varphi^k}{\partial x_j}$, in relation (11), vanishes, so

$$trace\nabla df\left(d\varphi\left(\frac{\partial}{\partial x_i}\right), d\varphi\left(\frac{\partial}{\partial x_j}\right)\right) = 0.$$

Conversely, consider $f : N \to \mathbf{C}$ a local complex holomorphic function, V a smooth vector field on M, and $\varphi : M \to N$ a smooth map, such that $\tau_V(f \circ \varphi) = 0$.

Applying the chain rule for V-harmonic maps (Lemma 2.3), we get:

(12) $$0 = df(\tau_V(\varphi)) + trace\nabla df(d\varphi, d\varphi)$$

We compute the above equality in local coordinates, using the local description of the V-tension field (4) and the equation (11):

(13) $$0 = \sum_{\alpha=1}^{n}\tau(\varphi)^\alpha\frac{\partial f}{\partial z_\alpha} + \sum_{\alpha=1}^{n}\sum_{i=1}^{m} V_i\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial f}{\partial z_\alpha} +$$
$$\sum_{\alpha,\beta=1}^{n}\left(\sum_{i,j=1}^{m} g^{ij}\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j}\right)\frac{\partial^2 f}{\partial z_\alpha\partial z_\beta} - \sum_{\alpha,\beta,\gamma=1}^{n}\left(\sum_{i,j=1}^{m} g^{ij}\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j}\right){}^N L^\gamma_{\alpha\beta}\frac{\partial f}{\partial z_\gamma}$$

Choosing in (13) particular holomorphic functions f, for example $f(z) = z_k, \forall k = \overline{1,n}$, and normal coordinates centred at p (respectively, at $\varphi(p)$), then $\frac{\partial^2 f}{\partial z_\alpha\partial z_\beta} = 0$, $\frac{\partial f}{\partial z_\gamma} = \delta_{k\gamma}$, and the Christoffel symbols of N vanish. Then,

$\tau_V(f \circ \varphi) = 0$ is equivalent to:

$$\tau(\varphi)^k + \sum_{i=1}^{m} V_i\frac{\partial\varphi^k}{\partial x_i} = 0, \forall k = \overline{1,n},$$

where the term $\tau(\varphi)^k + \sum_{i=1}^{m} V_i\frac{\partial\varphi^k}{\partial x_i} = 0$ is the k component of the $\tau_V(\varphi)$.

This proves that φ is a V-harmonic map. It follows that the equation (12) reduces to: $trace\nabla df(d\varphi, d\varphi) = 0$.

In this last equality, choosing $f(z) = z_\alpha z_\beta, \forall\alpha,\beta = \overline{1,n}$ and normal coordinates in M and N respectively, $\frac{\partial^2(z_\alpha z_\beta)}{\partial z_\alpha\partial z_\beta} = 1$, and ${}^N L^\gamma_{\alpha\beta} = 0$, implies

$$\sum_{i,j=1}^{m} g^{ij}\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j} = 0$$

which means that φ is PHWC. □

The following result gives a relation between V-harmonic morphisms, pseudo harmonic morphisms and pseudo V-harmonic morphisms.

Proposition 3.3. *Let (M, g) and (N, h) be two Riemannian manifolds, V a smooth vector field on M, (P, J, p) a Kähler manifold and $\psi : M \to N$ and $\varphi : N \to P$ two smooth maps. If ψ is V-harmonic morphism and φ is pseudo harmonic morphism (PHM), then $\varphi \circ \psi$ is pseudo V-harmonic morphism (PVHM).*

Proof. As φ is a pseudo harmonic morphism (see Section 1.2), for any $f : W \to \mathbf{C}$ a local complex valued holomorphic function defined in an open subset $W \subset P$, with $\varphi^{-1}(W) \subset N$ non-empty, $f \circ \varphi : \varphi^{-1}(W) \to \mathbf{C}$ is a local harmonic function on N.

Using Corollary 2.5 for the V-harmonic morphism ψ and the local harmonic function $f \circ \varphi : \varphi^{-1}(W) \to \mathbf{C}$, on N, we get $f \circ \varphi \circ \psi : \psi^{-1}(\varphi^{-1}(W)) \to \mathbf{C}$, a local V-harmonic function on M.

So, $\varphi \circ \psi : M \to P$ pulls-back local holomorphic functions on P to local V-harmonic functions on M. By Theorem 3.2, $\varphi \circ \psi$ is PVHM. □

3.2. **V-minimal submanifolds.** Let (M, g) be a Riemannian manifold, V a smooth vector field on M, and K a submanifold of M. For any $x \in M$, we have the orthogonal decomposition of the tangent bundle $T_xM = T_xK \oplus T_xK^\perp$, with respect to g_x. According to this decomposition, ${}^M\nabla_X Y = {}^K \nabla_X Y + A(X, Y), \forall X, Y \in \Gamma(TK)$. The symmetric bilinear map $A : TK \times TK \to TK^\perp$ is *the second fundamental form of the submanifold K*.

Definition 3.4. The submanifold K of M is called *V-minimal* if

$$trace(A) - V \in \Gamma(TK).$$

When K is the fibre of a map on M, the V-minimality condition translates to $trace(A) - V$ is a vertical vector.

Theorem 3.5. *Let $\varphi : (M^m, g) \to (N^{2n}, J, h), n \geq 2$, be a pseudo horizontally homothetic submersion (PHH). Then φ is V-harmonic if and only if φ has V-minimal fibres.*

Proof. Similar to Proposition 3.3, [1], we can choose in the pull-back bundle $\varphi^{-1}TN$ a local frame $\{e_1, e_2, \ldots, e_n, Je_1, Je_2, \ldots, Je_n\}$ such that

$$\{d\varphi^*(e_1), \ldots, d\varphi^*(e_n), d\varphi^*(Je_1), \ldots, d\varphi^*(Je_n)\}$$

is an orthogonal frame in the horizontal distribution: choose $e_1 \in \Gamma(\varphi^{-1}TN)$ a non-vanishing section and, using the PHWC property of φ and the fact that $d\varphi^*(e_1)$ is an horizontal vector, we get the orthogonality of $d\varphi^*(e_1)$ and $d\varphi^*(Je_1)$. At step k, take e_k orthogonal on both vectors $d\varphi \circ d\varphi^*(e_i)$ and $d\varphi \circ d\varphi^*(Je_i)$, $\forall i \leq k - 1$.

If we denote by $E_k = d\varphi^*(e_k), E'_k = d\varphi^*(Je_k)$, then we have an orthogonal frame in the horizontal distribution $\mathcal{H}(TM)$, $\{E_1, E_2, \ldots, E_n, E'_1, \ldots, E'_n\}$.

We pick up $\{u_1, u_2, \ldots, u_s\}$ an orthonormal basis for the vertical distribution $\mathcal{V}(TM)$.

By the PHWC property of φ, we observe that:

$$d\varphi(E'_i) = d\varphi(d\varphi^*(Je_i)) = J(d\varphi \circ d\varphi^*(e_i)) = Jd\varphi(E_i). \tag{14}$$

and

$$(15)\qquad \begin{aligned} g(E_i,E_i) &= g(d\varphi^*(e_i), d\varphi^*(e_i)) = h(e_i, (d\varphi\circ d\varphi^*)(e_i)) = \\ &= h(Je_i, J(d\varphi\circ d\varphi^*)(e_i)) = h(Je_i, (d\varphi\circ d\varphi^*)(Je_i)) = \\ &= g(d\varphi^*(Je_i), d\varphi^*(Je_i)) = g(E_i', E_i') \end{aligned}$$

Using the property of the induced connection ∇^φ on $\varphi^{-1}TN$ (see Lemma 1.16, [11]), (14) and (15), we compute:

$$\begin{aligned} \nabla^\varphi_{E_i'} d\varphi(E_i') = \nabla^\varphi_{E_i'} Jd\varphi(E_i) = J\nabla^\varphi_{E_i'} d\varphi(E_i) &= J(\nabla^\varphi_{E_i} d\varphi(E_i') + d\varphi(E_i', E_i)) = \\ &= \nabla^\varphi_{E_i} Jd\varphi(E_i') + Jd\varphi([E_i', E_i]) = \\ &= -\nabla^\varphi_{E_i} d\varphi(E_i) + Jd\varphi([E_i', E_i]) \end{aligned}$$

and

$$(16)\qquad \begin{aligned} \frac{1}{g(E_i,E_i)} Jd\varphi([E_i',E_i]) &= \frac{1}{g(E_i,E_i)}(\nabla^\varphi_{E_i'} d\varphi(E_i') + \nabla^\varphi_{E_i} d\varphi(E_i)) = \\ &= \frac{1}{g(E_i,E_i)}\nabla^\varphi_{E_i'} d\varphi(E_i') + \frac{1}{g(E_i,E_i)}\nabla^\varphi_{E_i} d\varphi(E_i) = \\ &= \frac{1}{g(E_i',E_i')}\nabla^\varphi_{E_i'} d\varphi(E_i') + \frac{1}{g(E_i,E_i)}\nabla^\varphi_{E_i} d\varphi(E_i) \end{aligned}$$

As φ is PHH, we get:

$$(17)\qquad \begin{aligned} d\varphi(\nabla^M_{E_i'} E_i') &= d\varphi(\nabla^M_{E_i'} d\varphi^*(Je_i)) \overset{\text{PHH}}{=} Jd\varphi(\nabla^M_{E_i'} d\varphi^*(e_i)) \\ &= Jd\varphi(\nabla^M_{E_i'} E_i) = Jd\varphi(\nabla^M_{E_i} E_i' + [E_i', E_i]) = \\ &= Jd\varphi(\nabla^M_{E_i} d\varphi^*(Je_i)) + Jd\varphi([E_i', E_i]) = \\ &= -d\varphi(\nabla^M_{E_i} E_i) + Jd\varphi([E_i', E_i]). \end{aligned}$$

so

$$(18)\qquad \frac{1}{g_M(E_i,E_i)} Jd\varphi([E_i',E_i]) = \frac{1}{g_M(E_i,E_i)} d\varphi(\nabla^M_{E_i} E_i) + \frac{1}{g_M(E_i',E_i')} d\varphi(\nabla^M_{E_i'} E_i')$$

Using equations (16) and (18) and the orthonormal basis of vertical vectors $\{u_i\}_{i=\overline{1,s}}$, the tension field of φ reads:

$$(19)\qquad \begin{aligned} \tau(\varphi) = &\sum_{i=1}^n \frac{1}{g_M(E_i,E_i)}(\nabla^\varphi_{E_i} d\varphi(E_i) - d\varphi(\nabla^M_{E_i} E_i)) + \\ &+ \sum_{i=1}^n \frac{1}{g_M(E_i',E_i')}(\nabla^\varphi_{E_i'} d\varphi(E_i') - d\varphi(\nabla^M_{E_i'} E_i')) + \\ &+ \sum_{j=1}^s \frac{1}{g_M(u_j,u_j)}(\nabla^\varphi_{u_j} d\varphi(u_j) - d\varphi(\nabla^M_{u_j} u_j)) = \\ &= -\sum_{j=1}^s d\varphi(\nabla^M_{u_j} u_j) = -d\varphi(\sum_{j=1}^s \nabla^M_{u_j} u_j) \end{aligned}$$

If φ is a V-harmonic map, then $\tau_V(\varphi) = 0$ and since $\tau(\varphi) = -d\varphi(\sum_{j=1}^s \nabla^M_{u_j} u_j)$ we get:

$$0 = -d\varphi(\sum_{j=1}^s \nabla^M_{u_j} u_j) + d\varphi(V)$$

which is equivalent with the V-minimality of the fibres.

Conversely, suppose φ has V-minimal fibres. As φ is a PHH submersions, in the above constructed frame $\{u_1, \ldots, u_s, E_1, \ldots, E_n, E'_1, \ldots, E'_n\}$, the V-tension field of φ reads:

$$\tau_V(\varphi) = -d\varphi(\sum_{j=1}^{s} \nabla^M_{u_j} u_j) + d\varphi(V)$$

The V-minimality of the fibres imply $d\varphi(trace(A) - V) = 0$.

Let us choose $y \in N$, and denote the fibre by $K = \varphi^{-1}(y)$.

Using (15),

$$trace(A) - V = \sum_{i=1}^{s} A(u_i, u_i) - V = \frac{1}{g_M(u_i,u_i)} \sum_{i=1}^{s} (\nabla^M_{u_i} u_i - \nabla^K_{u_i} u_i) - V$$

Since $\nabla^K_{u_i} u_i$ is vertical, $d\varphi(\nabla^K_{u_i} u_i) = 0$, and

$$d\varphi(trace(A) - V) = \sum_{i=1}^{s} d\varphi(\nabla^M_{u_i} u_i) - d\varphi(V),$$

so φ is V-harmonic. □

The construction of minimal submanifolds was done for horizontally homothetic harmonic morphisms (see [3]) and generalised for the case of pseudo-horizontally homothetic harmonic submersions (see [1]). Replacing harmonicity by V-harmonicity, a similar result can be proved.

Theorem 3.6. *Let (M^m, g) be a Riemannian manifold, (N^{2n}, J, h) be a Kähler manifold, V a smooth vector field on M and $\varphi : M \to N$ be a pseudo-horizontally homothetic (PHH), V-harmonic submersion.*

If $P^{2p} \subset N^{2n}$ is a complex submanifold of N, then $\varphi^{-1}(P) \subset M$ is a V-minimal submanifold of M.

Proof. As in ([1]), let us consider the decomposition of TM into the vertical distribution V and the horizontal one H, $T_xM = V_x \oplus H_x$, for any point $x \in M$. Denote $\varphi^{-1}(P)$, by K, $H_1 = TK \cap H$ and H_2 the orthogonal complement of H_1 in H. Also, $T_xK = V_x \oplus H_{1_x}$, $H_x = H_{1_x} \oplus H_{2_x}$, $H_{1_x} = \{u \in H_x, d\varphi_x(u) \in T_{\varphi(x)}P\}$, for any $x \in K$.

We can choose a linearly independent system of local sections $\{e_1, \ldots, e_p, Je_1, \ldots, Je_p\}$ in $\varphi^{-1}TN$, such that, if we denote by $E_i = d\varphi^*(e_i)$ and $E'_i = d\varphi^*(Je_i)$, the restriction of the system to K is a local orthonormal basis in H_1.

The way the system was chosen is the following: since $d\varphi_{x_{|H_x}} \circ d\varphi^*_x$ is a linear isomorphism, take v_1 a local section of $\varphi^{-1}TP$, non-vanishing along K and choose the section e_1 of $\varphi^{-1}TN$, such that $(d\varphi_{|H} \circ d\varphi^*)(e_1) = v_1$. Using the PHWC condition of φ, $d\varphi(d\varphi^*(Je_1))$ is also a section of $\varphi^{-1}TP$ and $d\varphi^*(e_1)$ and $d\varphi^*(Je_1)$ are orthogonal local vector fields in H and also in H_1. By induction, at step k, we take e_k perpendicular on both $(d\varphi \circ d\varphi^*)(e_i)$ and $(d\varphi \circ d\varphi^*)(Je_i), \forall 1 \leq i \leq k-1$, such that $(d\varphi \circ d\varphi^*)(e_i)$ is a section in $\varphi^{-1}TP$.

In the sequel the computations are done only on points of K.

Consider $\{u_1, u_2, \dots, u_r\}$ an orthonormal local basis in the vertical distribution V such that $\{u_1, u_2, \dots, u_r, E_1, \dots, E_p, E'_1, \dots, E'_p\}$ is an orthogonal local basis in TK. Denote by A the second fundamental form of the submanifold K.

The V-minimality of K is equivalent to $trace(A) - V \in \Gamma(TK)$ which reads: $\sum\limits_{i=1}^{r} A(u_i, u_i) + \sum\limits_{i=1}^{p} A(E_i, E_i) + \sum\limits_{i=1}^{p} A(E'_i, E'_i) - V \in \Gamma(TK)$ or equivalent:

$$\sum_{i=1}^{r} \left[\frac{1}{g_M(u_i,u_i)} d\varphi(\nabla^M_{u_i} u_i - \nabla^K_{u_i} u_i) \right] +$$

$$+ \sum_{i=1}^{p} \left[\frac{1}{g_M(E_i,E_i)} d\varphi(\nabla^M_{E_i} E_i - \nabla^K_{E_i} E_i) + \frac{1}{g_M(E'_i,E'_i)} d\varphi(\nabla^M_{E'_i} E'_i - \nabla^K_{E'_i} E'_i) \right] - d\varphi(V)$$

to be a section of $\varphi^{-1}TP$.

Using relations (15) and (17), we compute:

$$\begin{aligned}
&\sum_{i=1}^{p} \left[\frac{1}{g_M(u_i,u_i)} d\varphi(\nabla^M_{u_i} u_i - \nabla^K_{u_i} u_i) \right] + \\
&+ \sum_{i=1}^{p} \left[\frac{1}{g_M(E_i,E_i)} d\varphi(\nabla^M_{E_i} E_i) + \frac{1}{g_M(E'_i,E'_i)} d\varphi(\nabla^M_{E'_i} E'_i) \right] - \\
&- \sum_{i=1}^{p} \left[\frac{1}{g_M(E_i,E_i)} (\nabla^K_{E_i} E_i) + \frac{1}{g_M(E'_i,E'_i)} (\nabla^K_{E'_i} E'_i) \right] - d\varphi(V) = \\
&= \sum_{i=1}^{p} \left[\frac{1}{g_M(u_i,u_i)} d\varphi(\nabla^M_{u_i} u_i - \nabla^K_{u_i} u_i) \right] + \\
&+ \sum_{i=1}^{p} \frac{1}{g_M(E_i,E_i)} Jd\varphi[E'_i, E_i] - \sum_{i=1}^{p} \frac{1}{g_M(E_i,E_i)} Jd\varphi[E'_i, E_i] - d\varphi(V) = \\
&= \sum_{i=1}^{p} d\varphi(\nabla^M_{u_i} u_i) - \sum_{i=1}^{p} d\varphi(\nabla^K_{u_i} u_i) - d\varphi(V)
\end{aligned} \tag{20}$$

As φ is a pseudo-horizontally homothetic V-harmonic submersion, from Theorem (3.5), φ has V-minimal fibres, which is equivalent to: $d\varphi(\sum\limits_{i=1}^{p} \nabla^M_{u_i} u_i) = d\varphi(V)$.

Hence, $trace(A) - V \in \Gamma(TK)$ is equivalent to $\sum\limits_{i=1}^{p} d\varphi(\nabla^K_{u_i} u_i)$ is a section of $\varphi^{-1}TP$. $\square$

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