Free objects and coproducts in categories of posets and lattices

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Abstract

We investigate the existence of free objects and coproducts in the full subcategory of the category of lattices (posets) whose objects are all bounded lattices (posets).

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0 Introduction

A classical result in universal algebra, due to Birkhoff [2], says that for any type τ of an algebraic system, given by a set of operators and a set of identities, there exists a free τ-algebra over an arbitrary set, i.e., the forgetful functor from the category $\mathbf{Alg}_\tau$ of algebras of type τ (which is also called a variety of algebras) to the category **Set** of sets has a left adjoint, see for example [7, page 128]. We consider the category **Lat** of lattices and the category **bLat** of bounded lattices, i.e., lattices with a least element 0 and a greatest element 1. In the first one, morphisms are the mappings commuting with finite meets and joins, while in the latter

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one, morphisms also preserve 0 and 1. Both these categories are of the type $\mathbf{Alg}_\tau$, so the forgetful functors from these categories to **Set** have left adjoints. On the other hand, colimits, in particular coproducts, exist in any $\mathbf{Alg}_\tau$, see [7, Corollary 3, page 213].

In this paper we discuss the existence of free objects and coproducts in the full subcategory $\overline{\mathbf{bLat}}$ of **Lat** whose objects are all bounded lattices. The morphisms in $\overline{\mathbf{bLat}}$ are just lattice morphisms, not necessarily preserving 0 and 1. We also consider similar questions for posets. If **Pos** is the category of posets, with monotone mappings as morphisms, **bPos** is the category of bounded posets, i.e., posets with a least element 0 and a greatest element 1, and morphisms preserving 0 and 1, and $\overline{\mathbf{bPos}}$ is the full subcategory of **Pos** whose objects are the bounded posets, we discuss the existence of free objects and coproducts in $\overline{\mathbf{bPos}}$. Notice that **Pos** and **bPos** are not varieties of algebras, but they have coproducts and their forgetful functors to **Set** have left adjoints.

We show in Section 1 that a free object over any set with at least two elements does not exist in $\overline{\mathbf{bPos}}$, and a free object over a set X exists in $\overline{\mathbf{bLat}}$ if and only if X is finite.

In Section 2 we show that $\overline{\mathbf{bPos}}$ does not have any coproducts, while $\overline{\mathbf{bLat}}$ has finite coproducts, but it does not have any infinite coproduct. On the other hand, as we explained above, coproducts exist in **Lat**; their existence can be obtained as an application of the adjoint functor theorem. We briefly include a construction of such coproducts, since they are related to our result.

For all undefined notation and terminology on categories (respectively, lattices), the reader is referred to [7], [9] (respectively, [1], [3], [4], [5], [6], [9]).

1 Free objects

Let **bPos** be the category whose objects are the bounded posets, and the morphisms are the monotone functions preserving the least element and the greatest element.

It is easy to see that the forgetful functor $U_{\mathbf{bPos}} : \mathbf{bPos} \to \mathbf{Set}$ has a left adjoint F, defined as follows. If X is a set, let $F(X) = X \cup \{0,1\}$, where 0 and 1 are two objects not lying in X; $F(X)$ is a bounded poset with the least element 0 and the greatest element 1, while different elements of X are not comparable.

Let now $\overline{\mathbf{bPos}}$ be the full subcategory of **Pos** whose objects are all bounded posets. Thus the morphisms in $\overline{\mathbf{bPos}}$ are monotone functions, with no preservation condition on 0 or 1.

Proposition 1.1. *If X is a set with at least two elements, then a free object over X does not exist in $\overline{\mathbf{bPos}}$. In particular, the forgetful functor $U_{\overline{\mathbf{bPos}}} : \overline{\mathbf{bPos}} \longrightarrow \mathbf{Set}$ has no left adjoint.*

Proof. Assume that a free object $(F(X), \eta_X)$ over X exists in $\overline{\mathbf{bPos}}$. Thus $F(X)$ is a bounded lattice, $\eta_X : X \to F(X)$ is a map, and for any $A \in \overline{\mathbf{bPos}}$ and any map $f : X \to A$, there exists a unique morphism $\overline{f} : F(X) \to A$ in $\overline{\mathbf{bPos}}$ such that $\overline{f}\eta_X = f$.

We show that η_X is injective. Let $\overline{X} = X \cup \{0, 1\}$, where 0 and 1 are two different objects not lying in X. We consider the partial order on X such that any two different elements of X are not comparable, and $0 \leqslant x \leqslant 1$ for any $x \in X$. Then $\overline{X} \in \overline{\mathbf{bPos}}$, and let $i : X \hookrightarrow \overline{X}$ be the inclusion mapping. Using the universal property of $F(X)$ for $A = \overline{X}$, there exists a morphism $\bar{i} : F(X) \to \overline{X}$ in $\overline{\mathbf{bPos}}$ such that $\bar{i}\eta_X = i$. Since i is injective, so is η_X. Without loss of generality we may assume that η_X is the inclusion.

We claim that in $F(X)$ we have $1 \notin X$. Indeed, if 1 would lie in X, let $x \in X$, $x \neq 1$, and consider $A = \{0_A, 1_A\}$ an object in $\overline{\mathbf{bPos}}$ with just two elements. Let $f : X \to A$ be a map such that $f(x) = 1_A$ and $f(1) = 0_A$, and let $\overline{f}$ be its extension to $F(X)$ by the universal property. Then $x \leqslant 1$ in $F(X)$, so $\overline{f}(x) \leqslant \overline{f}(1)$, which means that $1_A \leqslant 0_A$, which is a contradiction. This proves our claim, i.e., $1 \notin X$.

Now let $B = F(X) \cup \{\overline{1}\}$, where $\overline{1}$ is an object not lying in $F(X)$. We extend the partial order of $F(X)$ to one of B by setting $t \leqslant \overline{1}$ for any $t \in F(X)$. Then $B \in \overline{\mathbf{bPos}}$, with the least element 0 and the greatest element $\overline{1}$. Let $f : X \hookrightarrow B$ be the inclusion mapping, and let $\overline{f} : F(X) \to B$ be its unique extension to a morphism in $\overline{\mathbf{bPos}}$ by the universal property. We are going to obtain a contradiction by showing that there are at least two such morphisms $\overline{f}$. The first one is just the inclusion mapping $\iota : F(X) \hookrightarrow B$. The second one is the mapping $j : F(X) \to B$ defined by $j(t) = t$ if $t \neq 1$, and $j(1) = \overline{1}$. Notice that j extends f because $1 \notin X$. Since $i(1) \neq j(1)$, we have $i \neq j$, a contradiction which ends the proof. □

As we explained in the Introduction, the forgetful functor $U_{\mathbf{Lat}} : \mathbf{Lat} \to \mathbf{Set}$ has a left adjoint G. The unit of this adjunction $(G, U_{\mathbf{Lat}})$ is injective, so we may assume that X is a subset of $G(X)$. In other words, $G(X)$ is the free lattice over the set X, satisfying the following universal property: for any lattice L and any mapping $f : X \to L$, there exists a unique lattice morphism $\overline{f} : G(X) \to L$ extending f. The lattice $G(X)$ was well understood long time ago, see [10], [11]. An explicit construction of $G(X)$ can be found for instance in [8, Chapter 6]. As an immediate consequence of the universal property, one sees that the sublattice B of $G(X)$ generated by X is the whole of $G(X)$. Indeed, the universal property of $G(X)$ applied for the inclusion mapping of X into B produces a lattice morphism $\gamma : G(X) \to B$ acting as identity on X. If $j : B \to G(X)$ is the inclusion mapping, then $j\gamma : G(X) \to G(X)$ is a lattice morphism acting as identity on X, and then the uniqueness part of the universal property shows that $j\gamma$ must be the identity mapping of $G(X)$. This shows that j is surjective, so then $B = G(X)$. As a consequence, any element of $G(X)$ is obtained from finitely many elements of X and by applying finitely many times the operations $\bigvee$ and $\bigwedge$. In particular, if $X = \{x_1, \ldots, x_n\}$ is finite, then $G(X)$ is bounded, with the least element $x_1 \wedge \ldots \wedge x_n$ and the greatest element $x_1 \vee \ldots \vee x_n$.

We think that the following two results are well known, but we include their proofs for the reader's convenience.

Lemma 1.2. *With the notations above, if $x, x_1, \ldots, x_n \in X$, where n is a positive integer, and $x \leqslant x_1 \vee \ldots \vee x_n$ in $G(X)$, then $x = x_i$ for some i.*

Proof. If x is not equal to any x_i, let $L = \{0, 1\}$ be a lattice with two elements $0 \leqslant 1$. Let $f : X \to L$ be a map such that $f(x) = 1$ and $f(x_1) = \ldots = f(x_n) = 0$, and let $\overline{f} : G(X) \to L$ be the unique lattice morphism extending f. Since $x \leqslant x_1 \vee \ldots \vee x_n$ in $G(X)$, we deduce that $1 = \overline{f}(x) \leqslant \overline{f}(x_1 \vee \ldots \vee x_n) = \overline{f}(x_1) \vee \ldots \vee \overline{f}(x_n) = 0 \vee \ldots \vee 0 = 0$, which is a contradiction. $\square$

Corollary 1.3. *If X is infinite, then $G(X)$ has neither a greatest element nor a least element.*

Proof. Assume that $G(X)$ has a 1. Then 1 is obtained from finitely many elements $x_1, \ldots, x_n$ of X, by applying several times the operations $\bigvee$ and $\bigwedge$, so $1 \leqslant x_1 \vee \ldots \vee x_n$. Thus $1 = x_1 \vee \ldots \vee x_n$, and then, if $x \in X \setminus \{x_1, \ldots, x_n\}$, we have $x \leqslant x_1 \vee \ldots \vee x_n$, so $x = x_i$ for some i by Lemma 1.2, which is a contradiction.

In a similar way one can prove that $G(X)$ does not have a 0. $\square$

Proposition 1.4. *If X is a finite set, then $G(X)$ is a free object over X in $\overline{\mathbf{bLat}}$. If X is an infinite set, then a free object over X does not exist in $\overline{\mathbf{bLat}}$. In particular, the forgetful functor $U_{\overline{\mathbf{bLat}}} : \overline{\mathbf{bLat}} \longrightarrow \mathbf{Set}$ does not have a left adjoint.*

Proof. If X is finite, then $G(X)$ is bounded, and its universal property in **Lat**, makes it also a free object over X in $\overline{\mathbf{bLat}}$.

Now let X be infinite, and assume that a free object $F(X)$ over X exists in $\overline{\mathbf{bLat}}$. As in Proposition 1.1, we see that the map $X \to F(X)$ associated with the free object is injective. Indeed, the same argument works as in the proof of the mentioned proposition, since for any set X, the set $\overline{X} = X \cup \{0, 1\}$, with the least element 0, the greatest element 1, and any different elements of X being incomparable, is a bounded lattice. Thus we may assume that X is a subset of $F(X)$. Moreover, using again the same argument as in the proof of Proposition 1.1, which works since $A = \{0_A, 1_A\}$ is a bounded lattice, we see that $1 \notin X$, and then, similarly, that $0 \notin X$.

Let us consider the lattice $G(X)$, where G is the left adjoint of $U_{\mathbf{Lat}}$. By Corollary 1.3, $G(X)$ does not have a least element and a greatest element, so we can adjoin two such elements $\overline{0}$ and $\overline{1}$, thus making $G(X) \cup \{\overline{0}, \overline{1}\}$ a bounded lattice. Let us consider the following diagram.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, hook] \arrow[ddr, hook] & G(X) \arrow[d, dotted, "\varphi"] \\ & F(X) \arrow[d, dotted, "\psi"] \\ & G(X) \cup \{\overline{0}, \overline{1}\} \end{tikzcd}$$

The universal property of $G(X)$ shows that there is a lattice morphism $\varphi : G(X) \to F(X)$ which fixes X pointwise. Also, the universal property of $F(X)$ produces a lattice morphism $\psi : F(X) \to G(X) \cup \{\overline{0}, \overline{1}\}$ acting as identity on X. Then $\psi\varphi : G(X) \to G(X) \cup \{\overline{0}, \overline{1}\}$ is a morphism of lattices acting as identity on X. The universal property of $G(X)$ shows that there is a unique such morphism, and then $\psi\varphi$ is necessarily the inclusion. We conclude that φ is an injective lattice morphism.

Now let B be the bounded sublattice of $F(X)$ generated by X, and let $j : B \hookrightarrow F(X)$ be the inclusion mapping. Denote by $\gamma : F(X) \to B$ the morphism obtained by the universal property of $F(X)$, as in the following diagram.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, hook] \arrow[ddr, hook] & F(X) \arrow[d, dotted, "\gamma"] \\ & B \arrow[d, hook, "j"] \\ & F(X) \end{tikzcd}$$

But $j\gamma : F(X) \to F(X)$ is a lattice morphism acting as identity on X, and the uniqueness part of the universal property shows that $j\gamma$ must be the identity map, so j is surjective, and then $B = F(X)$. We conclude that any element $p \in F(X) \setminus \{0, 1\}$ can be obtained from finitely many elements of X by applying finitely many times joins and unions. As a consequence, for any $p \in F(X) \setminus \{1\}$ there exist a positive integer m and $x_1, \ldots, x_m \in X$ such that $p \leqslant x_1 \vee \ldots \vee x_m$.

We show that for any $p, q \in F(X) \setminus \{1\}$ we have $p \vee q \neq 1$. Indeed, if $p, q \in F(X) \setminus \{1\}$, then $p \leqslant y_1 \vee \ldots \vee y_r$ and $q \leqslant z_1 \vee \ldots \vee z_s$ for some positive integers r, s and some $y_1, \ldots, y_r, z_1, \ldots, z_s \in X$. If $p \vee q = 1$, then by renoting, there are some $x_1, \ldots, x_n \in X$ such that $x_1 \vee \ldots \vee x_n = 1$ in $F(X)$. At this point, in order to avoid any danger of confusion, we denote by $\bigvee_F$ and $\bigvee_G$ the meet in $F(X)$ and $G(X)$, respectively. Thus we have $x_1 \vee_F \ldots \vee_F x_n = 1$. Since X is infinite, we can pick some $x \in X \setminus \{x_1, \ldots, x_n\}$. Then

$$\begin{aligned}
\varphi(x \vee_G x_1 \vee_G \ldots \vee_G x_n) &= \varphi(x) \vee_F \varphi(x_1) \vee_F \ldots \vee_F \varphi(x_n) \\
&= x \vee_F x_1 \vee_F \ldots \vee_F x_n \\
&= x \vee_F 1 \\
&= 1 \\
&= x_1 \vee_F \ldots \vee_F x_n \\
&= \varphi(x_1) \vee_F \ldots \vee_F \varphi(x_n) \\
&= \varphi(x_1 \vee_G \ldots \vee_G x_n),
\end{aligned}$$

and the injectivity of φ shows that $x \vee_G x_1 \vee_G \ldots \vee_G x_n = x_1 \vee_G \ldots \vee_G x_n$. It follows that $x \leqslant x_1 \vee_G \ldots \vee_G x_n$, and then $x = x_i$ for some i by Lemma 1.2, a contradiction with the choice of x. Thus $p \vee q \neq 1$.

Now let $L = F(X) \cup \{\overline{1}\}$, where $\overline{1}$ is an object not lying in $F(X)$. Then L is a lattice with the partial order extending the one of $F(X)$, and such that $t \leqslant \overline{1}$ for any $t \in F(X)$. The meet and join in L extend the ones in $F(X)$. Clearly, L is bounded, with the same least element as $F(X)$, and the greatest element $\overline{1}$. We proceed now for lattices as in the proof of Proposition 1.1 for posets. By the universal property, the inclusion map of X in L extends uniquely to a morphism in $\overline{\mathbf{bLat}}$ from $F(X)$ to L. On the other hand, we see that at least two different such morphism exist. One is the inclusion map $F(X) \hookrightarrow L$, and another one is $j : F(X) \to B$, such that $j(t) = t$ for any $t \neq 1$, and $j(1) = \overline{1}$.

Notice that j commutes with joins, i.e., $j(p \vee q) = j(p) \vee j(q)$ for any $p, q \in F(X)$. Indeed, if one of p and q is 1, then both sides are equal since $j(1) = \overline{1}$.

If none of p and q is 1, then $j(p) = p$, $j(q) = q$, and then $j(p) \vee j(q) = p \vee q$ in L is the same as in $F(X)$, since the join of L extends the one in $F(X)$. On the other hand, we showed above that $p \vee q \neq 1$ in $F(X)$, and then $j(p \vee q) = p \vee q$. Thus, also in this case, the two sides of the equality to be shown are equal.

Observe that if we would have $p \vee q = 1$ in $F(X)$ for some $p, q \neq 1$ then $j(p \vee q) = j(1) = \overline{1}$, while $j(p) \vee j(q) = p \vee q = 1 \in L$, so we wouldn't have $j(p \vee q) = j(p) \vee j(q)$. To avoid this situation, we had to show that $p \vee q \neq 1$ for $p, q \neq 1$.

This final contradiction shows that the assumption on the existence of $F(X)$ is false, and we are done. $\square$

Remark 1.5. We mentioned in the Introduction that the existence of a left adjoint of the forgetful functor $U_{\mathbf{bLat}} : \mathbf{bLat} \longrightarrow \mathbf{Set}$, or in other words, the existence of the free bounded lattice over a set, follows from the general theory of varieties of algebras. In fact, if X is a set, then the free bounded lattice over X can be obtained by taking the free lattice $G(X)$ over

X, and adding artificially a new greatest element and a new least element. Thus we take the set $G(X) \cup \{\overline{0}, \overline{1}\}$, where $\overline{0}, \overline{1}$ are two different objects not lying in $G(X)$, and we extend the partial order of $G(X)$ to $G(X) \cup \{\overline{0}, \overline{1}\}$ such that $\overline{0} \leqslant t \leqslant \overline{1}$ for any $t \in G(X) \cup \{\overline{0}, \overline{1}\}$. In this way $G(X) \cup \{\overline{0}, \overline{1}\}$ is a bounded lattice, and it is easy to check that it satisfies the universal property of the free bounded lattice over X. We noticed that in the case where X is finite, $G(X)$ is bounded. However, $G(X)$ is not the free bounded lattice over X, and we still have to adjoin artificially new 0 and 1 for obtaining such an object. □

2 Coproducts

It is easy to see that **Pos** has coproducts. Indeed, the coproduct of a family $(X_i)_{i \in I}$ of posets is just the coproduct of the family $(X_i)_{i \in I}$ of sets in **Set**, i.e., the disjoint union of the X_i's, endowed with a partial order extending the partial orders of each X_i, and such that elements from different X_i's are not comparable.

Also, **bPos** has coproducts. If $(X_i)_{i \in I}$ is a family of bounded posets, we consider the disjoint union of the family, with the partial order as above, and then we identify the 1's in all X_i's, and the 0's in all X_i's.

Proposition 2.1. *For any objects X and Y in $\overline{\mathbf{bPos}}$, a coproduct of X and Y does not exist in this category.*

Proof. Assume that a coproduct C of X and Y exists in $\overline{\mathbf{bPos}}$, with canonical morphisms $i_X : X \to C$ and $i_Y : Y \to C$.

We first show that i_X and i_Y are injective. Indeed, let $f : X \to X$ be the identity mapping, and let $g : Y \to X$ be the mapping defined by $g(y) = 0_X$ for any $y \in Y$, where 0_X is the least element of X. Then there is a unique morphism $\pi_X : C \to X$ in $\overline{\mathbf{bPos}}$ such that $\pi_X i_X = f$ and $\pi_X i_Y = g$, in particular i_X is injective.

Next we show that $i_X(X) \cap i_Y(Y) = \varnothing$. Indeed, let $A = \{0_A, 1_A\}$ be an object of $\overline{\mathbf{bPos}}$ with two elements. Consider the morphisms $u : X \to A$, $u(x) = 0_A$ for any $x \in X$, and $v : Y \to A$, $v(y) = 1_A$ for any $y \in Y$. Then there exists a morphism $\gamma : C \to A$ in $\overline{\mathbf{bPos}}$ such that $\gamma i_X = u$ and $\gamma i_Y = v$. If $i_X(x) = i_Y(y)$ for some $x \in X$ and $y \in Y$, then $0_A = \gamma(i_X(x)) = \gamma(i_Y(y)) = 1_A$, which is a contradiction. Thus $i_X(X) \cap i_Y(Y) = \varnothing$.

As a consequence, we may assume with no loss of generality that $X, Y \subseteq C$, $X \cap Y = \varnothing$, and i_X and i_Y are the inclusion mappings.

We claim that $1_C \notin X$ and $1_C \notin Y$. Indeed, if for instance $1_C \in X$, let $u : X \to C$, $u(x) = 0_C, \forall\, x \in X$, and $v : Y \to C$, $v(y) = 1_C, \forall\, y \in Y$. Of course, 0_C (respectively, 1_C) denotes the least (respectively, the greatest) element of C. Let $\delta : C \to C$ be the unique morphism in $\overline{\mathbf{bPos}}$ such that $\delta i_X = u$ and $\delta i_Y = v$. Then $\delta(1_C) = u(1_C) = 0_C$, so $\delta(t) = 0_C$ for any $t \in C$, since $\delta(t) \leqslant \delta(1_C)$. On the other hand $0_C = \delta(1_C) = 1_C$ since δ preserves 1_C, which is a contradiction. Thus δ cannot extend v, and this proves our claim.

Let now $B \in \overline{\mathbf{bPos}}$ be the object such that $B = C \cup \{\overline{1}\}$, where $\overline{1}$ is an object not lying in C; the partial order in B extends the one of C, and moreover $t \leqslant \overline{1}$ for any $t \in B$. Let $\alpha : X \hookrightarrow B$ and $\beta : Y \hookrightarrow B$ be the inclusion mappings, which are morphisms in $\overline{\mathbf{bPos}}$. The universal property shows that there is a unique morphism $\varphi : C \to B$ extending both α and β. On the other hand, both the inclusion mapping $\varphi_1 : C \hookrightarrow B$, and the mapping $\varphi_2 : C \to B$ defined by $\varphi_2(t) = t$ for any $t \neq 1_C$, and $\varphi_2(1_C) = \overline{1}$, extend α and β. As $\varphi_1 \neq \varphi_2$, we obtain a contradiction. $\square$

Now we briefly present an explicit construction of the coproduct of two lattices in the category **Lat**. Let L_1 and L_2 be lattices. With no loss of generality we may assume that $L_1 \cap L_2 = \varnothing$. We define recurrently an ascending chain $(T_n)_{n\geqslant 1}$ of sets and two binary operations $\bigvee$ and $\bigwedge$ on $T = \bigcup_{n\geqslant 1} T_n$ as follows.

Set $T_1 := L_1 \cup L_2$, and $p \vee q$, $p \wedge q$ are the ones in L_i, if both p and q are in the same L_i, $i = 1, 2$. Now let T_2 consists of all elements of T_1, and also some new objects, which do not lie in T_1, denoted by $p \vee q$ and $p \wedge q$, for any p, q in T_1, not both in the same L_i. For any new objects $p \vee q$, $p \wedge q$, $p' \vee q'$, $p' \wedge q'$ of this kind, we impose

$$p \vee q = p' \vee q' \Longleftrightarrow p \wedge q = p' \wedge q' \Longleftrightarrow p = p' \text{ and } q = q',$$

and

$$p \wedge q \neq p' \vee q'.$$

Assume that we constructed T_{n-1} for some $n \geqslant 3$, and we defined $\bigvee$ and $\bigwedge$ on T_{n-2}. Then we define T_n to consist of the elements of T_{n-1}, and also we add some new elements denoted by $p \vee q$ and $p \wedge q$, for any p, q in T_{n-1}, which are not both in T_{n-2}. Such $p \vee q$ and $p \wedge q$ are considered not to be in T_{n-1}, and for any new such elements $p \vee q$, $p \wedge q$, $p' \vee q'$, $p' \wedge q'$, we impose

$$p \vee q = p' \vee q' \Longleftrightarrow p \wedge q = p' \wedge q' \Longleftrightarrow p = p' \text{ and } q = q',$$

and

$$p \wedge q \neq p' \vee q'.$$

Now $T := \bigcup_{n\geqslant 1} T_n$ is an algebra with binary operations $\bigvee$ and $\bigwedge$ defined by collecting their meaning in each T_n. Moreover, L_1 and L_2 are subalgebras of T. Let $j_1 : L_1 \hookrightarrow T$ and $j_2 : L_2 \hookrightarrow T$ be the inclusion mappings.

The algebra T has the following universal property: if L is an algebra of the same type as T, and $f_1 : L_1 \to L, f_2 : L_2 \to L$ are two morphisms of algebras, then there is a unique morphism of algebras $\varphi : T \to L$ extending both f_1 and f_2. Indeed, we take φ to work as f_i on L_i, $i = 1, 2$, $\varphi(p \vee q) = \varphi(p) \vee \varphi(q)$ and $\varphi(p \wedge q) = \varphi(p) \wedge \varphi(q)$ for any $p, q \in T_1$, not both in the same L_i, and then for any $n \geqslant 3$ and any $p, q \in T_{n-1}$, not both in T_{n-2}, we take again $\varphi(p \vee q) = \varphi(p) \vee \varphi(q)$ and $\varphi(p \wedge q) = \varphi(p) \wedge \varphi(q)$. Notice that in all these cases $\varphi(p)$ and $\varphi(q)$ were defined previously. It is clear that φ is a morphism of algebras and that it is the only one with the required properties.

Now let $(\rho_i)_{i \in I}$ be the family of all congruences on T with the property that the factor algebras T/ρ_i are lattices. Then $\rho = \bigcap_{i \in I} \rho_i$ is a congruence on T, and the factor algebra T/ρ embeds in the product of algebras $\prod_{i \in I} T/\rho_i$. Now each T/ρ_i is a lattice, and then so is their product. We deduce that T/ρ is also a lattice, as a subalgebra of a lattice. Indeed, a lattice is defined as an algebra (with two binary operations $\bigvee$ and $\bigwedge$) with the property that it satisfies a set of equations (associativity of $\bigvee$ and $\bigwedge$, and the absorption law), and this property is inherited by any subalgebra. Let $\pi : T \to T/\rho$ be the natural projection.

We claim that the lattice T/ρ, together with the lattice morphisms $\pi j_1 : L_1 \to T/\rho$ and $\pi j_2 : L_2 \to T/\rho$, is a coproduct of L_1 and L_2 in the category **Lat**.

Indeed, if L is a lattice, and $f_1 : L_1 \to L, f_2 : L_2 \to L$ are lattice morphisms, then the universal property of T shows that there is a unique morphism of algebras $\varphi : T \to L$ such that $\varphi j_1 = f_1$ and $\varphi j_2 = f_2$. Since $T/\mathrm{Ker}\ \varphi \simeq \mathrm{Im}\ \varphi$, and $\mathrm{Im}\ \varphi$ is a subalgebra of a lattice, thus a lattice itself, we obtain that the congruence $\mathrm{Ker}\ \varphi$ is in fact one of the ρ_i's, so $\rho \subseteq \mathrm{Ker}\ \varphi$, and then there is a unique algebra (i.e., lattice) morphism $\psi : T/\rho \to L$ such that $\psi\pi = \phi$. Then $\psi\pi j_1 = f_1$ and $\psi\pi j_2 = f_2$, showing that indeed T/ρ is the coproduct of L_1 and L_2 in **Lat**. It is easy to see that πj_1 and πj_2 are in fact embedings of L_1 and L_2 in T/ρ, by applying the universal property for $L = L_1$, f_1 the identity map, and any f_2 (a constant function, for instance), and then similarly for L_2.

On the other hand, with an argument similar to the one in the proof of Proposition 2.1 we see that $\pi j_1(L_1) \cap \pi j_2(L_2) = \varnothing$. Indeed, let $A = \{0_A, 1_A\}$ be a lattice with two elements, and let $g_1 : L_1 \to A$, $g_1(x) = 0_A$ for any $x \in L_1$, and $g_2 : L_2 \to A$, $g_2(y) = 0_A$ for any $y \in L_2$. If $g : T/\rho \to A$ is the unique lattice morphism arising from the universal property of the coproduct, we have $g(\pi j_1(L_1)) = \{0_A\}$ and $g(\pi j_2(L_2)) = \{1_A\}$, so $\pi j_1(L_1)$ and $\pi j_2(L_2)$ cannot have common elements. Thus we may assume that L_1 and L_2 are disjoint subsets of their coproduct T/ρ.

Moreover, the construction shows that any element of T/ρ is obtained from finitely many elements of $\pi j_1(L_1) \cup \pi j_2(L_2)$ by applying finitely many meets and joins.

The coproduct of finitely many objects in **Lat** can be obtained by applying finitely many times the previous construction giving the coproduct of two lattices. As for the coproduct of an infinite family $(L_i)_{i\in I}$ of lattices, one first constructs the coproduct C_F of the finite family $(L_i)_{i\in F}$ for any finite subset F of I. These objects come with natural injective lattice morphisms from C_F to C_P for any $F \subseteq P$, with finite P. Now consider the filtered colimit C in **Set** of the family of sets $(C_F)_{F \text{ finite}}$, with the connecting maps described above. Then C can be endowed with a lattice structure induced by the lattice structure of all C_F's, such that the natural map from each C_F to C is a lattice morphism. One can see that this C is just the coproduct of the family $(L_i)_{i\in I}$ in **Lat**.

The construction of the coproduct in **bLat** can be adapted from the one in **Lat**, with a small change taking care of 0 and 1. First of all, we note that if L_0 is the trivial lattice, i.e., the lattice with just one element (thus in this lattice one has $0 = 1$), then for any bounded lattice $L \neq L_0$, there are no morphisms from L_0 to L in **bLat**, while there is just one morphism from L to L_0 (in fact just one function $L \to L_0$). As a consequence, for any family in **bLat** containing L_0, it is easily checked that the coproduct of the family is L_0. Therefore, we will consider coproducts of families not containing L_0.

Now let L_1 and L_2 be non-trivial bounded lattices. With no loss of generality we can assume that $L_1 \cap L_2 = \varnothing$. Firstly, we identify the 0's and 1's of the two lattices, more precisely, we consider the set $T_1 = (L_1 \setminus \{0_{L_1}, 1_{L_1}\}) \cup (L_2 \setminus \{0_{L_2}, 1_{L_2}\}) \cup \{0, 1\}$, where 0 and 1 are two objects not lying in $L_1 \cup L_2$. For each $i = 1, 2$ and any $p, q \in L_i \setminus \{0_{L_i}, 1_{L_i}\}$, we define $p \vee q$ and $p \wedge q$ as being the same as in L_i, with the exception of the case where $p \vee q = 1_{L_i}$, when we redefine $p \vee q = 1$, and the case where $p \wedge q = 0_{L_i}$, when we redefine $p \wedge q = 0$. Then we continue as in the construction described in **Lat**, by defining some new elements $p \vee q$ and $p \wedge q$ for any $p, q \in T_1$ which are not both in the same $L_i \setminus \{0_{L_i}, 1_{L_i}\}$, and imposing the same conditions as in the case of **Lat**. If we add these elements to T_1, we obtain a larger set T_2. Then for any $n \geqslant 3$, we define T_n by adding to T_{n-1} some new elements $p \vee q, p \wedge q$ for any $p, q \in T_{n-1}$, not both in T_{n-2}, with a similar set of conditions imposed. Now $T := \bigcup_{n\geqslant 1} T_n$ is an algebra with binary operations $\bigvee$ and $\bigwedge$ defined by collecting their meaning in each T_n, and two operations of arity zero, the constants 0 and 1. The maps $j_1 : L_1 \to T$ and $j_2 : L_2 \to T$, taking 0_{L_i}'s to 0, 1_{L_i}'s to 1, and acting as identity on the other elements, are morphisms of algebras of this type, and if L is an algebra of the same type, and $f_1 : L_1 \to L, f_2 : L_2 \to L$ are two morphisms of algebras, then there is a unique morphism of algebras $\varphi : T \to L$ such that $\varphi f_1 = j_1$ and $\varphi f_2 = j_2$. If we take $(\rho_i)_{i\in I}$ the family of all congruences on T with the property that the factor algebras T/ρ_i are bounded lattices with the greatest (respectively least) element being the class of 1 (respectively 0), then $\rho = \bigcap_{i\in I} \rho_i$ is a congruence on T, the factor algebra T/ρ is a bounded lattice, and one can show in a similar manner as above that T/ρ together the morphisms of bounded lattices $\pi j_1, \pi j_2$ is a coproduct of L_1 and L_2 in **bLat**. As above, we denoted by $\pi :\to T/\rho$ i the natural projection. The construction of the coproduct of an arbitrary family of non-trivial bounded lattices can be further continued as in the case of **Lat**.

For the readers' convenience, we collect the existence and effective constructions of coproducts in **Lat** and **bLat** in the next result.

Proposition 2.2. *The following statements hold for a family $(L_i)_{i\in I}$ of lattices.*

(1) *The coproduct $C := \coprod_{i\in I} L_i$ of $(L_i)_{i\in I}$ in* **Lat** *exists and described above.*

(2) *If all L_i, $i \in I$, are bounded, then their coproduct $B := \coprod_{i\in I} L_i$ in* **bLat** *exists and described above.* □

Proposition 2.3. $\overline{\mathbf{bLat}}$ *has finite coproducts, but it does not have any infinite coproduct.*

Proof. Let $L_1, L_2 \in \overline{\mathbf{bLat}}$, and let C be the coproduct of L_1 and L_2 in **Lat**. As we have seen above, we may assume that $L_1, L_2 \subseteq C$ and $L_1 \cap L_2 = \varnothing$. For each $i = 1, 2$, denote by 0_i and 1_i the least element and the greatest element of L_i, Since any element of C is obtained from finitely many elements of $L_1 \cup L_2$, by applying finitely many meets and joins, we deduce $0_1 \wedge 0_2 \leqslant c \leqslant 1_1 \vee 1_2$ for any $c \in C$. Thus $1_1 \vee 1_2$ is the greatest element of C, and $0_1 \wedge 0_2$ is the least element of C. We conclude that C is in fact a bounded lattice, and its universal property as a coproduct of L_1 and L_2 in **Lat** shows that it is also a coproduct of L_1 and L_2 in $\overline{\mathbf{bLat}}$.

We are now going to show that if $(L_\lambda)_{\lambda\in\Lambda}$ is a family of objects in $\overline{\mathbf{bLat}}$, where Λ is an infinite set, then this family does not have a coproduct in this category. Assume that C would be such a coproduct, with morphisms $i_\lambda : L_\lambda \to C$ for any $\lambda \in \Lambda$. Arguments as in the first part of the proof of Proposition 2.1 show that i_λ is injective for any λ, and $i_\lambda(L_\lambda) \cap i_\mu(L_\mu) = \varnothing$ for any $\lambda \neq \mu$. In the mentioned proof Λ was a set with just two elements, but the arguments work for any Λ. Again, we note that $A = \{0_A, 1_A\}$ is a bounded lattice, and all the morphisms considered in the proof of the mentioned proposition become morphisms in $\overline{\mathbf{bLat}}$ in the case of lattices.

Thus we may assume that $L_\lambda \subseteq C$ for any λ, and $L_\lambda \cap L_\mu = \varnothing$ for any $\lambda \neq \mu$. Denote by 0_λ and 1_λ the least element and the greatest element of L_λ, respectively. Let B be the sublattice of C (not necessarily with 0 and 1) generated by $\bigcup_{\lambda\in\Lambda} L_\lambda$. Each element of B is obtained from finitely many elements of $\bigcup_{\lambda\in\Lambda} L_\lambda$ and finitely many meets and joins, so for each $c \in B$ we have $0_{\lambda_1} \wedge \ldots \wedge 0_{\lambda_n} \leqslant c \leqslant 1_{\nu_1} \vee \ldots \vee 1_{\nu_p}$ for some positive integers n, p and some $\lambda_1, \ldots, \lambda_n, \nu_1, \ldots, \nu_p \in \Lambda$.

We claim that both 0_C and 1_C are not in B. Indeed, if $1_C \in B$, then we would obtain that

$$1_C \leqslant 1_{\mu_1} \vee \ldots \vee 1_{\mu_m} \leqslant 1_C$$

for some m and $\mu_1, \ldots, \mu_m$, and then

$$1_C = 1_{\mu_1} \vee \ldots \vee 1_{\mu_m}$$

Pick some $\mu \in \Lambda \setminus \{\mu_1, \ldots, \mu_m\}$, and let $A = \{0_A, 1_A\}$ be a bounded lattice with two elements, and the morphisms $g_\lambda : L_\lambda \to A$ in $\overline{\mathbf{bLat}}$ such that $g_\mu(x) = 1_A$ for any $x \in L_\mu$, and $g_\lambda(x) = 0_A$ for any $\lambda \neq \mu$ and any $x \in L_\lambda$. Let $g : C \to A$ be the lattice morphism following from the universal property of the coproduct C. Then

$$1_\mu \leqslant 1_C = 1_{\mu_1} \vee \ldots \vee 1_{\mu_m},$$

so

$$\begin{aligned} 1_A &= g_\mu(1_\mu) \\ &= g(1_\mu) \\ &\leqslant g(1_{\mu_1} \vee \ldots \vee 1_{\mu_m}) \\ &= g(1_{\mu_1}) \vee \ldots \vee g(1_{\mu_m}) \\ &= g_{\mu_1}(1_{\mu_1}) \vee \ldots \vee g_{\mu_m}(1_{\mu_m}) \\ &= 0_A \vee \ldots \vee 0_A \\ &= 0_A, \end{aligned}$$

which is a contradiction. Thus $1_C \notin B$, and similarly $0_C \notin B$. This proves our claim. In particular $p \vee q \neq 1_C$ and $p \wedge q \neq 0_C$ for any $p, q \in B$.

The set $B \cup \{0_C, 1_C\}$ is a bounded sublattice of C. Consider the diagram

$$\begin{tikzcd}
L_\lambda \arrow[r, hook] \arrow[dr, hook] \arrow[ddr, hook] & C \arrow[d, dotted, "\gamma"] \\
& B \cup \{0_C, 1_C\} \arrow[d, hook, "j"] \\
& C
\end{tikzcd}$$

where $\gamma : C \to B$ is the morphism in $\overline{\mathbf{bLat}}$ making for each λ commutative the diagram above obtained by the universal property of the coproduct C. Then $j\gamma$ must be the identity map, by the uniqueness part in the universal property of the coproduct, and so j is surjective. This shows that $B \cup \{0_C, 1_C\} = C$, and as a consequence, any element in $C \setminus \{0_C, 1_C\}$ is obtained from a finite number of elements of $L_1 \cup L_2$, by using finitely many times joins and unions.

Now let us add to C a new element $\overline{1}$, and make $C \cup \{\overline{1}\}$ a bounded lattice, whose $\bigvee$ and $\bigwedge$ extend the ones of C, and $\overline{1}$ is the greatest element. Let $j : C \hookrightarrow C \cup \{\overline{1}\}$ be the inclusion mapping, and $\varphi : C \to C \cup \{\overline{1}\}$ be defined by $\varphi(c) = c$ for any $c \in C \setminus \{1_C\}$, and $\varphi(1_C) = \overline{1}$. Then both j and φ are morphisms in $\overline{\mathbf{bLat}}$; this is obvious for j, while for φ we use, exactly as in the last lines of the proof of Proposition 1.4, the fact that $p \vee q \neq 1_C$ for any $p, q \in C \setminus \{1_C\}$.

Then the diagram

$$\begin{tikzcd} L_\lambda \arrow[r, hook] \arrow[dr, hook] & C \arrow[d, shift right, "j"'] \arrow[d, shift left, "\varphi"] \\ & C \cup \{\overline{1}\} \end{tikzcd}$$

is commutative for any λ, and this contradicts the uniqueness in the universal property of the coproduct C.

We conclude that the family $(L_\lambda)_{\lambda\in\Lambda}$ does not have a coproduct in $\overline{\mathbf{bLat}}$, as desired. $\square$

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