Submanifolds in Manifolds with Metric Mixed 3-Structures

Stere Ianuş, Liviu Ornea and Gabriel Eduard Vîlcu*

Abstract. Mixed 3-structures are odd-dimensional analogues of paraquaternionic structures. They appear naturally on lightlike hypersurfaces of almost paraquaternionic hermitian manifolds. We study invariant and anti-invariant submanifolds in a manifold endowed with a mixed 3-structure and a compatible (semi-Riemannian) metric. Particular attention is given to two cases of ambient space: mixed 3-Sasakian and mixed 3-cosymplectic.

Mathematics Subject Classification (2010). Primary 53C15; Secondary 53C50, 53C40, 53C12.

Keywords. Invariant submanifold, anti-invariant submanifold, mixed 3-structure, Einstein manifold.

1. Introduction

The counterpart in odd dimension of a paraquaternionic structure was introduced in [8]. It is called mixed 3-structure, and appears in a natural way on lightlike hypersurfaces in almost paraquaternionic hermitian manifolds. Such hypersurfaces inherit two almost paracontact structures and an almost contact structure, satisfying analogous conditions to those satisfied by almost contact 3-structures [14]. This concept has been refined in [3], where the authors have introduced positive and negative metric mixed 3-structures. The differential geometry of the semi-Riemannian hypersurfaces of co-index both 0 and 1 in such manifolds has been recently investigated in [11]. In the present paper, we discuss non-degenerate invariant and anti-invariant submanifolds in manifolds endowed with metric mixed 3-structures, the relevant ambients being mixed 3-Sasakian and mixed 3-cosymplectic.

Stere Ianuş passed away on April 8th, 2010.

*Corresponding author.

The paper is organized as follows. In Section 2 we recall definitions and basic properties of manifolds with metric mixed 3-structures. In Section 3 we establish several results concerning the existence of invariant and anti-invariant submanifolds in a manifold endowed with a metric mixed 3-structure, tangent or normal to the structure vector fields. Particularly, we show that an invariant submanifold is either tangent or normal to *all* the three structure vector fields. Moreover, we prove that a totally umbilical submanifold of a mixed 3-Sasakian manifold, tangent to the structure vector fields, is invariant and totally geodesic. This section ends with a wide range of examples. In Section 4 we study the anti-invariant submanifolds in a manifold endowed with a mixed 3-cosymplectic or mixed 3-Sasakian structure, normal to the structure vector fields. In particular, necessary and sufficient conditions are provided for the connection in the normal bundle to be trivial. We also provide an example of an anti-invariant flat minimal submanifold of S_{2n+1}^{4n+3}, normal to the structure vector fields. Section 5 discusses the distributions which naturally appear on invariant submanifolds of manifolds endowed with metric mixed 3-structures, tangent to the structure vector fields. Moreover, we obtain that a non-degenerate submanifold of a mixed 3-Sasakian manifold tangent to the structure vector fields is totally geodesic if and only if it is invariant. In the last Section we investigate the geometry of invariant submanifolds of mixed 3-cosymplectic manifolds, normal to the structure vector fields and prove that such a submanifold admits a para-hyper-Kähler structure.

2. Preliminaries

An almost product structure on a smooth manifold $\overline{M}$ is a tensor field P of type (1,1) on $\overline{M}$, $P \neq \pm Id$, such that

$$P^2 = Id.$$

where Id is the identity tensor field of type (1,1) on $\overline{M}$.

An almost complex structure on a smooth manifold $\overline{M}$ is a tensor field J of type (1,1) on $\overline{M}$ such that

$$J^2 = -Id.$$

An almost para-hypercomplex structure on a smooth manifold $\overline{M}$ is a triple $H = (J_\alpha)_{\alpha=\overline{1,3}}$, where J_1, J_2 are almost product structures on $\overline{M}$ and J_3 is an almost complex structure on $\overline{M}$, satisfying:

$$J_1 J_2 = -J_2 J_1 = J_3.$$

A semi-Riemannian metric $\overline{g}$ on $(\overline{M}, H)$ is said to be compatible or adapted to the almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=\overline{1,3}}$ if it satisfies:

$$\overline{g}(J_1 X, J_1 Y) = \overline{g}(J_2 X, J_2 Y) = -\overline{g}(J_3 X, J_3 Y) = -\overline{g}(X, Y)$$

for all vector fields X,Y on $\overline{M}$. Moreover, the triple $(\overline{M}, \overline{g}, H)$ is said to be an almost para-hyperhermitian manifold. If $\{J_1, J_2, J_3\}$ are parallel with respect to the Levi-Civita connection of $\overline{g}$, then the manifold is called para-hyper-Kähler. Note that, given a para-hypercomplex structure, compatible metrics might not exist at all, at least in real dimension 4, as recently shown in [4], using an Inoue surface.

An almost hermitian paraquaternionic manifold is a triple $(\overline{M}, \sigma, \overline{g})$, where $\overline{M}$ is a smooth manifold, σ is a rank 3-subbundle of $End(T\overline{M})$ which is locally spanned by an almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=\overline{1,3}}$ and $\overline{g}$ is a compatible metric with respect to H. Moreover, if the bundle σ is preserved by the Levi-Civita connection of $\overline{g}$, then $(\overline{M}, \sigma, \overline{g})$ is said to be a paraquaternionic Kähler manifold [6]. The prototype of paraquaternionic Kähler manifold is the paraquaternionic projective space $P^n(\mathbb{B})$ as described by Blažić [2].

A submanifold M of a quaternionic Kähler manifold $\overline{M}$ is called quaternionic (respectively totally real) if each tangent space of M is carried into itself (respectively into its orthogonal complement) by each section of σ. Several examples of paraquaternionic and totally real submanifolds of $P^n(\mathbb{B})$ are given in [7, 16].

Definition 2.1. Let $\overline{M}$ be a differentiable manifold equipped with a triple (φ, ξ, η), where φ is a field of endomorphisms of the tangent spaces, ξ is a vector field and η is a 1-form on $\overline{M}$. If we have:

$$\varphi^2 = \tau(-I + \eta \otimes \xi), \quad \eta(\xi) = 1 \qquad (2.1)$$

then we say that:

(i) (φ, ξ, η) is an almost contact structure on $\overline{M}$, if $\tau = 1$ ([17]).
(ii) (φ, ξ, η) is an almost paracontact structure on $\overline{M}$, if $\tau = -1$ ([18]).

We remark that many authors also include in the above definition the conditions that

$$\varphi\xi = 0, \ \eta \circ \varphi = 0, \qquad (2.2)$$

although these are deducible from (2.1) (see [1]).

Definition 2.2. A mixed 3-structure on a smooth manifold $\overline{M}$ is a triple of structures $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha \in \{1, 2, 3\}$, which are almost paracontact structures for $\alpha = 1, 2$ and almost contact structure for $\alpha = 3$, satisfying the following conditions:

$$\eta_\alpha(\xi_\beta) = 0, \qquad (2.3)$$
$$\varphi_\alpha(\xi_\beta) = \tau_\beta \xi_\gamma, \quad \varphi_\beta(\xi_\alpha) = -\tau_\alpha \xi_\gamma, \qquad (2.4)$$
$$\eta_\alpha \circ \varphi_\beta = -\eta_\beta \circ \varphi_\alpha = \tau_\gamma \eta_\gamma, \qquad (2.5)$$
$$\varphi_\alpha\varphi_\beta - \tau_\alpha\eta_\beta \otimes \xi_\alpha = -\varphi_\beta\varphi_\alpha + \tau_\beta\eta_\alpha \otimes \xi_\beta = \tau_\gamma\varphi_\gamma, \qquad (2.6)$$

where (α, β, γ) is an even permutation of $(1, 2, 3)$ and $\tau_1 = \tau_2 = -\tau_3 = -1$.

Moreover, if a manifold $\overline{M}$ with a mixed 3-structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$ admits a semi-Riemannian metric $\overline{g}$ such that:

$$\overline{g}(\varphi_\alpha X, \varphi_\alpha Y) = \tau_\alpha[\overline{g}(X, Y) - \varepsilon_\alpha\eta_\alpha(X)\eta_\alpha(Y)], \qquad (2.7)$$

for all $X, Y \in \Gamma(T\overline{M})$ and $\alpha = 1, 2, 3$, where $\varepsilon_\alpha = \overline{g}(\xi_\alpha, \xi_\alpha) = \pm 1$, then we say that $\overline{M}$ has a metric mixed 3-structure and $\overline{g}$ is called a compatible metric.

From (2.7) we obtain

$$\eta_\alpha(X) = \varepsilon_\alpha \overline{g}(X, \xi_\alpha), \quad \overline{g}(\varphi_\alpha X, Y) = -\overline{g}(X, \varphi_\alpha Y) \tag{2.8}$$

for all $X, Y \in \Gamma(T\overline{M})$ and $\alpha = 1, 2, 3$.

Note that if $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ is a manifold with a metric mixed 3-structure then from (2.8) it follows

$$\overline{g}(\xi_1, \xi_1) = \overline{g}(\xi_2, \xi_2) = -\overline{g}(\xi_3, \xi_3).$$

Hence the vector fields ξ_1 and ξ_2 are both either space-like or time-like and these force the causal character of the third vector field ξ_3. We may therefore distinguish between positive and negative metric mixed 3-structures, according as ξ_1 and ξ_2 are both space-like, or both time-like vector fields. Because one can check that, at each point of $\overline{M}$, there always exists a pseudo-orthonormal frame field given by $\{(E_i, \varphi_1 E_i, \varphi_2 E_i, \varphi_3 E_i)_{i=\overline{1,n}}, \xi_1, \xi_2, \xi_3\}$ we conclude that the dimension of the manifold is $4n + 3$ and the signature of $\overline{g}$ is $(2n + 1, 2n + 2)$, where we put first the minus signs, if the metric mixed 3-structure is positive (i.e. $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1$), or the signature of $\overline{g}$ is $(2n + 2, 2n + 1)$, if the metric mixed 3-structure is negative (i.e. $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -1$).

Remark 2.3. For the time being, it is not known wether a mixed 3-structure always admits both positive and negative compatible semi-Riemannian metrics or not. The cited result in [4] suggests a negative answer, but we do not have a proof.

Definition 2.4. Let $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ be a manifold with a metric mixed 3-structure.

(i) If $(\varphi_1, \xi_1, \eta_1, \overline{g})$, $(\varphi_2, \xi_2, \eta_2, \overline{g})$ are para-cosymplectic structures and $(\varphi_3, \xi_3, \eta_3, \overline{g})$ is a cosymplectic structure, i.e. the Levi-Civita connection $\overline{\nabla}$ of $\overline{g}$ satisfies

$$\overline{\nabla}\varphi_\alpha = 0 \tag{2.9}$$

for all $\alpha \in \{1, 2, 3\}$, then $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ is said to be a mixed 3-cosymplectic structure on $\overline{M}$.

(ii) If $(\varphi_1, \xi_1, \eta_1, \overline{g})$ and $(\varphi_2, \xi_2, \eta_2, \overline{g})$ are para-Sasakian structures and $(\varphi_3, \xi_3, \eta_3, \overline{g})$ is a Sasakian structure, i.e.

$$(\overline{\nabla}_X \varphi_\alpha)Y = \tau_\alpha[g(X, Y)\xi_\alpha - \varepsilon_\alpha \eta_\alpha(Y)X] \tag{2.10}$$

for all $X, Y \in \Gamma(T\overline{M})$ and $\alpha \in \{1, 2, 3\}$, then $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ is said to be a mixed 3-Sasakian structure on $\overline{M}$.

Note that from (2.9) it follows:

$$\overline{\nabla}\xi_\alpha = 0, \text{ (and hence } \overline{\nabla}\eta_\alpha = 0), \tag{2.11}$$

and from (2.10) we obtain

$$\overline{\nabla}_X \xi_\alpha = -\varepsilon_\alpha \varphi_\alpha X, \qquad (2.12)$$

for all $\alpha \in \{1, 2, 3\}$ and $X \in \Gamma(T\overline{M})$.

Like their Riemannian counterparts, mixed 3-Sasakian structures are Einstein, but now the scalar curvature can be either positive or negative (see [3, 10]):

Theorem 2.5. *Any $(4n + 3)$-dimensional manifold endowed with a mixed 3-Sasakian structure is an Einstein space with Einstein constant $\lambda = (4n+2)\varepsilon$, with $\varepsilon = \mp 1$, according as the metric mixed 3-structure is positive or negative, respectively.*

Several examples of manifolds endowed with metric mixed 3-structures are given in [9, 11]: $\mathbb{R}^{4n+3}_{2n+1}$ admits a positive mixed 3-cosymplectic structure, $\mathbb{R}^{4n+3}_{2n+2}$ admits a negative mixed 3-cosymplectic structure, the unit pseudo-sphere S^{4n+3}_{2n+1} and the real projective space $P^{4n+3}_{2n+1}(\mathbb{R})$ are the canonical examples of manifolds with positive mixed 3-Sasakian structures, while the unit pseudo-sphere S^{4n+3}_{2n+2} and the real projective space $P^{4n+3}_{2n+2}(\mathbb{R})$ can be endowed with negative mixed 3-Sasakian structures.

Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold and let M be an immersed submanifold of $\overline{M}$. Then M is said to be *non-degenerate* if the restriction of the semi-Riemannian metric $\overline{g}$ to TM is non-degenerate at each point of M. We denote by g the semi-Riemannian metric induced by $\overline{g}$ on M and by $TM^\perp$ the normal bundle to M. Then we have the following orthogonal decomposition:

$$T\overline{M} = TM \oplus TM^\perp.$$

Also, we denote by $\overline{\nabla}$ and ∇ the Levi-Civita connection on $\overline{M}$ and M, respectively. Then the Gauss formula is given by:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \qquad (2.13)$$

for all $X, Y \in \Gamma(TM)$, where $h : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^\perp)$ is the second fundamental form of M in $\overline{M}$.

On the other hand, the Weingarten formula is given by:

$$\overline{\nabla}_X N = -A_N X + \nabla^\perp_X N \qquad (2.14)$$

for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$, where $-A_N X$ is the tangential part of $\overline{\nabla}_X N$ and $\nabla^\perp_X N$ is the normal part of $\overline{\nabla}_X N$; A_N and $\nabla^\perp$ are called the shape operator of M with respect to N and the normal connection, respectively. Moreover, h and A_N are related by:

$$\overline{g}(h(X, Y), N) = g(A_N X, Y) \qquad (2.15)$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$.

For the rest of this paper we shall assume that the induced metric is non-degenerate.

3. Basic results

Definition 3.1. A non-degenerate submanifold M of a manifold $\overline{M}$ endowed with a metric mixed 3-structure $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ is said to be:

(i) *invariant* if $\varphi_\alpha(T_pM) \subset T_pM$, for all $p \in M$ and $\alpha = 1, 2, 3$;

(ii) *anti-invariant* if $\varphi_\alpha(T_pM) \subset T_pM^\perp$, for all $p \in M$ and $\alpha = 1, 2, 3$.

Lemma 3.2. *Manifolds with metric mixed 3-structure do not admit anti-invariant submanifolds tangent to the structure vector fields ξ_1, ξ_2, ξ_3.*

Proof. If we suppose that M is an anti-invariant submanifold of the manifold $\overline{M}$ endowed with a metric mixed 3-structure $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$, tangent to the structure vector fields, then it follows

$$\varphi_\alpha(\xi_\beta) \in T_pM^\perp, \ \alpha \neq \beta.$$

On the other hand, we have from (2.4) that

$$\varphi_\alpha(\xi_\beta) = \tau_\beta \xi_\gamma \in T_pM,$$

for any even permutation (α, β, γ) of $(1, 2, 3)$. So

$$\xi_\gamma \in T_pM \cap T_pM^\perp = \{0\},$$

which is a contradiction. $\qquad\square$

On the contrary, in mixed 3-Sasakian ambient, a submanifold normal to the structure fields is forced to be anti-invariant:

Lemma 3.3. *Let M be a non-degenerate m-dimensional submanifold of a $(4n+3)$-dimensional mixed 3-Sasakian manifold $((\overline{M}, \varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$. If the structure vector fields are normal to M, then M is anti-invariant and $m \leq n$.*

Proof. By using (2.12) and Weingarten formula, we obtain for all $X, Y \in \Gamma(TM)$:

$$\overline{g}(\varphi_\alpha X, Y) = -\varepsilon_\alpha \overline{g}(\overline{\nabla}_X \xi_\alpha, Y) = \varepsilon_\alpha g(A_{\xi_\alpha} X, Y)$$

and similarly we find

$$\overline{g}(\varphi_\alpha Y, X) = \varepsilon_\alpha g(A_{\xi_\alpha} Y, X).$$

But since A_ξ is a self-adjoint operator, it follows using also (2.8) that we have

$$\overline{g}(\varphi_\alpha X, Y) = 0, \forall X, Y \in \Gamma(TM), \ \alpha = 1, 2, 3.$$

Therefore M is anti-invariant and $m \leq n$ follows. $\qquad\square$

Corollary 3.4. *There do not exist invariant submanifolds in mixed 3-Sasakian manifolds normal to the structure vector fields. In particular, this is the case for the ambients: S_{2n+1}^{4n+3}, S_{2n+2}^{4n+3}, $P_{2n+1}^{4n+3}(\mathbb{R})$ and $P_{2n+2}^{4n+3}(\mathbb{R})$.*

Remark 3.5. Let $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ be a manifold endowed with a metric mixed 3-structure and let M be an anti-invariant submanifold of $\overline{M}$, such that the structure vector fields are not all normal to the submanifold. Hence we have $\xi_{\alpha p}^t \neq 0$, for $\alpha = 1, 2$ or 3, where $\xi_{\alpha p}^t$ denotes the tangential component of $\xi_{\alpha p}$, $p \in M$.

We consider the subspaces $\xi_p^t \subset T_pM$, $\xi_p^n \subset T_pM^\perp$, given by

$$\xi_p^t = Sp\{\xi_{1p}^t, \xi_{2p}^t, \xi_{3p}^t\}, \ \xi_p^n = Sp\{\xi_{1p}^n, \xi_{2p}^n, \xi_{3p}^n\},$$

where $\xi_{\alpha p}^n$ denotes the normal component of $\xi_{\alpha p}$, and let Q_p be the orthogonal complementary subspace to ξ_p^t in T_pM, $p \in M$. Therefore we have the decomposition $T_pM = \xi_p^t \oplus Q_p$.

Now, we put $\mathcal{D}_{ip} = \varphi_i(Q_p)$, $i \in \{1, 2, 3\}$, and note that $\mathcal{D}_{1p}, \mathcal{D}_{2p}, \mathcal{D}_{3p}$ are mutually orthogonal non-degenerate vector subspaces of $T_pM^\perp$. Moreover, if we let $\mathcal{D}_p = \mathcal{D}_{1p} \oplus \mathcal{D}_{2p} \oplus \mathcal{D}_{3p}$ we note that $\mathcal{D}_p$ and ξ_p^n are also mutually orthogonal non-degenerate vector subspaces of $T_pM^\perp$. Letting $\mathcal{D}_p^\perp$ be the orthogonal complementary subspace of $\xi_p^n \oplus \mathcal{D}_p$ in $T_pM^\perp$, we have the orthogonal decomposition $T_pM^\perp = \xi_p^n \oplus \mathcal{D}_p \oplus \mathcal{D}_p^\perp$. Note that $\mathcal{D}_p^\perp$ is invariant with respect to φ_i, $i \in \{1, 2, 3\}$.

We now prove a rather unexpected result concerning the dimensions of subspaces $\xi_p^t \subset T_pM$ and $\xi_p^n \subset T_pM^\perp$.

Proposition 3.6. *Let $(\overline{M}^{4n+3}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ be a manifold endowed with a metric mixed 3-structure and let M be an anti-invariant submanifold of $\overline{M}$, such that the structure vector fields are not all normal to the submanifold. Then $\dim \xi_p^t = 1$ and $\dim \xi_p^n = 2$.*

Proof. We put $q = \dim \xi_p^t$, $r = \dim \xi_p^n$. If the dimension of Q_p is s, then it is obvious that the dimension of $\mathcal{D}_p$ is $3s$. On the other hand, since the subspace $\mathcal{D}_p^\perp$ is invariant with respect to each φ_α, it follows that its dimension is $4t$. Taking into account that we have the decomposition

$$T_p\overline{M} = T_pM \oplus T_pM^\perp = \xi_p^t \oplus Q_p \oplus \xi_p^n \oplus \mathcal{D}_p \oplus \mathcal{D}_p^\perp$$

we obtain

$$4n + 3 = 4t + 4s + r + q$$

and so we deduce that $q + r \equiv 3 \mod 4$. In view of Lemma 3.2 and since $\xi_{\alpha p}^t \neq 0$, for $\alpha = 1, 2$ or 3, we have that $q, r \in \{1, 2, 3\}$ and so we conclude that $(q = 1, \ r = 2)$ or $(q = 2, \ r = 1)$.

We distinguish two cases.

Case I. If $\xi_{\alpha p}^n = 0$, then ξ_α is tangent to M and using (2.4) and taking into account that M is anti-invariant, we obtain that ξ_β and ξ_γ are both normal to M, where $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$. Therefore we have $q = 1$ and $r = 2$.

Case II. If $\xi_{\alpha p}^n \neq 0$, then we prove that is not possible to have $q = 2$ and $r = 1$. Indeed, if $r = 1$, then

$$\xi_{\beta p}^n = a\xi_{\alpha p}^n, \ \xi_{\gamma p}^n = b\xi_{\alpha p}^n,$$

where $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$, and from (2.8) we obtain

$$g(\varphi_\alpha \xi_{\alpha p}^n, \xi_{\alpha p}^n) = g(\varphi_\alpha \xi_{\alpha p}^n, \xi_{\beta p}^n) = g(\varphi_\alpha \xi_{\alpha p}^n, \xi_{\gamma p}^n) = 0. \qquad (3.1)$$

Since each η_i vanishes on Q_p, $i \in \{1, 2, 3\}$, making use of (2.6), (2.7) and (2.8) we derive for all $X \in Q_p$:

$$g(\varphi_\alpha \xi_{\alpha p}^n, \varphi_\alpha X) = g(\varphi_\alpha \xi_{\alpha p}^n, \varphi_\beta X) = g(\varphi_\alpha \xi_{\alpha p}^n, \varphi_\gamma X) = 0. \qquad (3.2)$$

On the other hand, since $\mathcal{D}_p^\perp$ is invariant with respect to φ_α, we also obtain using (2.8) that we have:

$$g(\varphi_\alpha \xi_{\alpha p}^n, U) = -g(\xi_{\alpha p}^n, \varphi_\alpha U) = 0, \qquad (3.3)$$

for all $U \in \mathcal{D}_p^\perp$.

From (3.1), (3.2) and (3.3) we deduce that $\varphi_\alpha \xi_{\alpha p}^n \in T_p M$. On the other hand, taking account of (2.2) and since M is anti-invariant, we obtain

$$\varphi_\alpha \xi_{\alpha p}^n = -\varphi_\alpha \xi_{\alpha p}^t \in T_p M^\perp.$$

Therefore it follows that $\varphi_\alpha \xi_{\alpha p}^n = 0$ and using (2.1) we get

$$0 = \varphi_\alpha^2 \xi_{\alpha p}^n = \tau_\alpha [\eta_\alpha(\xi_{\alpha p}^n)\xi_{\alpha p}^t + (\eta_\alpha(\xi_{\alpha p}^n) - 1)\xi_{\alpha p}^n],$$

which leads to a contradiction: $0 = \eta_\alpha(\xi_{\alpha p}^n) = 1$. Therefore it is not possible that $q = 2$ and $r = 1$. $\qquad \square$

Corollary 3.7. *Let $(\overline{M}^{4n+3}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ be a manifold endowed with a metric mixed 3-structure and let M be an anti-invariant submanifold of $\overline{M}$, such that $\xi_{\alpha p}^t \neq 0$, for all $p \in M$ and $\alpha = 1, 2$ or 3. Then it follows that the mapping $\xi : p \in M \mapsto \xi_p^t \subset T_p M$ defines a non-degenerate distribution of dimension 1 on M.*

In general, an invariant submanifold of a mixed 3-structure is either tangent or normal to *all* the three structure vector fields (this is the motivation for the analysis in the last two sections of the paper):

Proposition 3.8. *Let $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ be a manifold endowed with a metric mixed 3-structure and let M be an invariant submanifold of $\overline{M}$. Then the structure vector fields are all either tangent or normal to the submanifold.*

Proof. We suppose that we have the decomposition:

$$\xi_\alpha = \xi_\alpha^t + \xi_\alpha^n, \qquad (3.4)$$

where ξ_α^t denotes the tangential component of ξ_α and ξ_α^n is the normal component of ξ_α.

Applying now φ_α in (3.4) and taking account of (2.2) we obtain:

$$\varphi_\alpha \xi_\alpha^n = -\varphi_\alpha \xi_\alpha^t \in \Gamma(TM),$$

since M is an invariant submanifold of $\overline{M}$.

On the other hand, we derive from (2.8) that we have for all $X \in \Gamma(TM)$:

$$g(\varphi_\alpha \xi_\alpha^n, X) = -\overline{g}(\xi_\alpha^n, \varphi_\alpha X) = 0.$$

Therefore we deduce that $\varphi_\alpha \xi_\alpha^n = 0$ and so $\varphi_\alpha \xi_\alpha^t = 0$. Using now (2.1) and (3.4) we find

$$
\begin{aligned}
0 = \varphi_\alpha^2 \xi_\alpha^t &= \tau_\alpha[-\xi_\alpha^t + \eta_\alpha(\xi_\alpha^t)\xi_\alpha] \\
&= \tau_\alpha[(\eta_\alpha(\xi_\alpha^t) - 1)\xi_\alpha^t + \eta_\alpha(\xi_\alpha^t)\xi_\alpha^n].
\end{aligned}
$$

Consequently, if $\xi_\alpha^t \neq 0$ and $\xi_\alpha^n \neq 0$, we obtain a contradiction equating the tangential and normal components in the above relation. Hence we deduce that ξ_α is either tangent or normal to the submanifold. Finally, it is obvious that if one of the structure vector fields is tangent to the submanifold, then from (2.4) it follows that the next two structure vector fields are also tangent to the submanifold, because the tangent space of an invariant submanifold is closed under the action of $(\varphi_\alpha)_{\alpha=\overline{1,3}}$.   □

As in the Riemannian case, we have:

Proposition 3.9. *Let $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ be a mixed 3-cosymplectic or mixed 3-Sasakian manifold and let M be a totally umbilical submanifold tangent to the structure vector fields. Then M is totally geodesic.*

Proof. If $\overline{M}$ is a mixed 3-cosymplectic manifold, then from Gauss formula and (2.11) we obtain:

$$
0 = \overline{\nabla}_X \xi_\alpha = \nabla_X \xi_\alpha + h(X, \xi_\alpha)
$$

for all $X \in \Gamma(TM)$ and $\alpha = 1, 2, 3$. Therefore, equating the normal components we find:

$$
h(X, \xi_\alpha) = 0. \tag{3.5}
$$

If $\overline{M}$ is a mixed 3-Sasakian manifold, then from Gauss formula and (2.12) we similarly obtain:

$$
-\varepsilon_\alpha \varphi_\alpha X = \overline{\nabla}_X \xi_\alpha = \nabla_X \xi_\alpha + h(X, \xi_\alpha).
$$

Taking $X = \xi_\alpha$ in the above equality and using (2.2) we derive:

$$
0 = \nabla_{\xi_\alpha} \xi_\alpha + h(\xi_\alpha, \xi_\alpha)
$$

and so we get:

$$
h(\xi_\alpha, \xi_\alpha) = 0. \tag{3.6}
$$

On the other hand, since M is totally umbilical, its second fundamental form satisfies:

$$
h(X, Y) = g(X, Y)H \tag{3.7}
$$

for all $X, Y \in \Gamma(TM)$, where H is the mean curvature vector field on M.

Taking $X = Y = \xi_\alpha$ in (3.7) and using (3.5) - if the manifold $\overline{M}$ is mixed 3-cosymplectic, or (3.6) - if the manifold $\overline{M}$ is mixed 3-Sasakian, we obtain

$$
0 = \varepsilon_\alpha H
$$

and therefore $H = 0$. Using again (3.7) we obtain the assertion.   □

Corollary 3.10. *A totally geodesic submanifold of a mixed 3-Sasakian manifold, tangent to the structure vector fields, is invariant.*

Proof. From (2.12) we obtain that

$$\varphi_\alpha X = -\varepsilon_\alpha \nabla_X \xi_\alpha \in \Gamma(TM), \ \forall X \in \Gamma(TM)$$

and the conclusion follows. $\qquad \square$

3.1. Examples

3.1.1. Images of holomorphic maps. Let M, M' be manifolds endowed with metric mixed 3-structures $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$, $((\varphi'_\alpha, \xi'_\alpha, \eta'_\alpha)_{\alpha=\overline{1,3}}, g')$. We say that a smooth map $f : M \to N$ is holomorphic if the equation

$$f_* \circ \varphi_\alpha = \varphi'_\alpha \circ f_* \tag{3.8}$$

holds for all $\alpha \in \{1, 2, 3\}$.

We remark now that if f is an holomorphic embedding such that the image of f, denoted by $N' = f(M)$, is a non-degenerate submanifold, then it is an invariant submanifold. Indeed, if we consider $X_*, Y_* \in \Gamma(TN')$ such that $f_*X = X_*$ and $f_*Y = Y_*$, where $X, Y \in \Gamma(TM)$, we obtain using (3.8):

$$\varphi'_\alpha X_* = \varphi'_\alpha f_* X = f_*(\varphi_\alpha X) \in \Gamma(TN')$$

and therefore N' is an invariant submanifold of M'.

On the other hand, we can remark that if M is a manifold endowed with a metric mixed 3-structure $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ and M' is an invariant submanifold of M, tangent to the structure vector fields, then the restriction of $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ to M' is a metric mixed 3-structure and the inclusion map $i : M' \to M$ is holomorphic.

3.1.2. Correspondence between submanifolds of mixed 3-Sasakian manifolds and paraquaternionic Kähler manifolds via semi-Riemannian submersions. Consider the semi-Riemannian submersion $\pi : S^{4n+3}_{2n+1} \to P^n(\mathbb{B})$, with totally geodesic fibres S^3_1. It was used by Blažić in order to give a natural and geometrically oriented definition of the paraquaternionic projective space [2]. If $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ is the standard positive mixed 3-Sasakian structure on S^{4n+3}_{2n+1} (see [9]), then the semi-Riemannian metric g' of $P^n(\mathbb{B})$ is induced by

$$g'(X', Y') \circ \pi = g(X^h, Y^h),$$

for all vector fields $X', Y' \in \Gamma(P^n(\mathbb{B}))$, where X^h, Y^h are the unique horizontal lifts of X', Y' on S^{4n+3}_{2n+1}. Moreover, each canonical local basis $H = (J_\alpha)_{\alpha=\overline{1,3}}$ of $P^n(\mathbb{B})$ is related with structures $(\varphi_\alpha)_{\alpha=\overline{1,3}}$ of S^{4n+3}_{2n+1} by

$$J_\alpha X' = \pi_*(\varphi_\alpha X^h),$$

for any $X' \in \Gamma(P^n(\mathbb{B}))$.

Let now M be an immersed submanifold of S^{4n+3}_{2n+1} and let N be an immersed submanifold of $P^n(\mathbb{B})$ such that $\pi^{-1}(N) = M$. Then we have that N is a paraquaternionic (respectively totally real) submanifold of $P^n(\mathbb{B})$ if and only if M is an invariant (respectively anti-invariant) submanifold of S^{4n+3}_{2n+1}, tangent (respectively normal) to the structure vector fields.

In particular, if we consider the canonical paraquaternionic immersion $i : P^m(\mathbb{B}) \to P^n(\mathbb{B})$, where $m < n$, we obtain that $M = S^{4m+3}_{2m+1}$ is an

invariant totally geodesic submanifold of S_{2n+1}^{4n+3}, tangent to the structure vector fields. Similarly, if we take the standard totally real immersion $i :$ $P_\nu^m(\mathbb{R}) \to P^n(\mathbb{B})$, where $m \le n$ and $\nu \in \{0, ..., m\}$, we conclude that $M = S_\nu^m$ is an anti-invariant totally geodesic submanifold of S_{2n+1}^{4n+3}, normal to the structure vector fields.

Moreover, it can be proved that if $\pi : M \to N$ is a semi-Riemannian submersion from a mixed 3-Sasakian manifold onto a paraquaternionic Kähler manifold which commutes with the structure tensors of type $(1, 1)$ (we note that the corresponding notion in the Riemannian case was studied in [19]), and M', N' are immersed submanifolds of M and N respectively, such that $\pi^{-1}(N') = M'$, then M' is an invariant (respectively anti-invariant) submanifold of M, tangent (respectively normal) to the structure vector fields if and only if N' is a paraquaternionic (respectively totally real) submanifold of N.

3.1.3. Fibre submanifolds of a semi-Riemannian submersion. Let π be a semi-Riemannian submersion from a manifold M endowed with a metric mixed 3-structure $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ onto an almost hermitian paraquaternionic manifold (N, σ, g'), which commutes with the structure tensors of type $(1, 1)$. The horizontal and vertical distributions induced by π are closed under the action of φ_α, $\alpha = 1, 2, 3$, and therefore we conclude that the fibres are invariant submanifolds of M. Moreover, we have

$$J_\alpha \pi_* \xi_\alpha = \pi_* \varphi_\alpha \xi_\alpha = 0,$$

for $\alpha = 1, 2, 3$, and hence we deduce that ξ_1, ξ_2, ξ_3 are vertical vector fields.

In particular, since the semi-Riemannian submersion $\pi : S_{2n+1}^{4n+3} \to$ $P^n(\mathbb{B})$ given above commutes with the structure tensors of type $(1, 1)$, we have that S_1^3 is an invariant submanifold of S_{2n+1}^{4n+3}, tangent to the structure vector fields.

3.1.4. The Clifford torus $S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}}) \subset S_3^7$. Let S_3^7 be the 7-dimensional unit pseudo-sphere in $\mathbb{R}_4^8$, endowed with standard positive mixed 3-Sasakian structure $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ (see [9]). Let $H = \{J_1, J_2, J_3\}$ be the almost para-hypercomplex structure of $\mathbb{R}_4^8$ defined by

$$J_1((x_i)_{i=\overline{1,8}}) = (-x_7, x_8, -x_5, x_6, -x_3, x_4, -x_1, x_2),$$

$$J_2((x_i)_{i=\overline{1,8}}) = (x_8, x_7, x_6, x_5, x_4, x_3, x_2, x_1),$$

$$J_3((x_i)_{i=\overline{1,8}}) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7),$$

which is compatible with the semi-Riemannian $\overline{g}$ on $\mathbb{R}_4^8$, given by

$$\overline{g}((x_i)_{i=\overline{1,8}}, (y_i)_{i=\overline{1,8}}) = -\sum_{i=1}^{4} x_i y_i + \sum_{i=5}^{8} x_i y_i.$$

If $S^1(\frac{1}{\sqrt{2}})$ is a circle of radius $\frac{1}{\sqrt{2}}$, we consider the submanifold $M = S^1(\frac{1}{\sqrt{2}}) \times S^1(\frac{1}{\sqrt{2}})$ of S_3^7. The position vector X of M in S_3^7 in $\mathbb{R}_4^8$ has components given by

$$N = \frac{1}{\sqrt{2}}(0, 0, 0, 0, \cos u_1, \sin u_1, \cos u_2, \sin u_2),$$

u_1 and u_2 being parameters on each S^1.

The tangent space is spanned by $\{X_1, X_2\}$, where

$$X_1 = \frac{1}{\sqrt{2}}(0, 0, 0, 0, -\sin u_1, \cos u_1, 0, 0),$$

$$X_2 = \frac{1}{\sqrt{2}}(0, 0, 0, 0, 0, 0, -\sin u_2, \cos u_2)$$

and the structure vector fields ξ_1, ξ_2, ξ_3 of S_3^7 restricted to M are given by

$$\xi_1 = \frac{1}{\sqrt{2}}(\cos u_2, -\sin u_2, \cos u_1, -\sin u_1, 0, 0, 0, 0),$$

$$\xi_2 = \frac{1}{\sqrt{2}}(-\sin u_2, -\cos u_2, -\sin u_1, -\cos u_1, 0, 0, 0, 0),$$

$$\xi_3 = \frac{1}{\sqrt{2}}(0, 0, 0, 0, \sin u_1, -\cos u_1, \sin u_2, -\cos u_2).$$

Since $\varphi_\alpha X$ is the tangent part of $J_\alpha X$, for all $X \in \Gamma(TM)$ and $\alpha \in \{1, 2, 3\}$ (see [9]), we obtain:

$$g(\varphi_\alpha X_i, X_j) = \overline{g}(J_\alpha X_i, X_j) = 0,$$

for all $\alpha \in \{1, 2, 3\}$ and $i, j \in \{1, 2\}$. Therefore M is an anti-invariant submanifold of S_3^7. On the other hand, it is easy to verify that ξ_1, ξ_2 are normal to M and since $\xi_3 = -X_1 - X_2$, we deduce that ξ_3 is tangent to the submanifold.

4. Anti-invariant submanifolds of manifolds endowed with metric mixed 3-structures, normal to the structure vector fields

Let M be an n-dimensional anti-invariant submanifold of a manifold endowed with a metric mixed 3-structure $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$. From Lemma 3.2 it follows that the structure vector fields ξ_1, ξ_2, ξ_3 cannot be tangent to M, unlike the case of anti-invariant submanifolds in manifolds endowed with almost contact structures, where the structure vector field can be both tangent and normal (see [12, 13, 15, 21]). Next we suppose that the structure vector fields are normal to M.

Define the distribution $\xi = \{\xi_1\} \oplus \{\xi_2\} \oplus \{\xi_3\}$ and set $\mathcal{D}_{\alpha p} = \varphi_\alpha(T_p M)$, for $p \in M$ and $\alpha = 1, 2, 3$. We note that $\mathcal{D}_{1p}, \mathcal{D}_{2p}, \mathcal{D}_{3p}$ are mutually orthogonal non-degenerate vector subspaces of $T_p M^\perp$. Indeed, by using (2.6) and (2.8) we obtain

$$\overline{g}(\varphi_\alpha X, \varphi_\beta Y) = -\overline{g}(X, \varphi_\alpha \varphi_\beta Y) = -\tau_\gamma \overline{g}(X, \varphi_\gamma Y) = 0$$

for all $X, Y \in T_p M$, where (α, β, γ) is an even permutation of $(1, 2, 3)$.

Moreover, the subspaces

$$\mathcal{D}_p = \mathcal{D}_{1p} \oplus \mathcal{D}_{2p} \oplus \mathcal{D}_{3p}, \; p \in M$$

define a non-trivial subbundle of dimension $3n$ on $TM^\perp$. Note that $\mathcal{D}$ and ξ are mutually orthogonal subbundle of $TM^\perp$ and let $\mathcal{D}^\perp$ be the orthogonal complementary vector subbundle of $\mathcal{D} \oplus \xi$ in $TM^\perp$. So we have the orthogonal decomposition:

$$TM^\perp = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \xi.$$

Lemma 4.1.   (i) $\varphi_\alpha \mathcal{D}_{\alpha p} \subset T_p M$, $\forall p \in M$, $\alpha = 1, 2, 3$.

(ii) $\varphi_\alpha \mathcal{D}_{\beta p} \subset \mathcal{D}_{\gamma p}$, $\forall p \in M$, $\alpha = 1, 2, 3$.

(iii) *The subbundle $\mathcal{D}^\perp$ is invariant under the action of φ_α, $\alpha = 1, 2, 3$.*

(iv) $\varphi_\alpha^2 (TM^\perp) \subset TM^\perp$, $\forall \alpha = 1, 2, 3$.

Proof. (iv) is a consequence of the first three claims. (i) and (ii) follow, respectively, from (2.1) and (2.6). It remains to prove (iii). If $U \in \Gamma(\mathcal{D}^\perp)$, then using (2.2) and (2.8) we obtain

$$\overline{g}(\varphi_\alpha U, \xi_\alpha) = -\overline{g}(U, \varphi_\alpha \xi_\alpha) = 0, \; \alpha = 1, 2, 3.$$

Similarly, using (2.4) and (2.8) we get

$$\overline{g}(\varphi_\alpha U, \xi_\beta) = -\overline{g}(U, \varphi_\alpha \xi_\beta) = -\tau_\beta \overline{g}(U, \xi_\gamma) = 0$$

for any even permutation (α, β, γ) of $(1, 2, 3)$.

On the other hand, if $U \in \Gamma(\mathcal{D}^\perp)$ and $X \in \Gamma(TM)$, then using (2.1) and (2.8) we obtain:

$$\overline{g}(\varphi_\alpha U, \varphi_\alpha X) = -\overline{g}(U, \varphi_\alpha^2 X) = \tau_\alpha \overline{g}(U, X) = 0, \; \alpha = 1, 2, 3$$

and similarly, using (2.6) and (2.8) we have

$$\overline{g}(\varphi_\alpha U, \varphi_\beta X) = -\overline{g}(U, \varphi_\alpha \varphi_\beta X) = -\tau_\gamma \overline{g}(U, \varphi_\gamma X) = 0$$

for any even permutation (α, β, γ) of $(1, 2, 3)$. This ends the proof. $\square$

Lemma 4.2. *If M is an anti-invariant submanifold of a mixed 3-cosymplectic or mixed 3-Sasakian manifold $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$, normal to the structure vector fields, then the distribution ξ on $\overline{M}$ is integrable.*

Proof. If $\overline{M}$ is a mixed 3-cosymplectic manifold then the assertion is a direct consequence of (2.11). On the other hand, if M is a mixed 3-Sasakian manifold, then using (2.4) and (2.12) we obtain for any $N \in \Gamma(\mathcal{D} \oplus \mathcal{D}^\perp)$:

$$\overline{g}([\xi_\alpha, \xi_\beta], N) = (\varepsilon_\beta \tau_\alpha + \varepsilon_\alpha \tau_\beta) g(\xi_\gamma, N) = 0$$

for any even permutation (α, β, γ) of $(1, 2, 3)$. $\square$

Lemma 4.3. *If M is an anti-invariant submanifold of a mixed 3-cosymplectic or mixed 3-Sasakian manifold $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$, normal to the structure vector fields, then the following equation holds good:*

$$R^\perp(X, Y)\xi_\alpha = 0, \; \forall X, Y \in \Gamma(TM), \; \alpha = 1, 2, 3.$$

Proof. From the Weingarten formula we have for any $X \in \Gamma(TM)$ and $\alpha = 1, 2, 3$:

$$\overline{\nabla}_X \xi_\alpha = -A_{\xi_\alpha} X + \nabla_X^\perp \xi_\alpha. \tag{4.1}$$

If $\overline{M}$ is mixed 3-cosymplectic, then identifying the normal components in (2.11) and (4.1) we obtain:

$$\nabla_X^\perp \xi_\alpha = 0, \forall X \in \Gamma(TM), \ \alpha = 1, 2, 3,$$

and the conclusion follows.

If $\overline{M}$ is mixed 3-Sasakian, from (2.12) and (4.1) we obtain in a similar way that

$$\nabla_X^\perp \xi_\alpha = -\varepsilon_\alpha \varphi_\alpha X, \ \forall X \in \Gamma(TM), \ \alpha = 1, 2, 3. \tag{4.2}$$

Using now the Gauss and Weingarten formulas, we get

$$(\overline{\nabla}_X \varphi_\alpha)Y = -A_{\varphi_\alpha Y} X + \nabla_X^\perp \varphi_\alpha Y - \varphi_\alpha \nabla_X Y - \varphi_\alpha h(X, Y), \tag{4.3}$$

for $X, Y \in \Gamma(TM)$ and $\alpha = 1, 2, 3$.

On the other hand, from (2.10) we obtain

$$(\overline{\nabla}_X \varphi_\alpha)Y = \tau_\alpha g(X, Y)\xi_\alpha.$$

Identifying now the normal components in the last two equations we derive:

$$\nabla_X^\perp \varphi_\alpha Y = \tau_\alpha g(X, Y)\xi_\alpha + \varphi_\alpha \nabla_X Y + (\varphi_\alpha h(X, Y))^n \tag{4.4}$$

where $(\varphi_\alpha h(X, Y))^n$ denotes the normal component of $\varphi_\alpha h(X, Y)$.

Using now (4.2) and (4.4) we deduce

$$\nabla_X^\perp \nabla_Y^\perp \xi_\alpha = -\varepsilon_\alpha [\tau_\alpha g(X, Y)\xi_\alpha + \varphi_\alpha \nabla_X Y + (\varphi_\alpha h(X, Y))^n]$$

and

$$\nabla_Y^\perp \nabla_X^\perp \xi_\alpha = -\varepsilon_\alpha [\tau_\alpha g(Y, X)\xi_\alpha + \varphi_\alpha \nabla_Y X + (\varphi_\alpha h(Y, X))^n].$$

Finally we derive:

$$\begin{aligned} R^\perp(X, Y)\xi_\alpha &= \nabla_X^\perp \nabla_Y^\perp \xi_\alpha - \nabla_Y^\perp \nabla_X^\perp \xi_\alpha - \nabla_{[X,Y]}^\perp \xi_\alpha \\ &= \varepsilon_\alpha(\varphi_\alpha \nabla_Y X - \varphi_\alpha \nabla_X Y) + \varepsilon_\alpha \varphi_\alpha [X, Y] \\ &= 0. \qquad \qquad \square \end{aligned}$$

We can now prove the main result of this section: the flatness of the normal connection of an anti-invariant submanifold in a mixed 3-Sasakian or mixed 3-cosymplectic manifold implies strong restrictions on the behavior of the submanifold (compare with [20, Theorem] for totally real submanifolds in Kähler manifolds (where flat normal connection implies flatness of the submanifold) and with [15, Proposition 11] and [21, Corollary 2.1, page 126], for anti-invariant submanifolds in Sasakian manifolds).

Theorem 4.4. *Let M be an anti-invariant submanifold of minimal codimension in a manifold $\overline{M}$ with a metric mixed 3-structure $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$, such that the structure vector fields are normal to M.*

  (i) *If $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ is a mixed 3-cosymplectic manifold, then $R^\perp \equiv 0$ if and only if $R \equiv 0$.*

(ii) *If* $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ *is a mixed 3-Sasakian manifold, then the connection in the normal bundle is trivial if and only if M is of constant sectional curvature ∓ 1, according as the metric mixed 3-structure is positive or negative, respectively.*

Proof. If the dimension of $\overline{M}$ is $(4m + 3)$, since the submanifold M is of minimal codimension, then from Lemma 3.3 it follows that the dimension of M is m and so $\mathcal{D}^\perp = \{0\}$. Therefore we have the orthogonal decomposition

$$TM^\perp = \mathcal{D} \oplus \xi.$$

On the other hand, identifying the tangential components, from (2.11) and (4.1) - if $\overline{M}$ is a mixed 3-cosymplectic manifold, or from (2.12) and (4.1) - if $\overline{M}$ is mixed 3-Sasakian manifold, it follows that

$$A_{\xi_\alpha} X = 0, \ \forall X \in \Gamma(TM), \ \alpha = 1, 2, 3. \tag{4.5}$$

From (2.15) and (4.5) we can deduce

$$h(X, Y) \in \Gamma(\mathcal{D}), \ \forall X, Y \in \Gamma(TM)$$

and so we have

$$\varphi_\alpha h(X, Y) \in \Gamma(TM), \ \forall X, Y \in \Gamma(TM), \ \alpha = 1, 2, 3. \tag{4.6}$$

Suppose now that $\overline{M}$ is mixed 3-cosymplectic. Then, taking into account (2.9) and (4.6) in (4.3) and equating the normal components, we obtain

$$\nabla_X^\perp \varphi_\alpha Y = \varphi_\alpha \nabla_X Y, \ \forall X, Y \in \Gamma(TM), \ \alpha = 1, 2, 3. \tag{4.7}$$

Using now (4.7) we obtain for all $X, Y, Z \in \Gamma(TM)$ and $\alpha \in \{1, 2, 3\}$:

$$
\begin{aligned}
R^\perp(X, Y)\varphi_\alpha Z &= \nabla_X^\perp \nabla_Y^\perp \varphi_\alpha Z - \nabla_Y^\perp \nabla_X^\perp \varphi_\alpha Z - \nabla_{[X,Y]}^\perp \varphi_\alpha Z \\
&= \nabla_X^\perp (\varphi_\alpha \nabla_Y Z) - \nabla_Y^\perp (\varphi_\alpha \nabla_X Z) - \varphi_\alpha \nabla_{[X,Y]} Z \\
&= \varphi_\alpha \nabla_X \nabla_Y Z - \varphi_\alpha \nabla_Y \nabla_X Z - \varphi_\alpha \nabla_{[X,Y]} Z \\
&= \varphi_\alpha R(X, Y)Z
\end{aligned}
$$

and (i) follows from the above equation and Lemma 4.3.

For (ii), let $\overline{M}$ be mixed 3-Sasakian. Then from (4.4) and (4.6) we deduce that we have for all $X, Y \in \Gamma(TM)$ and $\alpha \in \{1, 2, 3\}$:

$$\nabla_X^\perp \varphi_\alpha Y = \tau_\alpha g(X, Y)\xi_\alpha + \varphi_\alpha \nabla_X Y. \tag{4.8}$$

Using now (4.2) and (4.8) we derive

$$
\begin{aligned}
R^{\perp}(X,Y)\varphi_\alpha Z &= \nabla_X^{\perp}\nabla_Y^{\perp}\varphi_\alpha Z - \nabla_Y^{\perp}\nabla_X^{\perp}\varphi_\alpha Z - \nabla_{[X,Y]}^{\perp}\varphi_\alpha Z \\
&= \nabla_X^{\perp}[\tau_\alpha g(Y,Z)\xi_\alpha + \varphi_\alpha \nabla_Y Z] \\
&\quad -\nabla_Y^{\perp}[\tau_\alpha g(X,Z)\xi_\alpha + \varphi_\alpha \nabla_X Z] \\
&\quad -[\tau_\alpha g([X,Y],Z)\xi_\alpha + \varphi_\alpha \nabla_{[X,Y]}Z] \\
&= \tau_\alpha X g(Y,Z)\xi_\alpha - \varepsilon_\alpha \tau_\alpha g(Y,Z)\varphi_\alpha X + \tau_\alpha g(X,\nabla_Y Z)\xi_\alpha \\
&\quad -\tau_\alpha Y g(X,Z)\xi_\alpha + \varepsilon_\alpha \tau_\alpha g(X,Z)\varphi_\alpha Y - \tau_\alpha g(Y,\nabla_X Z)\xi_\alpha \\
&\quad +\varphi_\alpha \nabla_X \nabla_Y Z - \varphi_\alpha \nabla_Y \nabla_X Z \\
&\quad -\tau_\alpha g([X,Y],Z)\xi_\alpha - \varphi_\alpha \nabla_{[X,Y]}Z) \\
&= \varphi_\alpha R(X,Y)Z - \varepsilon_\alpha \tau_\alpha [g(Y,Z)\varphi_\alpha X - g(X,Z)\varphi_\alpha Y] \\
&\quad +\tau_\alpha [X g(Y,Z) - Y g(X,Z) + g(X,\nabla_Y Z) \\
&\quad -g(Y,\nabla_X Z) - g(\nabla_X Y,Z) + g(\nabla_Y X,Z)]\xi_\alpha.
\end{aligned}
$$

Therefore, as ∇ is a Riemannian connection, we deduce

$$
R^{\perp}(X,Y)\varphi_\alpha Z = \varphi_\alpha R(X,Y)Z - \varepsilon_\alpha \tau_\alpha [g(Y,Z)\varphi_\alpha X - g(X,Z)\varphi_\alpha Y] \quad (4.9)
$$

for all $X,Y,Z \in \Gamma(TM)$ and $\alpha \in \{1,2,3\}$.

If the connection of the normal bundle is trivial, *i.e.* $R^{\perp} \equiv 0$, then from (4.9) we obtain that M has constant sectional curvature $\varepsilon_\alpha \tau_\alpha$. The conclusion follows now taking into account that $\varepsilon_\alpha \tau_\alpha = -1$ if the metric mixed 3-structure is positive, respectively $\varepsilon_\alpha \tau_\alpha = 1$ if the metric mixed 3-structure is negative.

Conversely, if M is of constant sectional curvature ∓ 1, according as the metric mixed 3-structure is positive or negative, then from (4.9) we obtain

$$
R^{\perp}(X,Y)\varphi_\alpha Z = 0, \ \forall X,Y,Z \in \Gamma(TM), \ \alpha = 1,2,3.
$$

On the other hand, from Lemma 4.3 we see that the curvature tensor of the normal bundle annihilates the structure vector fields. Therefore $R^{\perp} \equiv 0$, *i.e.* the connection in the normal bundle is trivial. $\qquad \square$

4.1. An example of an anti-invariant submanifold M of minimal codimension in a mixed 3-Sasakian manifold $\overline{M}$, such that the structure vector fields are normal to M

Let $H = \{J_1, J_2, J_3\}$ be the almost para-hypercomplex structure on $\mathbb{R}_{2n+2}^{4n+4}$, given by

$$
J_1((x_i)_{i=\overline{1,4n+4}}) = (-x_{4n+3}, x_{4n+4}, -x_{4n+1}, x_{4n+2}, ..., -x_3, x_4, -x_1, x_2),
$$

$$
J_2((x_i)_{i=\overline{1,4n+4}}) = (x_{4n+4}, x_{4n+3}, x_{4n+2}, x_{4n+1}, ..., x_4, x_3, x_2, x_1),
$$

$$
J_3((x_i)_{i=\overline{1,4n+4}}) = (-x_2, x_1, -x_4, x_3, ..., -x_{4n+2}, x_{4n+1}, -x_{4n+4}, x_{4n+3}).
$$

It is easily checked that the semi-Riemannian metric

$$
\overline{g}((x_i)_{i=\overline{1,4n+4}}, (y_i)_{i=\overline{1,4n+4}}) = -\sum_{i=1}^{2n+2} x_i y_i + \sum_{i=2n+3}^{4n+4} x_i y_i
$$

is adapted to the almost para-hypercomplex structure H given above.

Let S_{2n+1}^{4n+3} be the unit pseudo-sphere with standard positive mixed 3-Sasakian structure $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$. This structure is obtained by taking S_{2n+1}^{4n+3} as hypersurface of $(\mathbb{R}_{2n+2}^{4n+4}, \overline{g})$ (see [9]). Let T^n be the n-dimensional real torus $\underbrace{S^1 \times \ldots \times S^1}_{n}$, where S^1 is the unit circle. We can construct a minimal isometric immersion $f : T^n \to S_{2n+1}^{4n+3}$, defined by

$$f(u_1, \ldots, u_n) = \frac{1}{\sqrt{n+1}} (\underbrace{0, \ldots, 0}_{2n+2}, \cos x_1, \sin x_1, \ldots, \cos x_{n+1}, \sin x_{n+1}),$$

where

$$x_{n+1} = -\sum_{i=1}^{n} x_i, \ u_1 = (\cos x_1, \sin x_1), \ldots, u_n = (\cos x_n, \sin x_n).$$

The tangent space is spanned by $\{X_1, \ldots, X_n\}$, where:

$$X_1 = \frac{1}{\sqrt{n+1}} (\underbrace{0, \ldots, 0}_{2n+2}, -\sin x_1, \cos x_1, \underbrace{0, \ldots, 0}_{2n-2}, \sin x_{n+1}, -\cos x_{n+1}),$$

$$X_2 = \frac{1}{\sqrt{n+1}} (\underbrace{0, \ldots, 0}_{2n+4}, -\sin x_2, \cos x_2, \underbrace{0, \ldots, 0}_{2n-4}, \sin x_{n+1}, -\cos x_{n+1}),$$

$$\vdots$$

$$X_n = \frac{1}{\sqrt{n+1}} (\underbrace{0, \ldots, 0}_{4n}, -\sin x_n, \cos x_n, \sin x_{n+1}, -\cos x_{n+1}).$$

On the other hand, the position vector of T^n in $\mathbb{R}_{2n+2}^{4n+4}$ has components

$$N = \frac{1}{\sqrt{n+1}} (\underbrace{0, \ldots, 0}_{2n+2}, \cos x_1, \sin x_1, \ldots, \cos x_n, \sin x_n, \cos x_{n+1}, \sin x_{n+1})$$

and it is an outward unit spacelike normal vector field of the pseudo-sphere in $\mathbb{R}_{2n+2}^{4n+4}$. Therefore the structure vector fields ξ_1, ξ_2, ξ_3 of S_{2n+1}^{4n+3} restricted to T^n are given by

$$\xi_1 = \frac{1}{\sqrt{n+1}} (\cos x_{n+1}, -\sin x_{n+1}, \ldots, \cos x_1, -\sin x_1, \underbrace{0, \ldots, 0}_{2n+2}),$$

$$\xi_2 = \frac{1}{\sqrt{n+1}} (-\sin x_{n+1}, -\cos x_{n+1}, \ldots, -\sin x_1, -\cos x_1, \underbrace{0, \ldots, 0}_{2n+2}),$$

$$\xi_3 = \frac{1}{\sqrt{n+1}} (\underbrace{0, \ldots, 0}_{2n+2}, \sin x_1, -\cos x_1, \ldots, \sin x_{n+1}, -\cos x_{n+1}).$$

Finally, as the structure tensors $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$ of S_{2n+1}^{4n+3} satisfy

$$\varphi_\alpha X_i = J_\alpha X_i - \varepsilon_\alpha \eta_\alpha(X_i) N,$$

and

$$\eta_\alpha(X_i) = \varepsilon_\alpha \overline{g}(X_i, \xi_\alpha) = 0,$$

for all $i \in \{1, 2, \ldots, n\}$ and $\alpha \in \{1, 2, 3\}$, we conclude that the immersion f provides a non-trivial example of an anti-invariant flat minimal submanifold of S_{2n+1}^{4n+3}, normal to the structure vector fields.

5. Invariant submanifolds of manifolds endowed with metric mixed 3-structures, tangent to the structure vector fields

Let (M, g) be an invariant submanifold of a manifold endowed with a metric mixed 3-structure $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$, tangent to the structure vector fields ξ_1, ξ_2, ξ_3. As above, let $\xi = \{\xi_1\} \oplus \{\xi_2\} \oplus \{\xi_3\}$ and let $\mathcal{D}$ be the orthogonal complementary distribution to ξ in TM. Then we can state the following:

Lemma 5.1.   (i) $\varphi_\alpha(T_pM^\perp) \subset T_pM^\perp$, $\forall p \in M$, $\alpha = 1, 2, 3$.
  (ii) *The distribution $\mathcal{D}$ is invariant under the action of φ_α, $\alpha = 1, 2, 3$.*

Proof. (i) For any $N \in T_pM^\perp$ and $X \in T_pM$, taking account of (2.8) we obtain:
$$\overline{g}(\varphi_\alpha N, X) = -g(N, \varphi_\alpha X) = 0,$$
since M is an invariant submanifold.
    (ii) For any $X \in \Gamma(\mathcal{D})$, using (2.2) and (2.8) we obtain:
$$g(\varphi_\alpha X, \xi_\alpha) = -g(X, \varphi_\alpha \xi_\alpha) = 0$$
for $\alpha = 1, 2, 3$.
    Similarly, making use of (2.4) and (2.8), we deduce:
$$g(\varphi_\alpha X, \xi_\beta) = -g(X, \varphi_\alpha \xi_\beta) = -\tau_\beta g(X, \xi_\gamma) = 0$$
for any even permutation (α, β, γ) of $(1, 2, 3)$. $\square$

Proposition 5.2. *Let (M, g) be an invariant submanifold of a manifold endowed with a metric mixed 3-structure $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$, such that the structure vector fields ξ_1, ξ_2, ξ_3 are tangent to M. If $\overline{M}$ is mixed 3-cosymplectic or mixed 3-Sasakian, then M is mixed 3-cosymplectic and totally geodesic, respectively mixed 3-Sasakian and totally geodesic.*

Proof. Gauss equation implies:
$$(\overline{\nabla}_X \varphi_\alpha)Y = (\nabla_X \varphi_\alpha)Y + h(X, \varphi_\alpha Y) - \varphi_\alpha h(X, Y) \qquad (5.1)$$
for all $X, Y \in \Gamma(TM)$.
    If $\overline{M}$ is a mixed 3-cosymplectic manifold, then from (2.9) and (5.1) we deduce:
$$(\nabla_X \varphi_\alpha)Y + h(X, \varphi_\alpha Y) - \varphi_\alpha h(X, Y) = 0$$
and equating the normal and the tangential components we find
$$(\nabla_X \varphi_\alpha)Y = 0 \qquad (5.2)$$
and
$$h(X, \varphi_\alpha Y) = \varphi_\alpha h(X, Y), \ \alpha = 1, 2, 3.$$
    From (5.2) it follows that the induced metric mixed 3-structure on M is mixed 3-cosymplectic.

If $\overline{M}$ is a mixed 3-Sasakian manifold, then from (2.10) and (5.1) we deduce:

$$(\nabla_X\varphi_\alpha)Y + h(X, \varphi_\alpha Y) - \varphi_\alpha h(X, Y) = \tau_\alpha[g(X, Y)\xi_\alpha - \varepsilon_\alpha\eta_\alpha(Y)X]$$

and equating the normal and the tangential components we find

$$(\nabla_X\varphi_\alpha)Y = \tau_\alpha[g(X, Y)\xi_\alpha - \varepsilon_\alpha\eta_\alpha(Y)X] \tag{5.3}$$

and

$$h(X, \varphi_\alpha Y) = \varphi_\alpha h(X, Y), \ \alpha = 1, 2, 3.$$

From (5.3) it follows that the induced metric mixed 3-structure on M is mixed 3-Sasakian.

Moreover, making use of (2.6) and (2.8), in both cases we obtain

$$
\begin{aligned}
h(X, \varphi_1 Y) &= \varphi_1 h(X, Y) = \tau_1\varphi_2\varphi_3 h(X, Y) = \tau_1\varphi_2 h(X, \varphi_3 Y) \\
&= \tau_1\varphi_2 h(\varphi_3 Y, X) = \tau_1 h(\varphi_3 Y, \varphi_2 X) = \tau_1 h(\varphi_2 X, \varphi_3 Y) \\
&= \tau_1\varphi_3 h(\varphi_2 X, Y) = \tau_1\varphi_3 h(Y, \varphi_2 X) = \tau_1\varphi_3\varphi_2 h(Y, X) \\
&= \tau_1\varphi_3\varphi_2 h(X, Y) = -\varphi_1 h(X, Y).
\end{aligned}
$$

On the other hand, since $h(X, \varphi_1 Y) = \varphi_1 h(X, Y)$, it follows that

$$h(X, Y) = 0, \ \forall X, Y \in \Gamma(TM)$$

and therefore M is a totally geodesic submanifold of $\overline{M}$. $\qquad\square$

Corollary 5.3. *An invariant submanifold of a mixed 3-cosymplectic or mixed 3-Sasakian manifold, tangent to structure vector fields, has dimension $4k + 3$, $k \in \mathbb{N}$. Moreover, the induced metric has signature $(2k + 1, 2k + 2)$ or $(2k + 2, 2k + 1)$, according to the metric mixed 3-structure being positive or negative.*

Corollary 5.4. *An invariant submanifold of $\mathbb{R}_{2n+1}^{4n+3}$, $\mathbb{R}_{2n+2}^{4n+3}$, S_{2n+1}^{4n+3}, S_{2n+2}^{4n+3}, $P_{2n+1}^{4n+3}(\mathbb{R})$ and $P_{2n+2}^{4n+3}(\mathbb{R})$, tangent to the structure vector fields, is locally isometric with $\mathbb{R}_{2k+1}^{4k+3}$, $\mathbb{R}_{2k+2}^{4k+3}$, S_{2k+1}^{4k+3}, S_{2k+2}^{4k+3}, $P_{2k+1}^{4k+3}(\mathbb{R})$ and $P_{2k+2}^{4k+3}(\mathbb{R})$ respectively, where $0 \leq k \leq n$.*

Proposition 5.2 and Corollary 3.10 together imply the following result, which corresponds to a theorem of Cappelletti Montano, Di Terlizzi and Tripathi [5] for submanifolds in contact (κ, μ)-manifolds.

Proposition 5.5. *A non-degenerate submanifold of a mixed 3-Sasakian manifold, tangent to the structure vector fields, is totally geodesic if and only if it is invariant.*

Remark 5.6. The canonical immersions $S_\nu^n \hookrightarrow S_{2n+1}^{4n+3}$, $S_\nu^n \hookrightarrow S_{2n+2}^{4n+3}$, $P_\nu^n(\mathbb{R}) \hookrightarrow P_{2n+1}^{4n+3}(\mathbb{R})$ and $P_\nu^n(\mathbb{R}) \hookrightarrow P_{2n+2}^{4n+3}(\mathbb{R})$, where $\nu \in \{0, ..., n\}$, provide very natural examples of anti-invariant totally-geodesic submanifolds, but they are not tangent to the structure vector fields.

Lemma 5.7. *The distribution ξ of an invariant submanifold of a mixed 3-cosymplectic or mixed 3-Sasakian manifold tangent to the structure vector fields is integrable.*

Proof. If M is a mixed 3-cosymplectic manifold, then from (2.11) we obtain for any $X \in \Gamma(\mathcal{D})$:

$$\overline{g}([\xi_\alpha, \xi_\beta], X) = \overline{g}(\overline{\nabla}_{\xi_\alpha}\xi_\beta, X) - \overline{g}(\overline{\nabla}_{\xi_\beta}\xi_\alpha, X) = 0.$$

If M is a mixed 3-Sasakian manifold, then making use of (2.4) and (2.12) we obtain for any $X \in \Gamma(\mathcal{D})$:

$$
\begin{aligned}
\overline{g}([\xi_\alpha, \xi_\beta], X) &= \overline{g}(\overline{\nabla}_{\xi_\alpha}\xi_\beta, X) - \overline{g}(\overline{\nabla}_{\xi_\beta}\xi_\alpha, X) \\
&= -\varepsilon_\beta g(\varphi_\beta \xi_\alpha, X) + \varepsilon_\alpha g(\varphi_\alpha \xi_\beta, X) \\
&= (\varepsilon_\beta \tau_\alpha + \varepsilon_\alpha \tau_\beta)g(\xi_\gamma, X) \\
&= 0.
\end{aligned}
$$

Therefore, in both cases it follows that the distribution ξ is integrable. □

Proposition 5.8. *Let (M, g) be an invariant submanifold of a manifold $\overline{M}$ endowed with a metric mixed 3-structure, tangent to the structure vector fields.*

  (i) *If $\overline{M}$ is mixed 3-cosymplectic, then the distribution $\mathcal{D}$ is integrable. Moreover, the leaves of the foliation are mixed 3-cosymplectic manifold, totally geodesically immersed in $\overline{M}$.*
 (ii) *If $\overline{M}$ is mixed 3-Sasakian and $\dim M > 3$, then the distribution $\mathcal{D}$ is never integrable.*

Proof. (i) If $\overline{M}$ is mixed 3-cosymplectic, then using (2.11) we obtain for any $X, Y \in \Gamma(\mathcal{D})$ and $\alpha = 1, 2, 3$:

$$
\begin{aligned}
\overline{g}([X, Y], \xi_\alpha) &= \overline{g}(\overline{\nabla}_X Y, \xi_\alpha) - \overline{g}(\overline{\nabla}_Y X, \xi_\alpha) \\
&= -\overline{g}(Y, \overline{\nabla}_X \xi_\alpha) + \overline{g}(X, \overline{\nabla}_Y \xi_\alpha) \\
&= 0.
\end{aligned}
$$

Therefore the distribution $\mathcal{D}$ is integrable. Let M' be a leaf of $\mathcal{D}$. Then for any $X, Y \in \Gamma(TM')$ we have:

$$\overline{\nabla}_X Y = \nabla'_X Y + h'(X, Y),$$

where ∇' is the connection induced by $\overline{\nabla}$ on M' and h' is the second fundamental form of the immersion of M' in $\overline{M}$. Taking into account (2.11) we obtain:

$$
\begin{aligned}
h'(X, \varphi_\alpha Y) &= \overline{\nabla}_X \varphi_\alpha Y - \nabla'_X \varphi_\alpha Y \\
&= (\overline{\nabla}_X \varphi_\alpha)Y + \varphi_\alpha \overline{\nabla}_X Y - \nabla'_X \varphi_\alpha Y \\
&= \varphi_\alpha \nabla'_X Y + \varphi_\alpha h'(X, Y) - \nabla'_X \varphi_\alpha Y \\
&= -(\nabla'_X \varphi_\alpha)Y + \varphi_\alpha h'(X, Y).
\end{aligned}
$$

Therefore it follows $(\nabla'_X \varphi_\alpha)Y = 0$ and $h'(X, \varphi_\alpha Y) = \varphi_\alpha h'(X, Y)$, for $\alpha = 1, 2, 3$. From the last equality we deduce $h' = 0$ and the conclusion follows.

(ii) If $\overline{M}$ is a mixed 3-Sasakian manifold, then using (2.8) and (2.12), we obtain for any $X, Y \in \Gamma(\mathcal{D})$ and $\alpha = 1, 2, 3$:

$$
\begin{aligned}
\overline{g}([X,Y],\xi_\alpha) &= -\overline{g}(Y,\overline{\nabla}_X\xi_\alpha) + \overline{g}(X,\overline{\nabla}_Y\xi_\alpha) \\
&= \varepsilon_\alpha g(Y,\varphi_\alpha X) - \varepsilon_\alpha g(X,\varphi_\alpha Y) \\
&= 2\varepsilon_\alpha g(Y,\varphi_\alpha X).
\end{aligned}
$$

If we consider now X to be a non-lightlike vector field, then choosing $Y = \varphi_\alpha X$ in the last identity, we obtain using (2.7) and (2.8) that we have:

$$\overline{g}([X,\varphi_\alpha X],\xi_\alpha) = 2\varepsilon_\alpha \tau_\alpha g(X,X) \neq 0.$$

Therefore the distribution $\mathcal{D}$ is not integrable. $\square$

6. Invariant submanifolds of manifolds endowed with metric mixed 3-structures, normal to the structure vector fields

Let M be an invariant submanifold of a manifold endowed with a metric mixed 3-structure $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$, such that the structure vector fields ξ_1, ξ_2, ξ_3 are normal to M. We consider $\xi = \{\xi_1\} \oplus \{\xi_2\} \oplus \{\xi_3\}$ and we denote by $\mathcal{D}^\perp$ the orthogonal complementary subbundle to ξ in $TM^\perp$.

The following result is straightforward:

Lemma 6.1.   (i) $\varphi_\alpha(T_pM^\perp) \subset T_pM^\perp$, $\forall p \in M$, $\alpha = 1, 2, 3$.
 (ii) *The subbundle $\mathcal{D}^\perp$ is invariant under the action of φ_α, $\alpha = 1, 2, 3$.*

Remark 6.2. If $\overline{M}$ is mixed 3-cosymplectic, then (2.11) directly implies the integrability of ξ on $\overline{M}$.

Proposition 6.3. *Let M be an invariant submanifold of a manifold endowed with a metric mixed 3-structure $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$, such that the structure vector fields ξ_1, ξ_2, ξ_3 are normal to M. Then M admits an almost para-hyperhermitian structure.*

Proof. For any $X \in \Gamma(TM)$, we obtain from (2.8) that

$$\eta_\alpha(X) = \varepsilon_\alpha \overline{g}(X,\xi_\alpha) = 0.$$

Then from (2.1) it follows

$$\varphi_\alpha^2 X = -\tau_\alpha X, \ \alpha = 1, 2, 3$$

and if we denote by

$$J_\alpha = \varphi_{\alpha|M}, \alpha = 1, 2, 3$$

from (2.7) we obtain

$$J_\alpha J_\beta = -J_\beta J_\alpha = \tau_\gamma J_\gamma,$$

for any even permutation (α, β, γ) of $(1, 2, 3)$.

On the other hand, from (2.7) we get

$$g(\varphi_\alpha X, \varphi_\alpha Y) = \tau_\alpha g(X,Y), \ \forall X, Y \in \Gamma(TM), \ \alpha = 1, 2, 3.$$

Therefore $(M, (J_\alpha)_{\alpha=1,2,3}, g)$ is an almost para-hyperhermitian manifold. $\square$

Corollary 6.4. *Any invariant submanifold of a manifold endowed with a metric mixed 3-structure, normal to the structure vector fields, has the dimension $4k$, $k \in \mathbb{N}$, and the induced metric has signature $(2k, 2k)$.*

Proposition 6.5. *Let M be an invariant submanifold of a manifold endowed with a metric mixed 3-structure $(\overline{M}, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$, such that the structure vector fields ξ_1, ξ_2, ξ_3 are normal to M. If $\overline{M}$ is mixed 3-cosymplectic, then M is a para-hyper-Kähler manifold, totally geodesically immersed in $\overline{M}$.*

Proof. From Proposition 6.3 it follows that M can be endowed with an almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=1,2,3}$, which is para-hyperhermitian with respect to the induced metric g. On the other hand, from (2.9) and Gauss formula we obtain

$$0 = (\overline{\nabla}_X \varphi_\alpha)Y = (\nabla_X \varphi_\alpha)Y + h(X, \varphi_\alpha Y) - \varphi_\alpha h(X, Y)$$

for all $X, Y \in \Gamma(TM)$.

From the above identity, equating the normal and tangential components, it follows that we have:

$$h(X, \varphi_\alpha Y) = \varphi_\alpha h(X, Y) \tag{6.1}$$

and

$$(\nabla_X J_\alpha)Y = 0, \tag{6.2}$$

since $J_\alpha = \varphi_{\alpha|M}$.

From (6.2) we deduce that $(M, H = (J_\alpha)_{\alpha=1,2,3}, g)$ is a para-hyper-Kähler manifold and from (6.1) we obtain similarly as in the proof of Theorem 5.2 that M is totally geodesic immersed in $\overline{M}$. $\qquad\square$

Corollary 6.6. *The invariant submanifolds of $\mathbb{R}^{4n+3}_{2n+1}$ and $\mathbb{R}^{4n+3}_{2n+2}$, normal to the structure vector fields, are locally isometric with $\mathbb{R}^{4k}_{2k}$, where $0 \leq k \leq n$.*

Acknowledgement

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Stere Ianuş
University of Bucharest
Faculty of Mathematics and Computer Science
Str. Academiei 14
Bucharest 70109
Romania

Liviu Ornea
University of Bucharest
Faculty of Mathematics and Computer Science
Str. Academiei 14
Bucharest 70109
Romania
and
Institute of Mathematics "Simion Stoilow" of the Romanian Academy
Calea Griviţei 21
Bucharest 010702
Romania
e-mail: `lornea@gta.math.unibuc.ro`
        `liviu.ornea@imar.ro`

Gabriel Eduard Vîlcu
Petroleum-Gas University of Ploieşti
Department of Mathematics and Computer Science
Bulevardul Bucureşti 39
Ploieşti 100680
Romania
and
University of Bucharest
Research Center in Geometry, Topology and Algebra
Str. Academiei 14
Bucharest 70109
Romania
e-mail: `gvilcu@mail.upg-ploiesti.ro`