

# An immersion theorem for Vaisman manifolds

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**Abstract.** A locally conformally Kähler (LCK) manifold is a complex manifold admitting a Kähler covering  $\tilde{M}$ , with monodromy acting on  $\tilde{M}$  by Kähler homotheties. A compact LCK manifold is Vaisman if it admits a holomorphic flow acting by non-trivial homotheties on  $\tilde{M}$ . We prove that any compact Vaisman manifold admits a natural holomorphic immersion to a Hopf manifold  $(\mathbb{C}^n \setminus 0)/\mathbb{Z}$ . As an application, we obtain that any Sasakian manifold has a contact immersion to an odd-dimensional sphere.

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## 1. Introduction

Locally conformally Kähler (LCK) manifolds are, by definition, complex manifolds admitting a Kähler covering with deck transformations acting by Kähler homotheties. These manifolds appear naturally in complex geometry. Most examples of compact non-Kähler manifolds studied in complex geometry admit an LCK structure.

LCK structures appeared since 1954, studied by P. Libermann. They came again into attention since 1976, with the work of I. Vaisman who also produced the first compact examples: the (diagonal) Hopf manifolds (see [DO] and the references therein).

As any Hermitian geometry, LCK geometry is encoded in the properties of the Lee form (Definition 2.1) which, in this case, is closed. From the conformal

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viewpoint, the Lee form is the connection one-form in a real line bundle, called the weight bundle.

It became soon clear that the most tractable LCK manifolds are the ones with parallel Lee form. We call them Vaisman manifolds because it was I. Vaisman to introduce and study them extensively, under the name of generalized Hopf manifolds. With the exception of some compact surfaces (Inoue surfaces of the first kind, Hopf surfaces of Kähler rank 0) which are known to not admit Vaisman metrics (cf. [Bel]), all other known examples of compact LCK manifolds have parallel Lee form.

The universal covering space of a Vaisman manifold bears a Kähler metric of a very special kind: it is a conic metric over a Sasakian manifold. Sasakian spaces proved recently to be very important in physical theories (see the series of papers by C.P. Boyer, K. Galicki *et.al.*); this further motivated the interest for Vaisman manifolds.

The topology of (compact) Vaisman manifolds is very different from that of Kähler manifolds. While it was long ago known that the first Betti number must be odd, [Va1], it was proved only recently that Einstein-Weyl LCK manifolds have no non-trivial holomorphic forms, [AI]. Still, nothing is known about the topology of general LCK manifolds.

The parallelism of the Lee form is reflected in the properties of the weight bundle: indeed, we recently proved in [OV] that the weight bundle has monodromy  $\mathbb{Z}$ . This allowed us to obtain the structure theorem of compact Vaisman manifolds: they are Riemannian suspensions over the circle with fibre a Sasakian manifold. This result can be regarded as a kind of equivalence between Sasakian and Vaisman geometries.

Vaisman manifolds can also be viewed *via* the properties of their automorphism group. This was done separately by F.A. Belgun and Y. Kamishima. Finally, the existence of a complex flow of conformal transformations proved to be an equivalent definition of Vaisman structures, cf. [KO]. This was also the approach in [GOP] where the Hamiltonian actions were considered.

In the present paper we further exploit the techniques developed in [Ve]. We adapt algebraic geometrical methods to prove an analogue of the Kodaira embedding theorem in Vaisman geometry.

## 2. LCK and Vaisman geometry

In this Section we give a brief exposition of Vaisman geometry, following [DO], [KO], [Ve] and describe the ideas of the paper.

### 2.1. Differential and algebraic geometry of LCK manifolds

Throughout this paper  $(M, I, g)$  will denote a connected Hermitian manifold of complex dimension  $m \geq 2$ , with fundamental two-form defined by  $\omega(X, Y) = g(X, IY)$ .

**Definition 2.1.**  $(M, I, g)$  is called **locally conformal Kähler, LCK** for short, if there exists a global closed one-form  $\theta$  satisfying:

$$d\omega = \theta \wedge \omega. \tag{2.1}$$

The one-form  $\theta$  is called the **Lee form** and its metric dual field, denoted  $\theta^\sharp$ , is called the **Lee field**. Locally we may write  $\theta = df$ , hence the local metric  $e^{-f}g$  is Kähler, thus motivating the definition.

The pull-back of  $\theta$  becomes exact on the universal covering space  $\hat{M}$ . The deck transformations of  $\hat{M}$  induce homotheties of the Kähler manifold  $(\hat{M}, e^{-f}g)$ . We obtain that  $(M, I, g)$  is LCK if and only if its universal covering admits a Kähler metric, with deck transformations acting by homotheties. This is in fact true for any covering  $\tilde{M}$  such that the pullback of  $\theta$  is exact on  $\tilde{M}$ . For such a covering we let  $\mathcal{H}(\tilde{M})$  denote the group of all holomorphic homotheties of  $\tilde{M}$  and let

$$\rho : \mathcal{H}(\tilde{M}) \longrightarrow \mathbb{R}^{>0} \tag{2.2}$$

be the homomorphism which associates to each homothety its scale factor.

LCK manifolds are examples of a **Weyl manifolds**: Riemannian manifolds  $(M, g)$  admitting a torsion-free connection  $\nabla$  which satisfies preserves the conformal class of  $g$  ( $\nabla(g) = g \otimes \theta$ ). For a detailed overview of Weyl geometry, see [CP] and [Or]. A Weyl connection on an LCK manifold is provided by the Levi-Civita connection  $\nabla_{LC}$  on its Kähler covering  $\tilde{M}$ . The deck transforms of  $\tilde{M}$  multiply the Kähler metric by constant, hence commute with  $\nabla_{LC}$ . This allows one to pushdown  $\nabla_{LC}$  to  $M$ . Clearly,  $\nabla_{LC}$  preserves the conformal class of  $g$ .

From an algebro-geometric point of view, the most interesting feature of an LCK manifold is its weight bundle  $L$  (more precisely, its complexified  $L_C$  that we still call weight bundle), that is, the flat line bundle with the monodromy defined by  $\rho$ :

**Definition 2.2.** Let  $(M, \nabla, g)$  be a Weyl manifold. The real line bundle associated to the bundle of linear frames of  $(M, g)$  by the representation  $GL(n, \mathbb{R}) \ni A \mapsto |\det A|^{\frac{1}{n}}$  is called the **weight bundle** of  $M$ .

The Weyl connection induces a flat connection, also denoted  $\nabla$ , in  $L$ . Its connection 1-form can be identified with  $\theta$ , hence  $(L, \nabla)$  is a flat bundle.

Usually one has no way to choose a specific metric in a conformal class, hence to trivialize  $L$ . But on compact LCK manifolds one has such a possibility due to a result of Gauduchon:

*Claim 2.3.* [Ga] Let  $([g], \nabla)$  be a Weyl structure on a compact manifold. There exists a unique (up to homothety) metric in  $[g]$  with coclosed Lee form.

Clearly, any metric conformal with a LCK one is still LCK (with respect to a fixed complex structure), so that we can give:

**Definition 2.4.** *The unique (up to homothety) LCK metric in the conformal class  $[g]$  of  $(M, I, g)$  with harmonic associated Lee form is called the **Gauduchon metric**.*

Note that the Gauduchon metric endows  $L$  with a trivialization  $l$ .

Complex algebraic geometry deals mostly with the Kähler manifolds. On a compact Kähler manifold  $X$ , flat line bundles are not particularly interesting. They have no non-trivial holomorphic sections; moreover, every flat bundle on  $X$  is induced from the Albanese map  $X \rightarrow \text{Alb}(X)$  to the torus  $\text{Alb}(X)$ .

In non-Kähler complex geometry, the situation is completely different. The weight bundle  $L_{\mathbb{C}}$  on a compact LCK manifold  $M$  may admit quite a few non-trivial sections. In some cases there are so many sections that they provide an embedding from  $M$  to a model LCK manifold, called a Hopf manifold (Definition 2.6), in a manner of Kodaira embedding theorem.

To study the bundle  $L_{\mathbb{C}}$ , we fix a Hermitian metric (this is done using the Gauduchon's theorem, see Claim 2.3), and compute the curvature  $\Theta \in \Lambda^{1,1}(M)$  of the Chern connection on  $L_{\mathbb{C}}$ . It turns out that all eigenvalues of  $\Theta$  are positive except one.

J.-P. Demailly's holomorphic Morse inequalities ([De]) give strong estimates of the asymptotic behaviour of cohomology

$$H^i(L_{\mathbb{C}}^k), \quad k \rightarrow \infty$$

for such bundles.

Even more interesting, the weight covering  $\tilde{M}$  of  $M$  admits a function  $\psi : \tilde{M} \rightarrow \mathbb{R}^{>0}$  which is exhausting, and all eigenvalues of the form  $-\sqrt{-1} \partial \bar{\partial} \psi$  are positive, except one. This function is obtained by adding the Kähler potential  $r = e^{-f}$  of  $\tilde{M}$  (see (2.5)) and  $r^{-1}$ . Such a manifold is called 2-complete. There is a great body of work dedicated to the study of  $q$ -complete manifolds (see e.g. the survey [Co]).

The LCK manifolds are clearly an extremely interesting object of algebro-geometric study.

## 2.2. Vaisman manifolds

The typical example of LCK manifold, is the diagonal Hopf manifold  $H_{\alpha} := (\mathbb{C}^n \setminus 0)/\mathbb{Z}$  where  $\mathbb{Z}$  is generated by  $z \mapsto \alpha z$ ,  $|\alpha| \neq 0, 1$ . The LCK metric is here  $|z|^{-2} \sum dz_i \otimes d\bar{z}_i$  with Lee form given (locally) by  $-d \log |z|^2$  and one can check that it is parallel with respect to the Levi Civita connection of the LCK metric. Manifolds with such structure were intensively studied by Vaisman under the name of generalized Hopf manifolds which later proved to be inappropriate. So that we adopt the following:

**Definition 2.5.** *A LCK manifold  $(M, I, g)$  with parallel Lee form is called a **Vaisman manifold**.*

Among other compact LCK manifolds, the Vaisman manifolds are characterized as admitting a holomorphic flow which acts by non-trivial homotheties on its Kähler covering (see [KO]).

Every Hopf surface admits a LCK metric, [GO], but the Hopf surfaces with Kähler rank 0 do not admit Vaisman metrics, [Bel]. The construction in [GO] was generalized in [KO] where Vaisman metrics were found on manifolds  $H_\Lambda := (\mathbb{C}^n \setminus 0)/\mathbb{Z}$  with  $\Lambda = (\alpha_1, \dots, \alpha_n)$ ,  $\alpha_i \in \mathbb{C}$ ,  $|\alpha_1| \geq \dots \geq |\alpha_n| > 1$  and  $\mathbb{Z}$  generated by the transformations  $(z_j) \mapsto (\alpha_j z_j)$ .

**Definition 2.6.** *A linear operator on a complex vector space  $V$  is called **semisimple** if it can be diagonalized. Given a semisimple operator  $A : V \rightarrow V$  with all eigenvalues  $|\alpha_i| > 1$ , the quotient  $H_\Lambda := (V \setminus 0)/\langle A \rangle$  described above is called a **Hopf manifold**.*

The Vaisman structure on  $H_\Lambda$  is intimately related by a diffeomorphism between this manifold and  $S^1 \times S^{2n-1}$  to a specific Sasakian structure of the odd sphere. For an up to date introduction in Sasakian geometry we refer to [BG1] ; here we just recall that  $(X, h)$  is Sasakian if and only if the cone  $(\mathcal{C}(X) := X \times \mathbb{R}^{>0}, dt^2 + t^2h)$  is Kählerian. The example of the Hopf manifold is not casual: the total space of any flat  $S^1$  bundle over a compact Sasakian manifold can be given a Vaisman structure. Recently, the following structure theorem was proved:

**Theorem 2.7.** *[OV] Any compact Vaisman manifold is a Riemannian suspension with Sasakian fibers over a circle, and conversely, the total space of such a Riemannian suspension is a compact Vaisman manifold.*

The proof of this structure theorem relies on the following particular property of the weight bundle that we shall use also in the present paper:

**Theorem 2.8.** *[OV] The weight bundle of a compact Vaisman manifold has monodromy  $\mathbb{Z}$ .*

In [OV] it was also proven that the Kähler covering  $\tilde{M}$  of  $M$  admits an action of a commutative connected lie group  $\tilde{G}$ , such that the monodromy  $\Gamma$  lies inside  $\tilde{G}$ . We obtain that  $\Gamma$  belongs to the connected component of the group of holomorphic homotheties of  $\tilde{M}$ . The converse is also true, by [KO]: if  $\Gamma$  belongs to the connected component of the group of holomorphic homotheties of  $\tilde{M}$ , then  $M$  is Vaisman.

This gives the following characterization of Vaisman manifolds. Given a Kähler manifold  $\tilde{M}$ , denote by  $H$  the connected component of the Lie group of holomorphic homotheties, and let  $\Gamma \subset H$ ,  $\Gamma \cong \mathbb{Z}$  be a group acting by non-trivial homotheties in such a way that  $M := \tilde{M}/\Gamma$  is compact. Then  $M$  is Vaisman, and, moreover, all Vaisman manifolds are obtained this way.

An LCK metric with parallel Lee form on a compact manifold necessarily coincides with a Gauduchon metric. Moreover, on compact Vaisman manifolds  $(M, I, g)$ , the group  $\text{Aut}(M)$  of all conformal biholomorphisms coincides with  $\text{Iso}(M, g)$ , [MPPS], the isometry group of the Gauduchon metric, thus being compact.

We recall that on a Vaisman manifold, the flow of the Lee field (which we call Lee flow) is formed by holomorphic isometries of the Gauduchon metric. The same is true for the flow of the anti-Lee field  $I\theta^\sharp$ . We shall denote by  $G$  the real Lie subgroup of  $\text{Aut}(M)$  generated by the Lee flow.

**Definition 2.9.** *The covering  $\pi : \tilde{M} \longrightarrow M$  associated with the monodromy group of  $L$  is called the **weight covering** of  $M$ .*

We let  $\text{Aut}(\tilde{M}, M)$  be the group of all conformal automorphisms of  $\tilde{M}$  which make the following diagram commutative:

$$\begin{array}{ccc}
 \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{M} \\
 \pi \downarrow & & \downarrow \pi \\
 M & \xrightarrow{f} & M
 \end{array} \tag{2.3}$$

By the above,  $\text{Aut}(\tilde{M}, M)$  is nonempty, as it contains the Lee flow. For further use let

$$\Phi : \text{Aut}(\tilde{M}, M) \longrightarrow \text{Aut}(M) \tag{2.4}$$

be the forgetful map.

Much of the geometry of a LCK manifold, especially of a Vaisman manifold, can be expressed in terms of the two-form  $\omega_0 = d^c\theta$ , where  $d^c = -IdI$ , introduced and studied in [Ve]. It is semipositive, its only zero eigenvalue being in the direction of the Lee field. To further understand  $\omega_0$ , consider the complexified bundle  $L_{\mathbb{C}} := L \otimes_{\mathbb{R}} \mathbb{C}$ . The  $(0, 1)$  part of the complexified Weyl connection endows this complex line bundle with a holomorphic structure. We also equip  $L_{\mathbb{C}}$  with a Hermitian structure such that to normalize to 1 the length of the trivialization  $l$ . Whenever we refer to  $L_{\mathbb{C}}$ , we implicitly refer to this holomorphic and Hermitian structure. In this setting, the form  $\omega_0$  has the following geometric meaning:

*Claim 2.10.* [Ve, Th. 6.7] The curvature of the Chern connection of  $L_{\mathbb{C}}$  is equal to  $-2\sqrt{-1}\omega_0$ .

We end this subsection by recalling from [Ve, Pr. 4.4] that if the Lee form of a Vaisman manifold is exact,  $\theta = df$ , then the function  $r := e^{-f}$  is a potential for the Kähler metric  $r\omega$ , i.e.

$$r\omega = dd^c r. \tag{2.5}$$

### 2.3. Quasiregular Vaisman manifolds

Let  $M$  be a Vaisman manifold. In [Ve], the curvature  $\Theta$  of the Chern connection on the weight bundle was computed. It was shown that  $\Theta = -\sqrt{-1} \omega_0$ , where  $\omega_0$  is positive semidefinite form, with one zero eigenvalue.

The zero eigenvalue of  $\omega_0$  corresponds to a holomorphic foliation  $\Xi$  on  $M$ , called **the Lee foliation**. The leaves of this foliation are orbits of a holomorphic flow on  $M$ , called **the complex Lee flow** (see Subsection 3.1).

When this foliation has a Hausdorff leaf space  $Q$ , one would expect that the bundle  $L_{\mathbb{C}}$  is a pullback of a positive line bundle  $L_Q$  on  $Q$ . This is true (Theorem 3.4), at least when the leaves of  $\Xi$  are compact.

Vaisman manifolds with compact leaves of the Lee foliation are called **quasiregular**. The manifold  $Q$  is obtained as a quotient of a manifold by an action of a compact group, and therefore it is an orbifold. The form  $\omega_0$  on  $M$  defines a Kähler form on  $Q = M/\Xi$ . A compact orbifold equipped with a positive line bundle is actually projective, because Kodaira embedding theorem holds in the orbifold case ([Ba]).

This approach allows for a completely algebraic construction of quasiregular Vaisman manifolds. Consider a projective orbifold  $Q$  and a positive line bundle  $L_Q$  on  $Q$ . Let  $\tilde{M}$  be the total space of all non-zero vectors in  $L_Q^*$ ,  $q \in \mathbb{R}^{>0}$ , and  $\sigma_q : L_Q \rightarrow L_Q$  a map multiplying  $l \in L_Q|_{x \in M}$  by  $q$ . Denote by  $\Gamma \cong \mathbb{Z}$  the subgroup of  $\text{Aut}(\tilde{M})$  generated by  $\sigma_q$ . When  $\tilde{M}$  is smooth, it is Kähler, because the function  $\psi(b) = |b|^2$  gives a Kähler potential on  $\tilde{M}$ .<sup>1</sup> The quotient  $M := \tilde{M}/\sigma_q$  is clearly a Vaisman manifold; the Lee fibration  $M \rightarrow M/\Xi$  becomes the standard projection to  $Q$ , and the fibers of  $L_Q$  are identified with the leaves of Lee foliation. We obtain that  $M$  is quasiregular; moreover, all quasiregular Vaisman manifolds are obtained this way.

In a sense, the quasiregular Vaisman geometry is more algebraic than for instance Kähler geometry. Indeed, any quasiregular Vaisman manifold is constructed from a projective manifold and an ample bundle - purely algebraic set of data.

Consider the lifting  $\tilde{L}_{\mathbb{C}}^*$  of  $L_Q^*$  to

$$\tilde{M} \cong \text{Tot}(L_Q^* \setminus 0).$$

The points of  $\text{Tot}(L_Q^* \setminus 0)$  correspond to non-zero vectors in  $L_{\mathbb{C}}^*$ . Therefore, the bundle  $\tilde{L}_{\mathbb{C}}^*$  is equipped with a natural holomorphic non-degenerate section  $\tau$ .

Sections of  $L_Q$  give, after a pullback, sections of  $\tilde{L}_{\mathbb{C}}$ . Evaluating these sections on  $\tau$ , we obtain holomorphic functions on  $\tilde{M}$ . The map  $\sigma_q$  multiplies  $\tau$  by

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<sup>1</sup> This is true (and easy to prove) for a space of non-zero vectors in any negative bundle. For a proof, one may check e.g. [Bes], (15.19).

$q$ . Therefore, sections of  $L_Q$  give functions  $\mu$  on  $\tilde{M}$  which satisfy  $\sigma_q(\mu) = q\mu$ . Similarly, for sections of  $L_Q^k$ , we have

$$\sigma_q(\mu) = q^k \mu. \tag{2.6}$$

The bundle  $L_Q$  is ample, and therefore, for  $k$  sufficiently big, Kodaira embedding theorem gives an embedding  $Q \hookrightarrow \mathbb{P}(H^0(L_Q^k))$ . Associating to sections of  $L_Q^k$  functions on  $\tilde{M}$  as above, we obtain a morphism

$$\tilde{M} \longrightarrow H^0(L_Q^k) \setminus 0 \tag{2.7}$$

which maps the fibers of the Lee foliation to complex lines of  $H^0(L_Q^k)$  and induces an immersion. The automorphism  $\sigma_q$  acts on both sides of (2.7) as prescribed by (2.6), and this gives an immersion  $M \longrightarrow (H^0(L_Q^k) \setminus 0) / \langle q^k \rangle$ , where  $\langle q^k \rangle \cong \mathbb{Z}$  denotes the abelian subgroup of  $\text{Aut}(H^0(L_Q^k))$  generated by  $q^k$ .

We outlined the proof (for details see Subsection 3.3), of the following theorem.

**Theorem 2.11.** *Let  $M$  be a quasiregular Vaisman manifold. Then  $M$  admits a holomorphic immersion to  $\mathbb{C}^n \setminus 0 / q$ ,  $q \in \mathbb{R}^{>1}$ .*

#### 2.4. Immersion theorem for general Vaisman manifolds

To deal with a Vaisman manifold  $M$  which is not quasiregular, we look at the Lie group  $G$  within the group of isometries of  $M$  generated by the Lee flow (see 2.2). This group was studied at great length in [Ka] and [KO].

It is not very difficult to show that  $G$  is a compact abelian group ([KO]). Denote by  $\tilde{G}$  the group of automorphisms of the pair  $(\tilde{M}, M)$  and mapping to  $G$  under the natural forgetful map  $\text{Aut}(\tilde{M}, M) \longrightarrow \text{Aut}(M)$  (see Subsection 4.1). It was shown in [OV] that  $\tilde{G}$  is connected and isomorphic to  $\mathbb{R} \times (S^1)^k$ , and the natural forgetful map  $\tilde{G} \longrightarrow G$  is a covering.

The deck transformation group of  $\tilde{M}$  (denoted by  $\Gamma$ ) lies in  $\tilde{G}$  ([OV]; see also Subsection 4.1).

For all sufficiently small deformations  $\Gamma'$  of  $\Gamma$  within a complexification  $\tilde{G}_{\mathbb{C}} \subset \text{Aut}(\tilde{M}, M)$ , the quotient manifold  $\tilde{M} / \Gamma'$  remains Vaisman (Theorem 4.5). Moreover, one can choose  $\Gamma'$  in such a way that  $\tilde{M} / \Gamma'$  becomes quasiregular (Proposition 4.6).

This is remarkable because a parallel statement in Kähler geometry - whether any Kähler manifold is a deformation of a projective one - is still unknown.

The manifold  $\tilde{M}$  is not Stein; the existence of globally defined holomorphic functions on  $\tilde{M}$  is a priori unclear. However, from the existence of  $\Gamma' \subset \tilde{G}_{\mathbb{C}}$  with  $\tilde{M} / \Gamma'$  quasiregular, we obtain that  $\tilde{M}$  admits quite a few holomorphic functions. These functions are obtained from the sections of powers of an ample bundle  $L_Q$  on the leaf space  $M / \Xi$  (see Subsection 2.3).

Denote by  $\gamma' : \tilde{M} \rightarrow \tilde{M}$  the generator of  $\Gamma'$ , and let  $q \in \mathbb{R}^{>0}$  be the monodromy of the weight bundle  $L$ . Consider the space  $V$  of holomorphic functions  $\mu : \tilde{M} \rightarrow \mathbb{C}$ ,  $\gamma'(\mu) = q^k \mu$ . This space is identified with the space of sections of  $L_Q^k$  (Subsection 2.3). We shall think of  $V$  as of eigenspace of  $\gamma'$  acting on  $\mathcal{O}_{\tilde{M}}$ .

Clearly,  $M$  is equipped with a natural map

$$M \rightarrow V/\langle q^k \rangle. \tag{2.8}$$

For  $k$  sufficiently large, the functions from  $V$  have no common zeros, and the map (2.8) induces a holomorphic immersion  $M \rightarrow (V \setminus 0)/\langle q^k \rangle$  (Theorem 2.11).

We obtain that the natural map  $\tilde{M} \rightarrow (V \setminus 0)$  is an immersion. Denote by  $\gamma$  the generator of  $\Gamma$ . By construction, the maps  $\gamma, \gamma' : \tilde{M} \rightarrow \tilde{M}$  belong to the complexification  $\tilde{G}_{\mathbb{C}}$ . Since  $\tilde{G}$  is commutative,  $\gamma$  and  $\gamma'$  commute. Since  $V$  is an eigenspace of  $\gamma'$ ,  $\gamma$  acts on  $V$ . This gives a map

$$M = \tilde{M}/\langle \gamma \rangle \rightarrow (V \setminus 0)/\langle \gamma \rangle.$$

This map is an immersion.

This way we obtain an immersion from an arbitrary Vaisman manifold to a Hopf manifold  $(V \setminus 0)/\mathbb{Z}$ ; see Theorem 5.1 for details.

This Immersion Theorem is a Vaisman-geometric analogue of the Kodaira embedding theorem.

### 2.5. Category of Vaisman varieties

Vaisman manifolds naturally form a category. In [Ts] and [Ve] it was shown that any closed submanifold  $X \subset M$  of a compact Vaisman manifold is again Vaisman, assuming  $\dim X > 0$ .

Define a morphism of Vaisman manifolds as a holomorphic map commuting with the holomorphic Lee flow. The Immersion Theorem can be stated as follows (for the proof, see Theorem 5.1):

**Theorem 2.12.** *Any Vaisman manifold admits a Vaisman immersion to a Hopf manifold.*

We could define Vaisman varieties as varieties immersed in a Hopf manifold and equipped with the induced Lee flow, and define morphisms of Vaisman varieties as a holomorphic map commuting with the Lee flow. This is similar to defining projective varieties as complex subvarieties in a complex projective space.

The Vaisman geometry becomes then a legitimate chapter of complex algebraic geometry. In fact, the data needed to define a Vaisman submanifold in a Hopf manifold (a space  $V$  with a linear flow  $\psi_t = e^{tA}$ ,  $A \in \mathfrak{gl}(V)$  and an analytic subvariety  $X \subset V$ ) are classical and well known.

The Immersion Theorem suggests that Vaisman varieties have interesting intrinsic geometry, in many ways parallel to the usual algebraic geometry. One could ask, for instance, the following.

*Question 2.13.* What are the moduli spaces of Vaisman manifolds? They are, generally speaking, distinct from the moduli of complex analytic deformations: not every complex analytic deformation is Vaisman.

*Question 2.14.* Given a singular Vaisman variety (that is, a subvariety of a Hopf manifold), do we have a resolution of singularities within the Vaisman category? Given a holomorphic map of Vaisman varieties, is it always compatible with the Lee field? What are the birational maps of Vaisman varieties?

*Question 2.15.* Consider a Vaisman manifold  $M$ ,  $\dim M = n$ , with canonical bundle isomorphic to  $n$ -th degree of a weight bundle. Is there an Einstein-Weyl<sup>2</sup> LCK metric on  $M$ ? This will amount to a Vaisman analogue of the Calabi-Yau theorem.

### 3. Quasiregular Vaisman manifolds

#### 3.1. Quasiregular Vaisman manifolds and the weight bundle

As the Lee field is Killing and holomorphic on a Vaisman manifold, the distribution locally generated by  $\theta^\sharp$  and  $I\theta^\sharp$  defines a holomorphic, Riemannian, totally geodesic foliation  $\Xi$  that we call **the Lee foliation**. Its leaves are elliptic curves (cf. [Va1]).<sup>3</sup> The leaves of  $\Xi$  are identified with the orbits of a holomorphic flow generated by  $\theta^\sharp$ ; this flow is called **the complex Lee flow**.

**Definition 3.1.** *A Vaisman manifold is called:*

- **quasiregular** if  $\Xi$  has compact leaves. In this case the leaf space  $M/\Xi$  is a Hausdorff orbifold.
- **regular** if it is quasiregular and the quotient map  $M \rightarrow M/\Xi$  is smooth.

Combining results from [Va1] and [Ve], we can state:

*Claim 3.2.* Let  $M$  be a quasiregular Vaisman manifold and let  $Q$  be the leaf space of the Lee foliation. Then  $Q$  is a Kähler orbifold and the pull-back of its Kähler form by the natural projection coincides with  $\omega_0$ .

**Definition 3.3.** *Let  $M$  be a quasiregular Vaisman manifold and denote  $f : M \rightarrow Q$  the above associated fibration. Then  $f : M \rightarrow Q$  is called **the Lee fibration of  $M$** .*

Let  $f_*L_C \rightarrow Q$  be the push-forward of the bundle  $L_C$ . The main result of this Subsection is the following Theorem.

<sup>2</sup> Please consult [CP] or [Or] for a reference to Einstein-Weyl LCK structures.

<sup>3</sup> Some algebraic geometers prefer to distinguish between “compact 1-dimensional complex tori” and “elliptic curves” (compact 1-dimensional complex tori with a marked point). The compact fibers of the Lee foliation are, in this terminology, compact 1-dimensional complex tori.

**Theorem 3.4.** *Let  $M$  be a compact quasiregular Vaisman manifold, and let  $f : M \rightarrow Q$  its Lee fibration. Then  $L_{\mathbb{C}}$  is trivial along the fibers of  $f$ : the natural map*

$$L_{\mathbb{C}} \rightarrow f^* f_* L_{\mathbb{C}} \tag{3.1}$$

*is an isomorphism.*

*Proof.* The statement of Theorem 3.4 is local on  $Q$ . Since  $f$  is proper, it suffices to prove that  $L_{\mathbb{C}}$  is trivial on  $f^{-1}(U)$ , for a sufficiently small  $U \subset Q$ . Indeed, for any proper map with connected fibers, a pushforward of a structure sheaf is a structure sheaf, hence the equation (3.1) is true for trivial bundles.

Let  $\tilde{M}$  be the weight covering of  $M$ . The lifting  $\tilde{L}_{\mathbb{C}}$  of  $L_{\mathbb{C}}$  to  $\tilde{M}$  is flat and has trivial monodromy. Therefore,  $\tilde{L}_{\mathbb{C}}$  is trivialized, in a canonical way. This allows us to interpret sections of  $\tilde{L}_{\mathbb{C}}$  as holomorphic functions on  $\tilde{M}$ . A function  $\mu : \tilde{M} \rightarrow \mathbb{C}$  of  $\tilde{L}_{\mathbb{C}}$  is lifted from  $L_{\mathbb{C}}$  if and only if it satisfies the following monodromy condition:

$$\gamma(\mu) = q\mu, \tag{3.2}$$

where  $\gamma : \tilde{M} \rightarrow \tilde{M}$  is the generator of the deck transformation group of  $\tilde{M}$ , and  $q \in \text{End}(L) = \mathbb{R}^{>0}$  the corresponding monodromy action.

The fibers of  $f$  are elliptic curves and coincide with the orbits of  $G_{\mathbb{C}}$ , the group generated by the complex Lee flow. For any orbit  $C \subset M$ , the corresponding covering  $\tilde{C} \subset \tilde{M}$  is isomorphic to  $\mathbb{C}^*$ , with  $\gamma$  acting on  $\tilde{C} \cong \mathbb{C}^*$  as a multiplication by  $q$ . After this identification, the equation (3.2) is rewritten as

$$\mu(qz) = q\mu(z), \quad z \in \tilde{C} = \mathbb{C}^*. \tag{3.3}$$

A function on  $\mathbb{C}^*$  satisfying (3.3) must be linear. We obtain that the Lee field acts linearly on all functions satisfying (3.2):

$$\text{Lie}_{\theta^\#}(\mu) = \log q \cdot \mu. \tag{3.4}$$

This is an ordinary differential equation. Using the uniqueness and existence of solutions of ODE, we arrive at the following claim.

*Claim 3.5.* Let  $M$  be a quasiregular Vaisman manifold,  $G_{\mathbb{C}}$  the complex Lie group generated by the Lie flow,  $f : M \rightarrow Q$  the quotient space,  $Q = M/G_{\mathbb{C}}$ , and  $U \subset M$  the **slice** of  $f$ , that is, a submanifold of  $M$  (not necessarily closed in  $M$ ) such that the restriction  $f|_U : U \rightarrow Q$  is an open embedding. Consider the Vaisman manifold  $M_U := f^{-1}(f(U))$ . Pick a trivialization  $\mu_U$  of  $L_{\mathbb{C}}|_U$  (this is possible to do if  $U$  is sufficiently small). This gives a trivialization of  $L_{\mathbb{C}}$  on  $M_U$ .

*Proof.* We solve the ODE (3.4) with an initial condition  $\mu|_U = \mu_U$ . The solution gives a trivialization of  $L_{\mathbb{C}}$  as the above argument indicates.  $\square$

Return to the proof of Theorem 3.4. A slice of the action of  $G_{\mathbb{C}}$  exists at any point  $x \in M$  where  $G_{\mathbb{C}}$  acts smoothly, that is, for all points  $x \in M$  which are not critical points of  $f : M \rightarrow Q$ . Using Claim 3.5, we obtain a trivialization needed in Theorem 3.4 for all regular Vaisman manifolds (regular Vaisman manifolds are precisely those manifolds for which  $f$  is smooth). When  $M$  is only quasiregular, we use the orbifold covering to reduce Theorem 3.4 to the regular case, as follows.

Let  $x \in Q$  be a critical value of  $f$ . Since Theorem 3.4 is local, we may replace  $Q$  with a sufficiently small neighbourhood  $U$  of  $x$ , and  $M$  with  $f^{-1}(U)$ . Since  $f$  is an orbifold morphism, it admits a finite covering by a smooth morphism of manifolds  $M' \xrightarrow{f'} Q'$  such that the horizontal maps of the Cartesian square

$$\begin{array}{ccc} M' & \xrightarrow{\tau_M} & M \\ f' \downarrow & & \downarrow f \\ Q' & \xrightarrow{\tau} & Q \end{array}$$

are etale in the orbifold category.<sup>4</sup> The Vaisman manifold  $M'$  is by construction regular. Choosing  $Q$  sufficiently small, we may insure that  $f'$  admits a slice satisfying the assumptions of Claim 3.5. Then the lift  $L'_{\mathbb{C}}$  of  $L_{\mathbb{C}}$  to  $M'$  is a trivial bundle. Denote by  $\mathcal{G}$  the Galois group of  $\tau$ , that is, the orbifold deck transform group of the covering  $\tau : Q' \rightarrow Q$ . The group  $\mathcal{G}$  acts on the pushforward  $\tau_{M*}L'_{\mathbb{C}}$ , and we have

$$L_{\mathbb{C}} = (\tau_{M*}L'_{\mathbb{C}})^{\mathcal{G}},$$

where  $(\cdot)^{\mathcal{G}}$  denotes the sheaf of  $\mathcal{G}$ -invariants. Since  $L'_{\mathbb{C}}$  is trivial, the bundle  $L_{\mathbb{C}}$  is isomorphic to  $(\tau_{M*}\mathcal{O}_{M'})^{\mathcal{G}}$ . This bundle is trivial, because  $\tau_M$  is etale. We proved that  $L_{\mathbb{C}}$  is trivial, locally in  $Q$ . Theorem 3.4 is proven. The same argument shows that the natural Hermitian structure on  $L_{\mathbb{C}} \cong f^*f_*L_{\mathbb{C}}$  is lifted from the Hermitian structure on the bundle  $f_*L_{\mathbb{C}}$ .  $\square$

### 3.2. Kodaira-Nakano theorem for quasiregular Vaisman manifolds

Let  $M$  be a compact quasiregular Vaisman manifold,  $L_{\mathbb{C}}$  the weight bundle, and  $C = -2\sqrt{-1} \omega_0$  the curvature of the Chern connection in  $L_{\mathbb{C}}$  (Claim 2.10). It was shown in [Ve] that  $\omega_0$  is obtained as a pullback of a Kähler form on  $Q$ :

$$\omega_0 = f^* \omega_Q.$$

---

<sup>4</sup> Locally we have  $\tau : V'/G' \rightarrow V/G$  for some open sets  $V, V'$  and finite groups  $G', G$ . Moreover,  $V$  can be chosen simply connected. By etale in the orbifold category we understand that the induced map  $V \rightarrow V'$  is etale

**Proposition 3.6.** *Let  $M$  be a quasiregular Vaisman manifold,  $f : M \rightarrow Q$  the Lee fibration,  $L_{\mathbb{C}}$  the weight bundle and  $L_Q := f_*L_{\mathbb{C}}$  the corresponding line bundle on  $Q$ , equipped with the Chern connection. Then*

$$C_Q = -2\sqrt{-1} \omega_Q.$$

*In particular,  $L_Q$  is ample.*

*Proof.* The Hermitian bundle  $L_Q$  is equipped with a natural Hermitian metric  $h$ , such that the metric on  $L_{\mathbb{C}} = f^*L_Q$  is a pullback of  $h$  (see Theorem 3.4). Therefore, the Chern connection  $\nabla_C$  on  $L_{\mathbb{C}}$  is lifted from  $L_Q$ . The curvature of  $\nabla_C$  is  $-2\sqrt{-1} \omega_0$ , as Claim 2.10 implies. The form  $-2\sqrt{-1} \omega_0$  is a pullback of  $-2\sqrt{-1} \omega_Q$ . (Claim 3.2). This proves Proposition 3.6.  $\square$

From Proposition 3.6 and the Kodaira embedding theorem for orbifolds (cf. [Ba]), we obtain the following result

**Corollary 3.7.** *Let  $M$  be a compact quasiregular Vaisman manifold,  $f : M \rightarrow Q$  the Lee fibration and  $L_{\mathbb{C}}$  the weight bundle. Then, for  $k$  sufficiently big, the bundle  $L_{\mathbb{C}}^k$  has no base points, and defines a natural map  $l_k : M \rightarrow \mathbb{C}P^{n-1}$ , in a usual way ( $\dim H^0(L_{\mathbb{C}}^k) = n$ ). Moreover,  $l_k$  is factorized through  $f : M \rightarrow Q$ :*

$$l_k = \underline{l}_k \circ f, \quad \underline{l}_k : Q \rightarrow \mathbb{C}P^{n-1}.$$

*and  $\underline{l}_k : Q \rightarrow \mathbb{C}P^{n-1}$  is an embedding.*

### 3.3. Immersion theorem for quasiregular Vaisman manifolds

The Kodaira theorem (Corollary 3.7) implies the following Immersion Theorem for quasiregular Vaisman manifolds.

Recall that the weight bundle  $L$  is equipped with a flat connection. Consider the weight covering  $\tilde{M} \rightarrow M$  associated with the monodromy group of  $L$ . By definition,  $\tilde{M}$  is the smallest covering such that the pullback  $\tilde{L}$  of  $L$  to  $\tilde{M}$  has trivial monodromy. Since  $\tilde{L}$  is a flat bundle with trivial monodromy, it has a canonical trivialization.

Let  $l_1, l_2, \dots, l_n$  be a basis in the space  $H^0(L_{\mathbb{C}}^k)$ . Since  $\tilde{L}$  is equipped with a canonical trivialization,  $l_i$  gives a function  $\lambda_i : \tilde{M} \rightarrow \mathbb{C}$ . Since  $l_1, l_2, \dots, l_n$  induce an embedding  $Q \hookrightarrow \mathbb{C}P^{n-1}$ , the functions  $\lambda_i$  have no common zeros. Together these function define a map

$$\tilde{\lambda} : \tilde{M} \rightarrow \mathbb{C}^n \setminus 0. \tag{3.5}$$

The monodromy group  $\Gamma$  of  $L$  is isomorphic to  $\mathbb{Z}$  ([OV]; see also Subsection 4.1). Denote by  $\gamma$  its generator, and let  $q \in \text{Aut}(L) \cong \mathbb{R}^{>0}$  be the action of  $\gamma$  on

the fibers of  $L_{\mathbb{C}}$ . A holomorphic section  $\tilde{l}$  of  $\tilde{L}_{\mathbb{C}}^k$  is obtained as a pullback from a section of  $L_{\mathbb{C}}^k$  if and only if

$$\gamma(\tilde{l}) = q^k \tilde{l}, \tag{3.6}$$

where  $\gamma : H^0(\tilde{L}_{\mathbb{C}}^k) \rightarrow H^0(L_{\mathbb{C}}^k)$  is the natural equivariant action of  $\gamma \in \Gamma$  on  $\tilde{L}_{\mathbb{C}}^k$ .

Now return to the map (3.5). The eigenspace property (3.6) implies that the following square is commutative

$$\begin{CD} \tilde{M} @>\tilde{\lambda}>> \mathbb{C}^n \setminus 0 \\ @V\gamma VV @VV\text{mult. by } q^k V \\ \tilde{M} @>\tilde{\lambda}>> \mathbb{C}^n \setminus 0 \end{CD} \tag{3.7}$$

Therefore,  $\tilde{\lambda}$  is obtained as a covering for a map  $\lambda : M \rightarrow (\mathbb{C}^n \setminus 0 / \langle q^k \rangle)$  to a Hopf manifold. We also have a commutative square of Lee fibrations

$$\begin{CD} M @>\lambda>> (\mathbb{C}^n \setminus 0 / \langle q^k \rangle) \\ @Vf VV @VVV \\ Q @>l_k>> \mathbb{C}P^{n-1}. \end{CD} \tag{3.8}$$

The map  $l_k$  is by construction an embedding, hence  $\lambda$  maps different fibers of  $f$  to different Lee fibers of the Hopf manifold  $(\mathbb{C}^n \setminus 0 / \langle q^k \rangle)$ . Both sides of (3.8) are Lee fibrations, with fibers isomorphic to 1-dimensional compact tori, with the affine coordinates provided by the sections of the weight bundle. The map  $\lambda$  is by construction compatible with the affine structure on these fibers. Therefore,  $\lambda$  induces a finite covering on the fibers of Lee fibrations of (3.8). We obtain the following theorem.

**Theorem 3.8.** *Let  $M$  be a compact quasiregular Vaisman bundle. Then  $M$  admits a holomorphic immersion  $\lambda : M \rightarrow (\mathbb{C}^n \setminus 0 / \langle q^k \rangle)$  to a (diagonal) Hopf manifold. Moreover,  $\lambda$  is étale to its image, that is, induces a finite covering from  $M$  to  $\lambda(M)$ .*

In the rest of this paper we generalize this result to arbitrary (not necessarily quasiregular ) compact Vaisman manifolds.

### 4. The group generated by the Lee flow and its applications

#### 4.1. The groups $G, \tilde{G}$ generated by the Lee flow

**Definition 4.1.** *Let  $G$  be the smallest Lie subgroup of  $\text{Aut}(M)$  containing the Lee flow and let  $\tilde{G}$  be its lift to the weight covering  $\tilde{M}$ .*

It is easy to see that  $G$  is compact and abelian (see [KO]). Therefore,  $G \cong (S^1)^k$ . In [OV] it was shown that

$$\tilde{G} \cong (S^1)^{k-1} \times \mathbb{R}, \tag{4.1}$$

and the (restriction of) natural forgetful map  $\tilde{G} \xrightarrow{\Phi} G$  (cf. (2.4)) is a covering. The group  $\Gamma$  of deck transformations of  $\tilde{M}$  is clearly isomorphic to  $\ker \Phi$ . Therefore, (4.1) implies

$$\Gamma \cong \mathbb{Z}. \tag{4.2}$$

#### 4.2. Complexification of $\tilde{G}$

Consider the Lie algebra  $\text{Lie}(\tilde{G}) \subset H^0(T\tilde{M})$ . The vector fields in  $\text{Lie}(\tilde{G})$  are holomorphic, because the Lee flow is holomorphic. Let  $\tilde{G}_{\mathbb{C}} \subset \text{Aut}(\tilde{M})$  be the complex Lie group generated by  $\tilde{G}$  and acting on  $\tilde{M}$ . Clearly,  $\text{Lie}(\tilde{G}_{\mathbb{C}}) = \text{Lie}(\tilde{G}) + I(\text{Lie}(\tilde{G}))$ , where  $I \in \text{End}(T\tilde{M})$  is the complex structure operator on  $\tilde{M}$ .

This operation is called **complexification of the Lie group  $\tilde{G}$** . In the same way, a complexification can be performed for any real Lie group acting on a complex manifold by holomorphic diffeomorphisms.

By the Remmert-Morimoto theorem, a complex connected abelian Lie group is isomorphic to

$$(\mathbb{C}^*)^{l_1} \times \mathbb{C}^{l_2} \times T, \tag{4.3}$$

where  $T$  is a compact complex torus ([Mo]). A Lie group  $\mathcal{G}$  is called **linear** if it has an exact complex representation  $\mathcal{G} \hookrightarrow GL(n, \mathbb{C})$ . Clearly, an abelian group  $\mathcal{G}$  is linear if  $\mathcal{G} \cong (\mathbb{C}^*)^{l_1} \times \mathbb{C}^{l_2}$ .

**Lemma 4.2.** *Let  $M$  be a compact Vaisman manifold,  $\tilde{M}$  its weight covering, and  $\tilde{G}_{\mathbb{C}}$  the complex Lie group generated by the Lee flow as above. Then  $\tilde{G}_{\mathbb{C}}$  is linear.*

*Proof.* Consider the action of  $\tilde{G}_{\mathbb{C}}$  in  $\tilde{M}$ . Since  $\tilde{G}_{\mathbb{C}} \subset \text{Diff}(\tilde{M})$ , a general orbit  $X$  of  $\tilde{G}_{\mathbb{C}}$  is isomorphic to  $\tilde{G}_{\mathbb{C}}$ . Since  $\tilde{M}$  is equipped with a Kähler potential, the manifold  $X \cong \tilde{G}_{\mathbb{C}}$  is also equipped with a Kähler potential. Therefore,  $\tilde{G}_{\mathbb{C}}$  does not contain a non-trivial compact complex torus. □

**Proposition 4.3.** *Let  $M$  be a compact Vaisman manifold,  $\tilde{M}$  its weight covering, and  $\tilde{G}_{\mathbb{C}}$  the complex Lie group generated by the Lee flow as above. Then  $\tilde{G}_{\mathbb{C}} \cong (\mathbb{C}^*)^l$ , for some  $l$ .*

*Proof.* Denote by  $\tilde{G}_0$  the group of all  $g \in \tilde{G}$  satisfying  $\rho(g) = 1$ , where  $\rho : \tilde{G} \rightarrow \mathbb{R}^{>0}$  is the map defined in (2.2). Since  $\tilde{G} \cong (S^1)^{k-1} \times \mathbb{R}$ , and  $\rho : \tilde{G} \rightarrow \mathbb{R}$

is surjective,<sup>5</sup> we have  $\tilde{G}_0 \cong (S^1)^{k-1}$ . Denote by  $\tilde{G}_K$  the Lie subgroup of  $\tilde{G}_\mathbb{C}$  generated by  $\tilde{G}_0$  and  $e^{tI(\theta^\sharp)}$ ,  $t \in \mathbb{R}$ , where  $I(\theta^\sharp)$  is the complex conjugate of the Lee field. It is easy to see (see e.g. [DO, Pr. 4.3]) that  $e^{tI(\theta^\sharp)}$  acts on  $M$  by holomorphic isometries and preserves the Kähler metric on  $\tilde{M}$ .<sup>6</sup> Therefore, the forgetful map  $\Phi : \tilde{G}_K \rightarrow \text{Aut}(M)$  sends  $\tilde{G}_K$  to isometries of  $M$ .

The following elementary claim is used to prove Proposition 4.3.

*Claim 4.4.* Let  $M$  be a compact Vaisman manifold,

$$\tilde{G}_K \subset \text{Aut}(\tilde{M}, M)$$

the group defined above, and

$$\Phi : \tilde{G}_K \rightarrow \text{Iso}(M) \tag{4.4}$$

the forgetful map from  $\tilde{G}_K$  to the group of isometries of  $M$ . Then  $\Phi : \tilde{G}_K \rightarrow \text{Iso}(M)$  is a monomorphism.

*Proof.* Let  $\nu \in \ker \Phi$ . Since  $\nu$  does not act on  $M$ ,  $\nu$  belongs to the deck transformation group of  $\tilde{M}$ . Unless  $\nu$  is trivial,  $\nu$  acts on  $\tilde{M}$  by non-trivial homotheties of the Kähler form. However,  $\tilde{G}_K$  is generated by  $\tilde{G}_0$  and  $e^{tI(\theta^\sharp)}$ , and these diffeomorphisms preserve the Kähler metric on  $\tilde{M}$ . Therefore, the group  $\tilde{G}_K$  preserves the Kähler metric on  $\tilde{M}$ , and  $\nu$  is trivial.  $\square$

Return to the proof of Proposition 4.3. The group  $\tilde{G} \cong (S^1)^k \times \mathbb{R}$  is generated by  $\tilde{G}_0 \cong (S^1)^k$  and  $e^{t\theta^\sharp}$ , and  $\tilde{G}_K$  is generated by  $\tilde{G}_0$  and  $e^{tI(\theta^\sharp)}$ . Therefore the complexification of  $\tilde{G}_K$  coincides with the complexification of  $\tilde{G}$ .

By Claim 4.4,  $\tilde{G}_K$  is a subgroup of the group of isometries of  $M$ , and therefore it is compact. A compact subgroup of a complex linear Lie group is **totally real**, that is, its Lie algebra does not intersect its complex conjugate. Now,  $\tilde{G}_\mathbb{C}$  is a complexification of its totally real subgroup  $\tilde{G}_K$ ; therefore,  $\tilde{G}_K$  is a real form of  $\tilde{G}_\mathbb{C}$ . A complex linear connected abelian Lie group with compact real form is isomorphic to  $(\mathbb{C}^*)^n$ , as the Morimoto-Remmert classification theorem implies (see (4.3)). This proves Proposition 4.3.  $\square$

### 4.3. LCK manifolds associated with $\Gamma' \subset \tilde{G}_\mathbb{C}$

Let  $M$  be a compact Vaisman manifold,  $\tilde{M}$  its weight covering,  $G, \tilde{G}, \tilde{G}_\mathbb{C}$  the Lie groups defined in Subsection 4.1 and Subsection 4.2. By construction,  $M = \tilde{M}/\Gamma$ , where  $\Gamma \cong \mathbb{Z}$  is a subgroup of  $\tilde{G} \subset \text{Aut}(\tilde{M}, M)$ . Let  $\gamma \in \Gamma$  be a generator of  $\Gamma$ .

<sup>5</sup> The Lee flow acts by non-trivial homotheties on the Kähler form.

<sup>6</sup> The last statement is clear because  $I\theta^\sharp$  is Killing and if  $\theta = df$ , then  $\mathcal{L}_{I\theta^\sharp}(e^{-f}g) = I\theta^\sharp(e^{-f}g) = -e^{-f}\theta(I\theta^\sharp)g = 0$ .

**Theorem 4.5.** *In the above assumptions, consider an element  $\gamma' \in \tilde{G}_{\mathbb{C}}$  generating an abelian group  $\Gamma' \subset \tilde{G}_{\mathbb{C}}$ . Then, for  $\gamma'$  sufficiently close to  $\gamma \in \tilde{G}_{\mathbb{C}}$ , the quotient space  $\tilde{M}/\Gamma'$  is a compact Vaisman manifold. Moreover, the Lee field of  $\tilde{M}/\Gamma'$  is proportional to  $v' := \log \gamma' \in \text{Lie}(\tilde{G}_{\mathbb{C}})$ .<sup>7</sup>*

*Proof.* Let  $v$  denote the Lee field  $\theta^{\sharp}$  of  $M$  lifted to  $\tilde{M}$ . The Kähler form on  $\tilde{M}$  has a potential  $\varphi$  (see [Ve, Pr. 4.4] and the end of Section 2), which can be written as  $\varphi := |v|^2$ . If  $\gamma$  is sufficiently close to  $\gamma'$ , then  $\varphi' := |v'|^2$  is also plurisubharmonic and gives a Kähler potential on  $\tilde{M}$ . Denote by  $\omega'$  the corresponding Kähler metric on  $\tilde{M}$ :

$$\omega' := \sqrt{-1} \partial \bar{\partial} |v'|^2$$

Consider the differential flow  $\psi'_t$  associated with the vector field  $v'$ . Clearly,  $\psi'_t = e^{tv'}$  multiplies  $v'$  by a number  $e^t$ . Therefore,  $\psi'_t$  maps  $|v'|^2$  to  $e^{2t}|v'|^2$ . We obtain that  $\psi'_t$  preserves the conformal class of the Kähler metric  $\omega'$ . This implies that the quotient  $\tilde{M}/\Gamma' = \tilde{M}/\psi'_t$  is LCK. By construction,  $v'$  is the Lee field of  $\tilde{M}/\Gamma'$ , hence  $\tilde{M}/\Gamma'$  is Vaisman. This proves Theorem 4.5.

We also obtain that  $\tilde{M}/\Gamma'$  is Vaisman (in particular LCK) whenever the function

$$\psi' := |\log \gamma'|^2$$

is strictly plurisubharmonic.<sup>8</sup>

□

#### 4.4. Quasiregular Vaisman manifolds obtained as deformations

Let  $M$  be a compact Vaisman manifold and  $\tilde{G}_{\mathbb{C}} \cong (\mathbb{C}^*)^l$  the complex Lie group constructed in Subsection 4.2. As we have shown,  $\tilde{G}_{\mathbb{C}}$  has a compact real form  $\tilde{G}_K$ . This allows us to equip the Lie algebra

$$\text{Lie}(\tilde{G}_{\mathbb{C}}) = \text{Lie}(\tilde{G}_K) \otimes \mathbb{C}$$

with a rational structure, as follows. Given  $\delta \in \text{Lie}(\tilde{G}_K)$ , we say that  $\delta$  is **rational** if the corresponding 1-parametric subgroup  $e^{t\delta} \subset \tilde{G}_K$  is compact. Clearly, this gives a rational lattice in  $\text{Lie}(\tilde{G}_K)$ , and hence in  $\text{Lie}(\tilde{G}_{\mathbb{C}})$ . This rational lattice corresponds to an integer lattice of all

$$\{\delta \in \text{Lie}(\tilde{G}_K) \mid e^{\delta} = 1\}.$$

<sup>7</sup> The group  $\tilde{G}_{\mathbb{C}}$  is connected and its exponential map is surjective, hence a definition of the logarithm is possible.

<sup>8</sup> A function  $f$  is called **strictly plurisubharmonic** if and only if the 2-form  $\sqrt{-1} \partial \bar{\partial} f$  is Kähler.

A complex 1-parametric subgroup  $e^{\lambda\delta} \subset \tilde{G}_{\mathbb{C}}$  is isomorphic to  $\mathbb{C}^*$  if and only if the line  $\lambda\delta \subset \text{Lie}(\tilde{G}_{\mathbb{C}})$  is rational (contains a rational point); otherwise,  $e^{\lambda\delta}$  is isomorphic to  $\mathbb{C}$ . This is clear, because the kernel of the exponential map consists exactly of integer elements within  $\text{Lie}(\tilde{G}_K)$ .

**Proposition 4.6.** *Let  $M$  be a compact Vaisman manifold,  $\tilde{M}$  its weight covering and  $G, \tilde{G}, \tilde{G}_{\mathbb{C}}$  the Lie groups defined above. Denote by  $\Gamma \subset \tilde{G}$  the weight monodromy group generated by  $\gamma \in \tilde{G}$ . Take  $\gamma' \in \tilde{G}_{\mathbb{C}}$ , sufficiently close to  $\gamma$ , in such a way that the quotient  $\tilde{M}/\langle\gamma'\rangle$  is a Vaisman manifold (see Theorem 4.5). Assume that the line  $\mathbb{C}\log \gamma'$  is rational.<sup>9</sup> Then  $\tilde{M}/\langle\gamma'\rangle$  is quasiregular.*

*Proof.* Since  $\gamma'$  is close to  $\gamma \notin \tilde{G}_K$ , we may assume that  $\gamma'$  does not lie in the compact subgroup  $\tilde{G}_K \subset \tilde{G}_{\mathbb{C}}$ . Denote by  $v'$  the vector  $\log \gamma'$ , and let  $v'' = \lambda_0 v'$  be the integer point on the line  $\mathbb{C}v'$  which exists by our assumptions. Since  $v'' \in \text{Lie}(\tilde{G}_K)$ , the coefficient  $\lambda_0$  is not real.

Consider the holomorphic Lee flow  $\psi'_\lambda = e^{a v'}$  on  $\tilde{M}/\langle\gamma'\rangle$ . Clearly,  $\psi'_a$  acts on  $M$  trivially for  $a = 1$  (because  $e^{v'} = \gamma'$ ) and  $a = a_0$  (because  $v'' = a_0 v'$  is integer in  $\text{Lie}(\tilde{G}_K)$ ). Therefore, the Lee flow is factorized through the action of a compact Lie group  $\mathbb{C}/\langle 1, a_0 \rangle$ , and has compact orbits. We proved that  $\tilde{M}/\langle\gamma'\rangle$  is quasiregular. □

### 5. Immersion theorem for general Vaisman manifolds

The main result of this paper is the following

**Theorem 5.1.** *Let  $M$  be a compact Vaisman manifold. Then  $M$  admits an immersion  $M \xrightarrow{\lambda} H_\Lambda$  to a Hopf manifold. Moreover,  $\lambda$  is compatible with the action of the Lee flow on  $M$  and  $H_\Lambda$ , and induces a finite covering  $\lambda : M \rightarrow \lambda(M)$  from  $M$  to its image.*

*Remark 5.2.* The converse statement is also true: given a compact submanifold  $X \subset H_\Lambda$ ,  $\dim X > 0$ , the manifold  $X$  is LCK (which is clear from the definitions) and Vaisman (see [Ve], Proposition 6.5).

*Remark 5.3.* Theorem 3.8 (Immersion Theorem for quasiregular Vaisman manifolds) is a special case of Theorem 5.1.

*Proof of Theorem 5.1.* Let  $\tilde{M}$  be the weight covering of  $M$ ,  $\Gamma \cong \mathbb{Z}$  its deck transform group,  $M \cong \tilde{M}/\Gamma$ , and let  $\gamma$  be a generator of  $\Gamma$ . By Proposition 4.6, there exists  $\gamma' \in \tilde{G}_{\mathbb{C}} \subset \text{Aut}(\tilde{M}, M)$  sufficiently close to  $\gamma$ , such that the quotient  $M' := \tilde{M}/\langle\gamma'\rangle$  is a quasiregular Vaisman manifold. Applying the same argument we used to construct (3.7), we find an immersion

$$\lambda' : \tilde{M} \rightarrow \mathbb{C}^n \setminus 0$$

with the following properties.

---

<sup>9</sup> Such  $\gamma'$  are clearly dense in  $\tilde{G}_{\mathbb{C}}$ , because the set of rational points is dense.

- (i)  $\lambda'$  is a finite covering of its image
- (ii)  $\lambda'$  is a map associated with the space  $H^0(M', L_{\mathbb{C}}^k)$  of all holomorphic functions on  $\tilde{M}$  which satisfy the following condition

$$H^0(M', L_{\mathbb{C}}^k) = \{\chi : \tilde{M} \rightarrow \mathbb{C} \mid \chi_i \circ \gamma' = q^k \chi\} \tag{5.1}$$

where  $q > 1$  is monodromy action of  $\gamma'$  in the weight bundle on  $M'$ .

By construction,  $\gamma$  and  $\gamma'$  commute (they both sit in the same commutative group  $\tilde{G}_{\mathbb{C}}$ ; see Subsection 4.4). Therefore, the action of  $\gamma$  on  $\tilde{M}$  preserves the space (5.1). This gives a commutative square

$$\begin{CD} \tilde{M} @>\gamma>> \tilde{M} \\ @V\lambda'VV @VV\lambda'V \\ H^0(M', L_{\mathbb{C}}^k) @>\lambda'(\gamma)>> H^0(M', L_{\mathbb{C}}^k). \end{CD} \tag{5.2}$$

If  $\gamma'$  is chosen sufficiently close to  $\gamma$ , the action of  $\lambda'(\gamma)$  on  $H^0(M', L_{\mathbb{C}}^k)$  will be sufficiently close to  $q^k$ , which can be chosen arbitrarily big. Therefore, we may assume that all eigenvalues of  $\lambda'(\gamma)$  are  $> 1$ . This implies that

$$H := (H^0(M', L_{\mathbb{C}}^k) \setminus \{0\}) / \lambda'(\gamma)$$

is a well-defined complex manifold. From (5.2) we obtain an immersion

$$\tilde{M} / \langle \gamma \rangle \rightarrow (H^0(M', L_{\mathbb{C}}^k) \setminus \{0\}) / \lambda'(\gamma),$$

that is, an immersion of  $M$  to  $H$  which is a finite covering of its image.

To finish the proof of Theorem 5.1, it remains to show that  $H$  is a Hopf manifold, that is, to show that the operator  $\lambda'(\gamma)$  is semisimple (Definition 2.6). This is implied by the following claim.

*Claim 5.4.* Let  $M$  be a compact Vaisman manifold,  $\tilde{M}$  its weight covering, and  $\tilde{G}_{\mathbb{C}}$  the complex Lie group acting on  $\tilde{M}$  as in Subsection 4.2. Take arbitrary elements  $\gamma, \gamma' \in \tilde{G}_{\mathbb{C}}$ , and denote by  $V_{\gamma'}$  the vector space of all holomorphic functions  $f : \tilde{M} \rightarrow \mathbb{C}$  satisfying

$$f \circ \gamma' = q^k f.$$

Assume that  $V_{\gamma'}$  is finite-dimensional. Then the natural action of  $\gamma$  on  $V_{\gamma'}$  is semisimple.

*Proof.* Consider the natural action of  $\tilde{G}_{\mathbb{C}}$  on  $\tilde{M}$ . This action is holomorphic and commutes with  $\gamma'$ , hence  $\tilde{G}_{\mathbb{C}}$  acts on  $V_{\gamma'}$ . Now,  $\gamma \in \tilde{G}_{\mathbb{C}}$  is an element of a group  $\tilde{G}_{\mathbb{C}} \cong (\mathbb{C}^*)^n$  acting on a finite-dimensional vector space. All elements of a subgroup  $G \subset GL(n, \mathbb{C})$ ,  $G \cong (\mathbb{C}^*)^n$  are semisimple, as an elementary group-theoretic argument shows. Therefore,  $\gamma$  is semisimple. Claim 5.4 is proven and  $H$  is in fact a Hopf manifold  $H_{\Lambda}$ . This proves Theorem 5.1. □

*Remark 5.5.* It follows from [Va1, Th. 5.1], [Ts, Th. 3.2] and [Ve, Pr. 6.5] that the metric induced on  $M$  by the above holomorphic immersion is a Vaisman metric. But it doesn't necessarily coincide with the initial one. Hence the immersion is not isometric.

### 6. Applications to Sasakian geometry

As a by-product of the immersion theorem for compact Vaisman manifolds, we derive a similar result for compact Sasakian manifolds, the model space now being the odd sphere equipped with a generally non-round metric and with a Sasakian structure obtained by deforming the standard one by means of an  $S^1$  action (see [KO]). We briefly recall this construction:

Let  $S^{2n-1}$  be the unit sphere of  $\mathbb{C}^n$  endowed with its standard round metric and contact structure  $\eta_0 = \sum(x_j dy_j - y_j dx_j)$ . Let also  $J$  denote the almost complex structure of the contact distribution. Deform  $\eta_0$  by means of an action of  $S^1$  by setting  $\eta_A = \frac{1}{\sum a_j |z_j|^2} \eta_0$ , for  $0 < a_1 \leq a_2 \cdots \leq a_n$ . Its Reeb field is  $R_A = \sum a_j (x_j \partial y_j - y_j \partial x_j)$ . Clearly,  $\eta_0$  and  $\eta_A$  underly the same contact structure. Finally, define the metric  $g_A$  as follows:

- $g_A(X, Y) = d\eta_A(IX, Y)$  on the contact distribution ;
- $R_A$  is normal to the contact distribution and has unit length.

It can be seen that  $S_A^{2n-1} := (S^{2n-1}, g_A)$  is a Sasakian manifold. If all the  $a_j$  are equal, this structure corresponds to a homothetic transformation along the contact distribution (a D-homothetic transformation in the terms of S. Tanno [Ta]).

The Hopf manifold  $H_\Lambda$ , with  $\lambda_j = e^{-a_j}$ , is obtained by identifying  $S_A^{2n-1} \times \mathbb{R}$  with  $\mathbb{C}^n \setminus 0$  by means of the diffeomorphism  $(z_j, t) \mapsto (e^{-a_j t} z_j)$ . In particular, the Lee form of  $H_\Lambda$  can now be identified with  $-dt$ , hence the Lee flow corresponds to the action of the  $S^1$  factor.

Let now  $(X, h)$  be a compact Sasakian manifold. Then the trivial bundle  $M = X \times S^1$  has a Vaisman structure  $(I, g)$  which is quasiregular if the Reeb field  $\xi$  of  $X$  (which, on  $M$ , corresponds to  $I\theta^\sharp$ ), determines a regular foliation. From Theorem 5.1,  $M$  is immersed in a Hopf manifold  $H_\Lambda$  and the immersion commutes with the action of the respective Lee flows. As the Lee field of  $H_\Lambda$  is induced from the action of the  $S^1$  factor, the immersion descends to an immersion of Sasakian manifolds  $f : X \rightarrow S_A^{2n-1}$ . If the Reeb field on  $X$  is regular, then  $M$  is quasiregular and  $H_\Lambda$  is a diagonal Hopf manifold, hence all  $a_j$  are equal.

Note that the Reeb field of  $X$  corresponds on the Vaisman manifold  $M$  to the anti-Lee field  $I\theta^\sharp$ . As the holomorphic immersion of Vaisman manifolds preserves the Lee field, the induced immersion at the Sasakian level preserves the Reeb fields (still not being an isometry).

On the other hand, recall that, by construction, the restriction of the complex structure  $I$  to the contact distribution  $D$  of  $X$  coincides with the natural CR-

structure of  $X$  (see [DO]). Hence, any  $X \in \Gamma(D)$  is of the form  $X = IY$  with  $Y \in \Gamma(D)$  and, as  $df$  commutes with  $I$ , we have  $g_A(dfX, R_A) = g_A(dfIY, R_A) = g_A(I_0dfY, R_A) = 0$ , because  $I_0R_A = 0$  by on  $S_A^{2n-1}$ . Thus  $dfX$  belongs to the contact distribution of  $S_A^{2n-1}$ .

Summing up we proved:

**Theorem 6.1.** *Let  $X$  be a compact Sasakian manifold. Then  $X$  admits an immersion into a Sasakian odd sphere  $S_A^{2n-1}$  as above. This immersion is compatible with the contact structure on  $X$ ,  $S_A^{2n-1}$ , and preserves the Reeb field.*

The existence of an effective  $U(1)$  action on a compact Riemannian manifold imposes severe geometric and topological restrictions. From Proposition 4.6 we derive that any compact Sasakian manifold admits such an action. More precisely:

**Proposition 6.2.** *Any compact Sasakian manifold  $X$  has a deformation which is isomorphic to a circle bundle over a projective orbifold. In particular, any Sasakian manifold admits an effective circle action.*

We note that this result is in fact true in a more general context, for  $K$ -contact manifolds, as proved in [Ru].

Recall that a compact manifold which has an effective circle action necessarily has zero minimal volume (cf. [Gr]; we are thankful to Gilles Courtois for explaining us this point). On the other hand, a compact manifold which admits a metric of strictly negative sectional curvature must have non-zero minimal volume, *loc. cit.* We obtain the following corollary, previously proved in [BG2, Pr. 2.7] for  $K$ -contact manifolds and, using harmonic maps, for Sasakian manifolds, in [Pe]:

**Corollary 6.3.** *A compact Sasakian manifold cannot bear any Riemannian metric of strictly negative sectional curvature.*

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