Induced Hopf bundles and Einstein metrics

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Abstract

We give a natural construction of an Einstein metric g on the products $S^3 \times S^2$ and $S^7 \times S^6$, total spaces of some induced Hopf bundles. Since g is also a Sasakian metric, a locally conformal Kähler and conformally Ricci-flat metric h is induced by g on the products $S^3 \times S^2 \times S^1$ and $S^7 \times S^6 \times S^1$, that fiber also as twistor spaces over the hypercomplex and the Cayley Hopf manifolds $S^3 \times S^1$ and $S^7 \times S^1$. An extension of this construction is given to some Stiefel manifolds and induced Hopf bundles over Segre manifolds.

The product of spheres $S^3 \times S^2$ is an example of manifold whose moduli space of Einstein structures has infinitely many components, cf. [1], p. 472. Among all these possible choices, a very special Einstein metric g - non homothetic to the standard product $2g_0 \times g_0$ - has been considered in several contexts: [9], p. 404, [16], p. 291, [5], p. 277, [2], pp. 95-96, and indeed a general framework for the existence of g can be traced back to a theorem of S. Kobayashi (cf. [11], p. 136, as well as its generalization in [1], pp. 255-256).

The simple construction of g we are presenting here is obtained by a natural imbedding of $S^3 \times S^2$ into S^7, after a deformation of the standard metric of S^7 in the direction of one of its Sasakian structure vector fields. This procedure can be extended to obtain similar Einstein metrics on $S^7 \times S^6$, on the Stiefel manifolds $V_2(\mathbf{R}^{n+1})$ of the oriented orthonormal 2-frames, and on the induced Hopf bundles over some Segre complex projective manifolds.

Our motivation comes from studying diagrams like the following:

$$\begin{array}{ccccccc} Z_{S^{4n+3}\times S^1} & \to & Z_{S^{4n+3}} & \to & Z_{\mathbf{C}P^{2n+1}} & \to & \mathbf{C}P^{2n+1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S^{4n+3}\times S^1 & \to & S^{4n+3} & \to & \mathbf{C}P^{2n+1} & \to & \mathbf{H}P^n, \end{array}$$

whose lower horizontal arrows are the prototypes of well known fibrations appearing in both 3-Sasakian and quaternion Hermitian-Weyl geometry: [4], [14], [15]. The vertical arrows, fibrations in spheres S^2, can be looked at as "twistor spaces" over the base manifolds, with respect to their structures - from the left to the right of the diagram - of hyperhermitian-Weyl, 3-Sasakian, Kähler-Einstein and quaternion Kähler manifold. The fibers of the above diagram, over a point of $\mathbf{H}P^n$ - and in fact of any positive quaternion Kähler manifold - are describes as:

$$\begin{array}{ccccccc} S^3\times S^2\times S^1 & \to & S^3\times S^2 & \to & \mathbf{C}P^1\times\mathbf{C}P^1 & \to & \mathbf{C}P^1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S^3\times S^1 & \to & S^3 & \to & \mathbf{C}P^1 & \to & pt, \end{array}$$

suggesting to study structures on $S^3\times S^2$ and on $S^3\times S^2\times S^1$ related to the geometries appearing in both the diagrams.

1 Preliminaries

We start by collecting some basic definitions and facts about Sasakian and 3-Sasakian geometry (cf. for example [3], [4]).

Definition 1 .

(i) Let (N, g^N) be a $(2n+1)$-dimensional Riemannian manifold endowed with a unitary Killing vector field ξ whose dual 1-form is denoted by η. The Levi-Civita connection ∇^N of g^N defines the smooth section $\varphi = \nabla^N\xi$ of $End(TN)$. If the equation:

$$(\nabla^N_Y\varphi)Z = \eta(Z)Y - g^N(Y,Z)\xi$$

holds on N, then ξ defines a Sasakian structure *on (N, g).*

(ii) A $(4n+3)$-dimensional Riemannian manifold (P, g^P) is 3-Sasakian *if a triple ξ^1, ξ^2, ξ^3 of orthonormal Sasakian structures are defined on P and they satisfy the identities $[\xi^\alpha, \xi^\beta] = \xi^\gamma$ for $(\alpha,\beta,\gamma) = (1,2,3)$ and cyclic permutations.*

We call ξ or ξ^1, ξ^2, ξ^3 the *structure vector fields*, and note that each dual one-form η, η^1, η^2, η^3 is a contact form.

The following formulae are easily proved:

Proposition 1 .

(i) On any Sasakian manifold:

$$\varphi\xi = 0, \quad g^N(\varphi Y, \varphi Z) = g^N(Y,Z) - \eta(Y)\eta(Z),$$

for all the tangent vector fields Y, Z. Moreover the sectional curvature K of sections containing ξ satisfies the following normalization condition:

$$K(Y,\xi) = 1.$$

(ii) On any 3-Sasakian manifold, besides the above formulae for each $\alpha = 1, 2, 3$, *the following holds:*

$$\varphi^\alpha \xi^\beta = -\xi^\gamma,$$

for $(\alpha, \beta, \gamma) = (1, 2, 3)$ *and cyclic permutations.*

The spheres S^{2n+1} and S^{4n+3}, with their standard metric g_0, are examples of Sasakian and 3-Sasakian manifolds, respectively: their structure vector fields are $-JN$ or $-I_1N$, $-I_2N$, $-I_3N$, where J and I_1,I_2,I_3 are the canonical complex and hypercomplex structure of the respective Euclidean spaces E^{2n+2} or E^{4n+4}, and N is the unit normal.

There are fibrations relating Sasakian and 3-Sasakian manifolds respectively with Kähler and quaternion Kähler geometry. Namely (cf. [20], pp. 286-291, and [4]):

Proposition 2 .

(i) Let N^{2n+1} *be a compact Sasakian manifold whose structure vector field* ξ *generates a regular foliation. Then the projection* $\pi : N \to M = N/\xi$ *is a principal circle bundle, the metric* g^N *of* N *projects to a Kähler metric on* M*, whose Kähler form* Ω *has integral values.* M *is thus a complex projective algebraic manifold, and* η *is a connection form in* $N \to M$ *with curvature the pull-back of* Ω.

(ii) Let P^{4n+3} *be a compact 3-Sasakian manifold whose structure vector fields* ξ^1, ξ^2, ξ^3 *generate a regular foliation. Then the projection* $\pi : P \to M = P/\xi$ *is a bundle of 3-dimensional homogeneous spherical space forms over the positive quaternion Kähler manifold* M.

The map $\pi : N \to M = N/\xi$ is known as a *Boothby-Wang fibration*, and we are concerned with the Einstein property of its total space N. We need in this respect the following notion:

Definition 2 . *A Sasakian manifold* (N, g^N, ξ) *is* η-Einstein *if its Ricci tensor satisfies* $Ric = \lambda g^N + \mu\eta \otimes \eta$.

If N has dimension ≥ 5, then λ and μ can be shown to be constant (cf. [20], p. 285): they can be called the *Einstein constants* of the Sasaki η-Einstein manifold N.

Lemma 1 . *On any Boothby-Wang fibration* $N^{2n+1} \to M^{2n}$, N *is* η*-Einstein with Einstein constants* $(\lambda, \mu) = (\alpha - 2, 2n + 2 - \alpha)$ *if and only if* M *is Kähler-Einstein with Einstein constant* α.

Proof. Use the following relation between the Ricci tensors of M and N, total and base spaces of a Riemannian submersion $N \to M$ with geodesic S^1 fibres ([1], p. 244):

$$Ric^M(Y, Z) = Ric^N(Y^*, Z^*) + 2g^N(A_{Y^*}V, A_{Z^*}V).$$

Here V is tangent to the fibre, A is the O'Neill tensor: $A_{Y^*}V$ = horizontal part of $\nabla^N_{Y^*}V$, and Y^* is the horizontal lift of the vector field Y of M. In our case $V = \xi$, $A_{Y^*}V =$

φY^* and from prop. 1 we see that $g^N(\varphi Y^*, \varphi Z^*) = g^N(Y^*, Z^*) - \eta(Y^*)\eta(Z^*)$. Since the horizontal distribution is the kernel of η, it follows:

$$Ric^M(Y, Z) = Ric^N(Y^*, Z^*) + 2g^N(Y^*, Z^*).$$

Then the conclusion follows from the normalization property $K(Y, \xi) = 1$ of prop. 1, giving $Ric(\xi, \xi) = 2n$ on any $2n + 1$ dimensional Sasakian manifold.

2 Induced Hopf bundles and $S^3 \times S^2$

Denote now by $(\overline{N}, g^{\overline{N}}, \xi)$ a compact Sasakian manifold, and assume that its structure Killing vector field ξ generates a regular foliation, so that the Boothby-Wang fibration $\overline{\pi} : \overline{N} \to \overline{M} = \overline{N}/\xi$ projects $\overline{N}$ over the complex projective algebraic manifold $\overline{M}$. Any isometric immersion $i_M : M \hookrightarrow \overline{M}$ gives rise to both a corresponding isometric immersion $i_N : N \hookrightarrow \overline{N}$ of the induced total space N and to an induced S^1 bundle $\pi : N \to M$. If i_M is a complex immersion and ξ is tangent to the immersed manifold $i_N(N)$, then i_N is invariant to $\varphi = \nabla^{\overline{N}}\xi$, thus N inherits a Sasakian structure (cf. [19], p. 102). We shall denote by g^M, g^N the induced metrics.

Lemma 2 . *In the above setting, if i_M is minimal, also i_N is minimal.*

Proof. With the above notations one has ([19], p.100):

$$(JY)^* = \varphi Y^*, \quad g^{\overline{N}}(Y^*, Z^*) = g^{\overline{M}}(Y, Z),$$

$$(\nabla^{\overline{M}}_Y Z)^* = \nabla^{\overline{N}}_{Y^*} Z^* + g^{\overline{N}}(\varphi Y^*, Z^*)\xi,$$

for any vector fields Y, Z on $\overline{M}$. The Gauss formula then yields the following relation between the second fundamental forms of i_M and i_N:

$$(B^M(Y, Z))^* = B^N(Y^*, Z^*).$$

Use local bases of vector fields on N as $\{e_1^*, e_2^*, ..., e_k^*, \xi\}$, $\{e_i^*\}$ $(i = 1, ...k)$ projecting to local bases $\{e_i\}$ on M: then, if i_M is minimal, in order for i_N to be minimal it is enough to show that $B^N(\xi, \xi) = 0$. This follows from prop. 1, since $B^N(\xi, \xi)$ is the normal part of $\nabla^{\overline{N}}_\xi \xi = \varphi\xi$.

Consider now the following induced Hopf bundle:

$$\begin{array}{ccc} V & \to & S^7 \\ \downarrow & & \downarrow \\ \mathbf{C}P^1 \times \mathbf{C}P^1 & \to & \mathbf{C}P^3, \end{array}$$

where the lower horizontal arrow is the Segre map $\Psi : ([x_0 : x_1], [y_0 : y_1]) \to [z_0 : z_1 : z_2 : z_3] = [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1]$ imbedding the product of two complex projective lines as the non singular quadric $Q_2 : \ z_0z_3 = z_1z_2$ of $\mathbf{C}P^3$.

V can be identified by looking at both the Hopf fibrations $S^7 \to \mathbf{C}P^3 \to \mathbf{H}P^1$, $(z_0, z_1, z_2, z_3) \to [z_0 : z_1 : z_2 : z_3] \to [h_0 : h_1]$, representing points of S^7 also by pairs of quaternions $h_0 = z_0 + z_2 j,\ h_1 = z_1 + z_3 j$. Thus the restriction of $S^7 \to \mathbf{C}P^3$ to a projective line, say $l : \Psi([x_0 : x_1], [1 : 0])$, in one of the two families that rule the quadric Q_2, is the $S^3 \subset S^7$ over the point $[1 : 0] \in \mathbf{H}P^1$. By letting the line l vary in all its family, spanned by $[y_0 : y_1]$, the diffeomorphism $V \cong S^3 \times S^2$ is recognized. Observe now that Ψ is a complex and isometric immersion with respect to the product metric of $\mathbf{C}P^1 \times \mathbf{C}P^1$ and the Fubini-Study metric of $\mathbf{C}P^3$, fixed on both $\mathbf{C}P^1$ and $\mathbf{C}P^3$ to have holomorphic sectional curvature 4. Thus $S^7 \to \mathbf{C}P^3$ is a Riemannian submersion and the manifolds S^7, $\mathbf{C}P^3$, $\mathbf{C}P^1 \times \mathbf{C}P^1$ in the diagram have Einstein constants respectively $\alpha = 6, 8, 4$. Hence lemmas 1 and 2 give:

Proposition 3 . *$V \cong S^2 \times S^3 \subset S^7$ is a minimal Sasakian and η-Einstein submanifold with Einstein constants $(\lambda, \mu) = (2, 2)$.*

We now prove that the induced Hopf bundle $\rho : V \to \mathbf{C}P^1 \times \mathbf{C}P^1$ can be identified with the Stiefel bundle of the oriented orthonormal 2-frames $\sigma : V_2(\mathbf{R}^4) \to Gr_2(\mathbf{R}^4)$ over the Grassmannian of the oriented 2-planes in $\mathbf{R}^4$. Compare in fact the Chern class of the two S^1-bundles. First $c_1(\rho) = \Psi^\star \alpha = a_1 + a_2$, where Ψ is the Segre map and α, a_1, a_2 are the canonical generators of the $H^2(\mathbf{C}P^3)$ and of the H^2 of the two factors $\mathbf{C}P^1$ in the quadric surface Q_2. On the other hand $c_1(\sigma)$ is $1/2$ of the first Chern class $c_1^\star$ of the quadric surface Q_2 ([5], pp. 276-277). Since $c_1^\star = 2(a_1 + a_2)$, it follows $c_1(\rho) = c_1(\sigma)$, and $\rho \cong \sigma$.

Denote by g_1 the metric induced on V by the standard metric g_0 of S^7. By prop. 3, g_1 is Sasakian with respect to the unit tangent vector ξ_1 to the fibers and $Ric_1 = 2g_1 + 2\eta_1 \otimes \eta_1$. Then a straightforward computation shows that

$$g = \frac{2}{3} g_1 - \frac{2}{9} \eta_1 \otimes \eta_1,$$

Sasakian with respect to $\xi = \frac{3}{2}\xi_1$, satisfies the Einstein condition $Ric = 4g$ (cf. [16], p. 290). Therefore:

Theorem 1 . *The product of spheres $S^3 \times S^2$, imbedded in S^7 as the total space of the induced Hopf bundle over the quadric $Q_2 \subset \mathbf{C}P^3$, inherits from the standard metric of S^7 an η-Einstein Sasakian metric g_1, allowing to define the Sasakian Einstein metric $g = \frac{2}{3} g_1 - \frac{2}{9} \eta_1 \otimes \eta_1$.*

3 The product $S^3 \times S^2 \times S^1$

Recall that a complex Hermitian manifold (W^{2n+2}, h, J) is a *generalized Hopf manifold* if it is locally conformal Kähler, i. e. if there is an open covering $\{U_i\}$ such that $h_{|U_i} = e^{f_i} h'_i$ with h'_i Kähler on U_i, and moreover its *Lee form* ω, locally defined as $\omega_{|U_i} = df_i$, is parallel with respect to the Levi-Civita connection of h. The structure of compact *regular* generalized Hopf manifold (W^{2n+2}, h, J) - i. e. such that the foliation $\mathcal{B}$ generated by the dual vector field B of ω is regular - has been established by I. Vaisman. In particular the following results are proved in [17], [18]:

a) *the class of compact regular generalized Hopf manifolds coincides with the class of principal flat S^1-bundles over compact Sasakian manifolds;*

b) *the conformally flat compact regular generalized Hopf manifolds reduce, up to finite coverings, to products $S^{2n+1} \times S^1 \to S^{2n+1}$.*

As an intermediate situation between the above cases a) and b) we prove here the following:

Proposition 4 . *A compact regular generalized Hopf manifold W^{2n+2} is locally conformally Ricci-flat if and only if it fibers in circles over a compact Sasakian-Einstein manifold.*

Proof. Look at the projection $p : W \to M = W/\mathcal{B}$ as a Riemannian submersion with totally geodesic fibers, and denote by $\widetilde{Ric}^W$ the Ricci tensor of the local Kähler metrics on W. Assume, as always possible up to normalization, that ω is unitary. A well known formula ([1], p.59) then gives $\widetilde{Ric}^W = Ric^W - 2n[h - \omega \otimes \omega]$, and observe that $\widetilde{Ric}^W(B, B) = Ric^W(B, B) = 0$. On the other hand, by the formula connecting the Ricci tensors for Riemannian submersions (cf. proof of lemma 1), we have $Ric^M(Y, Z) = Ric^W(Y^*, Z^*)$. Thus:

$$\widetilde{Ric}^W = Ric^M - 2n[g - \omega \otimes \omega],$$

where g is the projection of h to the Sasakian manifold $M = W/\mathcal{B}$. Then the conclusion follows from the fact that Sasakian Einstein metrics in dimension $2n + 1$ have Einstein constant $2n$ (cf. prop. 1).

Recall now from [17] the definition of the (integrable) complex structure J and of the generalized Hopf metric h on the total space W of any flat principal S^1-bundle $\pi : W \to M$ over a Sasakian M:

$$h = \pi^* g + u \otimes u, \quad JY = -\varphi(Y) - \eta(Y)B, \quad JB = \xi.$$

Here u is the flat connection, Y is any horizontal vector field and B the dual vector field with respect to h. Thus, by applying prop. 4 to the metric g of theorem 1, we obtain:

Corollary 1 . *The product $S^3 \times S^2 \times S^1$ admits a complex structure and a Hermitian metric h, making it a conformally Ricci flat and non conformally flat generalized Hopf manifold.*

Remark 1 . It is worth to observe that the product of a locally conformal Kähler manifold with a Kähler manifold is *not* locally conformal Kähler. Thus the generalized Hopf structure on $S^3 \times S^1 \times S^2$ established by cor. 1 cannot be obtained as a product.

On the other hand, it is natural to compare the $(S^3 \times S^2 \times S^1, J, h)$ of cor. 1 with some natural Hermitian structure related to the twistor fibration $S^3 \times S^2 \times S^1 \to S^3 \times S^1$. Indeed, the properties of Hermitian metrics on twistor spaces over oriented Riemannian 4-manifolds exclude the locally conformal Kähler possibility, at least by looking at metrics defined by means of the Levi Civita connection [13]. Another natural connection on standard Hopf surfaces $S^3 \times S^1$ is the Weyl connection, that glues together the Levi Civita connections of the local standard Kähler metrics (cf. [17]). However, by choosing the Weyl connection, the lifted Hermitian metric on $S^3 \times S^2 \times S^1$ turns out to be *locally conformal semikähler, but not locally conformal Kähler.* This is obtained from formulae in the appendix of [6], namely from its lemma 12 and corollary 2, pp. 618-619. We wish to thank Paul Gauduchon for a very helpful conversation about this point.

4 The products $S^7 \times S^6$, $S^7 \times S^6 \times S^1$ and more examples

The construction of the metrics g and h expressed in theorem 1 and corollary 1 can be pursued also in the following similar context.

Consider the Hopf fibration $S^{15} \to \mathbf{C}P^7$ and the induced Hopf bundle $\rho : V \to Gr_2(\mathbf{R}^8)$ over the Grassmannian $Gr_2(\mathbf{R}^8)$ of the oriented 2-planes in $\mathbf{R}^8$, isometrically immersed in $\mathbf{C}P^7$ as a non singular quadric complex hypersurface Q_6. A comparison of the first Chern classes shows that the bundle ρ is isomorphic to the Stiefel bundle $\sigma : V_2(\mathbf{R}^8) \to Gr_2(\mathbf{R}^8)$ of oriented orthonormal 2-frames in $\mathbf{R}^8$ (cf. the discussion following prop. 3 as well as [2], pp.84-86). Since $Gr_2(\mathbf{R}^8) \cong Q_6$ is Kähler-Einstein with Einstein constant 12 ([12], p.282), by lemma 1 its total space $S^7 \times S^6 \cong V_2(\mathbf{R}^8)$ inherits from S^{15} an η-Einstein Sasakian metric g_1. Its Ricci tensor satisfies $Ric_1 = 10 g_1 + 2\eta_1 \otimes \eta_1$, where η_1 is the dual of the unit Killing vector field ξ_1, induced by the Sasakian structure of S^{15}. Then the metric:

$$g = \frac{6}{7} g_1 - \frac{6}{49} \eta_1 \otimes \eta_1,$$

Sasakian with respect to $\xi = \frac{7}{6}\xi_1$, satisfies $Ric = 12g$.

This, together with proposition 4 applied to it, yields the following:

Corollary 2 . *The product $S^7 \times S^6$, total space of the induced Hopf bundle over the quadric $Q_6 \subset \mathbf{C}P^7$, inherits from the standard metric of S^{15} an η-Einstein Sasakian metric g_1, and then the metric $g = \frac{6}{7} g_1 - \frac{6}{49} \eta_1 \otimes \eta_1$ is Sasakian Einstein. Accordingly, the product $S^7 \times S^6 \times S^1$ admits a structure of generalized Hopf manifold, whose metric h is conformally Ricci flat and non conformally flat.*

It can be observed that both $S^3 \times S^2$ and $S^7 \times S^6$ are examples of *Brieskorn manifolds* ([20], pp. 291-305), as one recognizes from the equations: $z_0 z_3 = z_1 z_2$, $\sum_i z_i \overline{z}_i = 1$ of $S^3 \times S^2$ in $\mathbf{C}^4$, and the similar equations of $S^7 \times S^6$ in $\mathbf{C}^8$.

Remark 2 . The product $S^7 \times S^6 \times S^1$ can be looked at as a "twistor space" of the Hopf manifold $S^7 \times S^1$ with respect to the structure induced on it by the Cayley numbers. Indeed, the diffeomorphism $S^7 \times S^1 \cong (\mathbf{R}^8 - 0)/(x_\alpha \to 2x_\alpha)$ and the seven almost complex structures $I_1, I_2, I_3, E, EI_1, EI_2, EI_2$, defined on $\mathbf{R}^8$ by its identification with the Cayley numbers **Ca**, show that $S^7 \times S^1$ is naturally equipped by such a "Cayley structure". The space of the compatible almost complex structures on $S^7 \times S^1$, diffeomorphic to $S^7 \times S^6 \times S^1$, can be endowed with a natural almost complex structure J: this can be done through the Weyl connection in the usual tautological twistorial way. However, J turns out (even on the fibers) to be non integrable. Thus *the complex structure obtained on $S^7 \times S^6 \times S^1$ by corollary 2 (that is integrable) is different from that defined by looking at it as the twistor space of the almost complex structures compatible with the Cayley structure of $S^7 \times S^1$.*

More examples of induced Hopf bundles carrying a Sasakian Einstein metric can be given by extending the above constructions in the following two cases.

Let $M = \mathbf{C}P^k \times \mathbf{C}P^k$ with its product Fubini Study metric, that is Einstein with Einstein constant $2k+2$. Let $\Psi : ([x_0 : ... : x_k], [y_0 : ... : y_k]) \to [z_0 : ... : z_{(k+1)^2-1}] = [x_0 y_0 : x_0 y_1 ... : x_k y_k]$ be the Segre map, isometrically imbedding $\mathbf{C}P^k \times \mathbf{C}P^k$ into $\mathbf{C}P^{(k+1)^2-1}$. Then the Hopf bundle $S^{2(k+1)^2-1} \to \mathbf{C}P^{(k+1)^2-1}$, restricted to the Segre manifold $S_{k,k} = \mathbf{C}P^k \times \mathbf{C}P^k$, carries an induced metric g_1, which is Sasakian and η-Einstein. Denote by $V_{k,k}$ the total space of this induced Hopf bundle, on which g_1 is defined. By lemma 1 the Ricci tensor of g_1 satisfies $Ric_1 = 2kg_1 + (4k^2 - 2k)\eta_1 \otimes \eta_1$. Next, a Sasakian Einstein metric g is constructed on $V_{k,k}$ by following the procedure used for thm. 1 and inspired by formulae in [16]. Thus define on $V_{k,k}$ the metric

$$g = \frac{2k+2}{4k^2+2} g_1 + \frac{k(k+1)(1-2k)}{(2k^2+1)^2} \eta_1 \otimes \eta_1,$$

and observe that g is Sasakian with respect to $\xi = \frac{4k^2+2}{2k+2}\xi_1$. Moreover the computation of the Ricci tensor of g gives $Ric = 4k^2 g$. Thus:

Corollary 3 . *The induced Hopf bundle $V_{k,k}$ carries the Sasakian Einstein metric $g = \frac{2k+2}{4k^2+2} g_1 + \frac{k(k+1)(1-2k)}{(2k^2+1)^2} \eta_1 \otimes \eta_1$. The metric g_1 is induced by the imbedding of $V_{k,k}$ into $S^{2(k+1)^2-1}$, defined through the Segre map $\mathbf{C}P^k \times \mathbf{C}P^k \to \mathbf{C}P^{(k+1)^2-1}$.*

Another setting for our construction is that of a complex non singular hyperquadric $Q_{n-1} \subset \mathbf{C}P^n$, Kähler Einstein submanifold with Einstein constant $2n-2$. The induced Hopf bundle over Q_{n-1} can be identified with the Stiefel bundle $V_2(\mathbf{R}^{n+1}) \to Gr_2(\mathbf{R}^{n+1})$ of the oriented orthonormal 2-frames in $\mathbf{R}^{n+1}$. Thus the standard metric of S^{2n+1} induces the Sasakian η-Einstein metric g_1 on $V_2(\mathbf{R}^{n+1})$ and $Ric_1 = (2n-4)g_1 + 2\eta_1 \otimes \eta_1$. Here the associated Einstein metric is:

$$g = \frac{2n-2}{2n} g_1 + \frac{-2(2n-2)}{4n^2} \eta_1 \otimes \eta_1,$$

Sasakian with respect to $\xi = \frac{2n}{2n-2}\xi_1$ and satisfying $Ric = (2n-2)g$.

Corollary 4 . *The Stiefel manifold $V_2(\mathbf{R}^{n+1})$ carries the Sasakian Einstein metric $g = \frac{2n-2}{2n}g_1 + \frac{-2(2n-2)}{4n^2}\eta_1 \otimes \eta_1$. Here g_1 is induced by looking at $V_2(\mathbf{R}^{n+1})$ as the total space of the Hopf bundle over the quadric $Q_{n-1} \subset \mathbf{C}P^n$.*

Remark 3 . The Stiefel manifold $V_2(\mathbf{R}^7)$ is diffeomorphic to the (unique) homogeneous 3-Sasakian manifold $G_2/Sp(1)$ over the exceptional positive quaternion Kähler manifold $G_2/SO(4)$. This diffeomorphism (stated in [8], p. 115) can be recognized as follows. Look at $G_2/SO(4)$ as the Grassmannian of the quaternionic 4-planes in $\mathbf{R}^7 \cong Im\ \mathbf{Ca}$, and at $G_2/Sp(1)$ as the space of the same quaternionic 4-planes together with a hypercomplex structure on them. The latter fibers in circles over $G_2/U(2)$, twistor space of $G_2/SO(4)$, and space of the same quaternionic 4-planes together with a complex structure on them. For any oriented orthonormal 2-frame $\{\mathbf{i}, \mathbf{j}\}$ of $\mathbf{R}^7$ the cross product $\mathbf{k} = \mathbf{i} \times \mathbf{j}$ defines, through the Cayley multiplication, a hypercomplex structure on the oriented 4-plane L orthogonal to $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and since this construction can be reversed, the diffeomorphism is obtained. A similar argument shows that $G_2/U(2) \cong Gr_2(\mathbf{R}^7)$. Thus the natural question arises of comparing on $V_2(\mathbf{R}^7)$ the Sasakian Einstein metric g of cor. 4 with the (also Einstein) 3-Sasakian metric $\tilde{g}$ as defined through the fibration $G_2/Sp(1) \to G_2/SO(4)$, following [4]. We are not able to give the answer, involving a comparison between the Kähler Einstein structure of $Gr_2(\mathbf{R}^7) \cong Q_5 \subset \mathbf{C}P^6$ and the structure of contact Fano manifold coming from the stated diffeomorphism $Gr_2(\mathbf{R}^7) \cong G_2/U(2)$ with the twistor space of $G_2/SO(4)$. Note that both Q_5 and $G_2/U(2)$ appear in the study of nilpotent orbits in the complexified Lie algebra of G_2, cf. [10], pp. 29-30.

Finally, as a consequence of prop. 4 and cor. 4, we have:

Corollary 5 . *The products $V_{k,k} \times S^1$ and $V_2(\mathbf{R}^{n+1}) \times S^1$ carry a structure of conformally Ricci flat and non conformally flat generalized Hopf manifold.*

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