

# ON THE LOCAL STRUCTURE OF GENERALIZED KÄHLER MANIFOLDS

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## ABSTRACT

Let  $(g, b, J_+, J_-)$  be the bihermitian structure corresponding to a generalized Kähler structure. We find natural integrability conditions, in terms of the eigendistributions of  $J_+J_- + J_-J_+$ , under which  $db = 0$ .

## INTRODUCTION

A *generalized almost complex structure* on a smooth (connected) manifold is given by a vector subbundle  $L \subset (TM \oplus T^*M)^{\mathbb{C}}$  such that  $L \cap \bar{L} = \{0\}$  and which is maximally isotropic with respect to the canonical inner product

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(X)) .$$

If  $E = \pi_{TM}(L)$  is a bundle, where  $\pi_{TM} : TM \oplus T^*M \rightarrow TM$  is the projection, then there exists a unique complex two-form  $\varepsilon \in \Gamma(\Lambda^2 E^*)$  such that  $L = L(E, \varepsilon)$ , where

$$L(E, \varepsilon) = \{X + \alpha \mid X \in E, \alpha|_E = \varepsilon(X)\} .$$

Furthermore, by [4], to which we refer for all of the facts on generalized complex structures recalled here, the condition  $L \cap \bar{L} = \{0\}$  is equivalent to  $E + \bar{E} = T^{\mathbb{C}}M$  and  $\text{Im}(\varepsilon|_{E \cap \bar{E}})$  is non-degenerate.

A generalized almost complex structure  $L$  is *integrable* if its space of sections is closed under the *Courant bracket*, defined by

$$[X + \alpha, Y + \beta] = [X, Y] + \mathcal{L}_X \beta - \mathcal{L}_Y \alpha - \frac{1}{2} d(\iota_X \beta - \iota_Y \alpha) ,$$

for any  $X + \alpha, Y + \beta \in \Gamma(L)$ .

A *generalized complex structure* is an integrable generalized almost complex structure. Obviously, any generalized complex structure corresponds to a linear complex structure on  $TM \oplus T^*M$  whose eigenbundle, corresponding to  $i$ , is isotropic, with respect to the canonical inner product, and its space of sections is

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closed under the Courant bracket.

A generalized almost complex structure of the form  $L = L(E, \varepsilon)$  is integrable if and only if the space of sections of  $E$  is closed under the (Lie) bracket and  $d\varepsilon(X, Y, Z) = 0$ , for any  $X, Y, Z \in E$ .

A particular feature of Generalized Complex Geometry is that imposing Hermitian compatibility to a generalized almost complex structure and a Riemannian metric on  $TM \oplus T^*M$ , compatible with the canonical inner product, forces the manifold to admit a second generalized almost complex structure, commuting with the first one. One arrives to the notion of *generalized Kähler structure*, as a couple of commuting generalized complex structures  $\mathcal{J}_1$  and  $\mathcal{J}_2$  such that  $\mathcal{J}_1\mathcal{J}_2$  is negative definite; furthermore, in [4] it is explained the correspondence between generalized Kähler structures and a special type of bihermitian structures which appeared in Theoretical Physics, over twenty years ago [3].

More precisely, any generalized almost Kähler structure on  $M$  corresponds to a quadruple  $(g, b, J_+, J_-)$ , where  $g$  is a Riemannian metric,  $b$  is a two-form and  $J_{\pm}$  are almost Hermitian structures on  $(M, g)$ . Furthermore, the corresponding generalized almost Kähler structure is integrable if and only if  $J_{\pm}$  are integrable and parallel with respect to  $\nabla^{\pm}$ , where  $\nabla^{\pm} = \nabla \pm \frac{1}{2}g^{-1}h$ , with  $\nabla$  the Levi-Civita connection of  $g$  and  $h = db$  (equivalently,  $J_{\pm}$  are integrable and  $d_{\pm}^c\omega_{\pm} = \mp h$ , where  $\omega_{\pm}$  are the Kähler forms of  $J_{\pm}$ ).

Classification results for compact bihermitian manifolds were given, mainly in dimension 4, in several papers (see, for example, [1], [2]).

In higher dimensions, a natural case to consider is when  $J_+$  and  $J_-$  are admissible for an almost quaternionic structure. This condition was, essentially, considered by physicists who have shown that it holds if and only if the bihermitian structure is part of a hyperkähler one [7, Theorem 1] (see Theorem 1.1, below).

By combining this fact with results of [9] and [8], we study the ‘eigendistributions’ of the operator  $J_+J_- + J_-J_+$ . Thus, we obtain natural integrability conditions under which  $db = 0$  (Theorem 2.3, Corollary 2.4).

## 1. THE ALMOST QUATERNIONIC GENERALIZED KÄHLER MANIFOLDS ARE HYPERKÄHLER

A *bundle of associative algebras* is a vector bundle whose typical fibre is an associative algebra  $\mathcal{A}$  and whose structural group is the group of automorphisms of  $\mathcal{A}$ .

An *almost quaternionic structure* on  $M$  is a morphism of bundles of associative algebras  $\sigma : A \rightarrow \text{End}(TM)$ , where the typical fibre of  $A$  is  $\mathbb{H}$ . Then,  $\sigma(\text{Im}A)$

is an oriented Riemannian vector bundle of rank 3 and the (local) sections of its sphere bundle are the *admissible almost complex structures* of  $\sigma$  (see [6]).

The following result reformulates [7, Theorem 1]. For the reader's convenience, we supply a proof.

**Theorem 1.1.** *Let  $(M, L_1, L_2)$  be a generalized almost Kähler manifold of dimension at least eight and let  $(g, b, J_+, J_-)$  be the corresponding almost bihermitian structure. Suppose that  $J_+$  and  $J_-$  are admissible almost complex structures of an almost quaternionic structure on  $M$ .*

*Then the following assertions are equivalent:*

- (i)  $(M, L_1, L_2)$  is generalized Kähler.
- (ii)  $(M, g, J_\pm)$  are Kähler manifolds.

*Furthermore, if (i) or (ii) holds and  $J_+ \neq \pm J_-$  then the almost quaternionic structure is hyperkähler, with respect to  $(M, g)$ .*

*Proof.* As (ii) $\implies$ (i) is trivial, it is sufficient to prove that (i) $\implies$ (ii).

By hypothesis, there exists  $a : M \rightarrow [-1, 1]$  such that  $J_+J_- + J_-J_+ = -2a$  on  $M$ . If  $J_+ = \pm J_-$  there is nothing to be proved. Hence, we may suppose that  $a^{-1}((-1, 1)) \neq \emptyset$ .

Moreover, as we have to prove that  $(M, g, J_\pm)$  are Kähler and, consequently,  $a$  is constant, we may assume  $a(M) \subseteq (-1, 1)$ .

Then  $L_1 = L(T^{\mathbb{C}}M, \varepsilon_+)$  and  $L_2 = L(T^{\mathbb{C}}M, \varepsilon_-)$ , where  $\varepsilon_\pm$  are closed complex two-forms on  $M$ . From [4, (6.4) and (6.5)], it quickly follows that

$$(1.1) \quad \begin{aligned} (\operatorname{Im} \varepsilon_\pm)(J_+ \mp J_-) &= 2g, \\ (\operatorname{Re} \varepsilon_\pm)(J_+ \mp J_-) &= b(J_+ \mp J_-) + g(J_+ \pm J_-). \end{aligned}$$

On multiplying, to the right, both relations of (1.1) by  $J_+ \mp J_-$  we obtain

$$\begin{aligned} (-2 \pm 2a)(\operatorname{Im} \varepsilon_\pm) &= 2g(J_+ \mp J_-), \\ (-2 \pm 2a)(\operatorname{Re} \varepsilon_\pm) &= (-2 \pm 2a)b \mp g(J_+J_- - J_-J_+) \end{aligned}$$

and, consequently,  $(a - 1)\operatorname{Re} \varepsilon_+ - (a + 1)\operatorname{Re} \varepsilon_- = -2b$ .

Therefore

$$(1.2) \quad d \left[ \frac{1}{1 \pm a} g(J_+ \pm J_-) \right] = 0.$$

Also, as, up to a  $B$ -field transformation, we may suppose  $\operatorname{Re} \varepsilon_- = 0$ , we deduce that the two-form  $\frac{1}{a-1}b$  is closed; equivalently,

$$(1.3) \quad db = \frac{1}{a-1} da \wedge b.$$

Note that, the condition  $\nabla^\pm J_\pm = 0$  is equivalent to

$$(1.4) \quad g((\nabla_X J_\pm)(Y), Z) = \mp \frac{1}{2} [(db)(X, J_\pm Y, Z) + (db)(X, Y, J_\pm Z)] ,$$

for any  $X, Y, Z \in TM$ .

From (1.3) and (1.4) we obtain

$$(1.5) \quad g((\nabla_X J_\pm)(Y), Z) = \pm \frac{1}{2(1-a)} (da \wedge b) (X \wedge J_\pm Y \wedge Z + X \wedge Y \wedge J_\pm Z) ,$$

for any  $X, Y, Z \in TM$ .

Obviously,

$$K_\pm = \frac{1}{\sqrt{2(1 \pm a)}} (J_+ \pm J_-) .$$

are anti-commuting almost Hermitian structures on  $(M, g)$ . Furthermore, (1.5) gives

$$(1.6) \quad \begin{aligned} g((\nabla_X K_\pm)(Y), Z) &= \mp \frac{1}{2(1 \pm a)} X(a) g(K_\pm Y, Z) \\ &+ \frac{1}{2(1-a)} \left( \frac{1-a}{1+a} \right)^{\pm \frac{1}{2}} (da \wedge b) (X \wedge K_\mp Y \wedge Z + X \wedge Y \wedge K_\mp Z) , \end{aligned}$$

for any  $X, Y, Z \in TM$ .

On the other hand, by (1.2), the almost Hermitian manifolds  $(M, e^{2f_\pm} g, K_\pm)$  are  $(1, 2)$ -symplectic, where  $f_\pm = -\frac{1}{4} \log 2(1 \pm a)$ . A straightforward calculation shows that this is equivalent to

$$(1.7) \quad \begin{aligned} &g((\nabla_{K_\pm X} K_\pm)(Y), Z) - g((\nabla_X K_\pm)(Y), K_\pm Z) = \\ &\pm \frac{1}{2(1 \pm a)} [(K_\pm Y)(a) g(K_\pm X, Z) - (K_\pm Z)(a) g(K_\pm X, Y) \\ &\quad + Y(a) g(X, Z) - Z(a) g(X, Y)] , \end{aligned}$$

for any  $X, Y, Z \in TM$ .

Now, (1.6) and (1.7) imply

$$(1.8) \quad \begin{aligned} &(K_\pm X)(a) g(K_\pm Y, Z) + (K_\pm Y)(a) g(K_\pm X, Z) - (K_\pm Z)(a) g(K_\pm X, Y) \\ &\quad - X(a) g(Y, Z) + Y(a) g(X, Z) - Z(a) g(X, Y) = \\ &\pm \left( \frac{1-a}{1+a} \right)^{-\frac{1}{2}} (da \wedge b) (K_\pm X \wedge K_\mp Y \wedge Z + K_\pm X \wedge Y \wedge K_\mp Z \\ &\quad - X \wedge K_\mp Y \wedge K_\pm Z - X \wedge Y \wedge K_\mp K_\pm Z) , \end{aligned}$$

for any  $X, Y, Z \in TM$ .

In (1.8), if from the first relation we subtract the second one, with the roles of

$X$  and  $Y$  interchanged, then we obtain

$$(1.9) \quad \begin{aligned} & (K_+X)(a)g(K_+Y, Z) + (K_+Y)(a)g(K_+X, Z) - (K_+Z)(a)g(K_+X, Y) \\ & + (K_-X)(a)g(K_-Y, Z) + (K_-Y)(a)g(K_-X, Z) + (K_-Z)(a)g(K_-X, Y) \\ & - 2Z(a)g(X, Y) = 2 \left( \frac{1-a}{1+a} \right)^{-\frac{1}{2}} (da \wedge b)(K_+X \wedge K_-Y \wedge Z), \end{aligned}$$

for any  $X, Y, Z \in TM$ .

From (1.9), with  $Z = K_+X$ , it quickly follows that  $\text{grad}_g a$  is zero on the orthogonal complement of each quaternionic line. As  $\dim M \geq 8$ , we obtain that  $a$  is constant. Together with (1.6), this gives that  $K_\pm$  generate a hyperkähler structure on  $(M, g)$ , whilst, together with (1.3), this implies  $db = 0$ . The proof is complete.  $\square$

**Remark 1.2.** In dimension four, the hypothesis of Theorem 1.1 is equivalent to the condition that  $J_+$  and  $J_-$  induce the same orientation on  $M$ , whilst if  $J_+$  and  $J_-$  induce different orientations on  $M$  then, up to a unique  $B$ -field transformation,  $M$  is locally given by a product of two Kähler manifolds (consequence of [8, Corollary 5.7]). Furthermore, there exist four-dimensional generalized Kähler manifolds with  $J_+$  and  $J_-$  inducing the same orientation and which are not given by a hyperkähler structure (see [5]).

The next result follows quickly from (1.3) and (1.9).

**Corollary 1.3.** *Let  $(M, L_1, L_2)$  be a four-dimensional generalized Kähler manifold with  $J_+, J_-$  inducing the same orientation on  $M$  and linearly independent, at each point.*

*Then, up to a unique  $B$ -field transformation, the following relations hold:*

$$(1.10) \quad \begin{aligned} db &= -\frac{1}{1-a} da \wedge b. \\ *(da \wedge b) &= \frac{1}{2(1+a)} [J_+, J_-](da), \end{aligned}$$

where  $*$  is the Hodge star operator of  $(M, g)$  and the function  $a : M \rightarrow (-1, 1)$  is characterised by  $J_+J_- + J_-J_+ = -2a$ .

We end this section by showing how equations (1.10) can be slightly simplified.

**Remark 1.4.** Let  $(M, L_1, L_2)$  be a four-dimensional generalized Kähler manifold with  $J_+, J_-$  inducing the same orientation on  $M$  and linearly independent, at each point.

With the same notations as in Theorem 1.1, let  $K = K_+K_-$ ,  $k = \left(\frac{1+a}{1-a}\right)^{\frac{1}{2}}g$  and

$u = \log(1 - a)$ .

Then (1.10) is equivalent to

$$(1.11) \quad db = du \wedge b = - *_{\mathcal{K}} K du .$$

If  $du$  is nowhere zero, then the second equality of (1.11) is equivalent to

$$b = cv_{\mathcal{E}} + v_{\mathcal{F}} + du \wedge \alpha ,$$

where  $c$  is a function,  $\mathcal{E}$  is generated by  $\{\text{grad } u, K(\text{grad } u)\}$ ,  $\mathcal{F} = \mathcal{E}^{\perp}$ ,  $\alpha$  is a section of  $\mathcal{F}^*$ , and  $v_{\mathcal{E}}, v_{\mathcal{F}}$  are the volume forms of  $\mathcal{E}, \mathcal{F}$ , respectively.

## 2. FACTORISATION RESULTS FOR GENERALIZED KÄHLER MANIFOLDS

Let  $(M, L_1, L_2)$  be a generalized Kähler manifold and let  $(g, b, J_+, J_-)$  be the corresponding bihermitian structure. For any  $a \in [-1, 1]$ , we (pointwisely) denote by  $\mathcal{H}^a$  the eigenspace of  $J_+J_- + J_-J_+$  corresponding to  $-2a$ ; also, we denote  $\mathcal{H}^{\pm} = \mathcal{H}^{\pm 1}$  and  $\mathcal{V} = (\mathcal{H}^+ \oplus \mathcal{H}^-)^{\perp}$ . Then, at each point of  $M$ , we have that  $\mathcal{H}^a$  are preserved by  $J_{\pm}$  and there exist (finite) orthogonal decompositions  $TM = \bigoplus_a \mathcal{H}^a$  and  $\mathcal{V} = \bigoplus_{|a| < 1} \mathcal{H}^a$ .

**Corollary 2.1.** *Let  $N$  be a complex submanifold of  $(M, J_{\pm})$ , of complex dimension at least four, endowed with a function  $a : N \rightarrow (-1, 1)$  such that  $T_x N \subseteq \mathcal{H}_x^{a(x)}$ , ( $x \in N$ ).*

*Then  $a$  is constant and  $N$  is endowed with a natural hyperkähler structure whose underlying Riemannian metric is  $g|_N$  and for which  $J_+|_N$  and  $J_-|_N$  are admissible complex structures.*

*Proof.* As, obviously,  $(g, b, J_+, J_-)$  induces a generalized Kähler structure on  $N$ , this follows quickly from Theorem 1.1.  $\square$

From [9, Lemma 2.3] it follows that in an open neighbourhood  $U$  of each point of a dense open subset of  $M$  there exist (smooth) functions  $a_j : M \rightarrow [-1, 1]$ , ( $j = 1, \dots, r$ ), such that  $\mathcal{H}^{a_j}$  are distributions on  $U$  and  $TM = \bigoplus_j \mathcal{H}^{a_j}$ ; we call the  $\mathcal{H}^{a_j}$  the (local) eigendistributions of  $J_+J_- + J_-J_+$ . Furthermore, if  $a$  is a function on  $U$  such that, at each point,  $-2a$  is an eigenvalue of  $J_+J_- + J_-J_+$  then there exists an open subset of  $U$  on which  $a = a_j$ , for some  $j$ ; thus, if we assume real-analyticity then  $a = a_j$  on  $U$ .

We point out the following facts:

- The functions  $a_j$  are constant along the integrable manifolds, of dimensions at least eight, of  $\mathcal{H}^{a_j}$ , ( $j = 1, \dots, r$ ); this is a consequence of Corollary 2.1.
- If  $J_+ \pm J_-$  are invertible then the holomorphic diffeomorphisms of  $(M, L_1, L_2)$  preserve each  $\mathcal{H}^{a_j}$ , ( $j = 1, \dots, r$ ); this is a consequence of [8, Corollary 6.7].

**Remark 2.2.** Let  $(M, L_1, L_2)$  be a generalized Kähler manifold with  $db = 0$ . Then  $(M, g, J_\pm)$  are Kähler and there exists a nonempty finite subset  $A$  of  $[-1, 1]$  such that, for any  $a \in A$ , we have that  $\mathcal{H}^a$  is a parallel foliation which is holomorphic with respect to both  $J_+$  and  $J_-$ . Therefore  $(g, J_\pm)$  induce Kähler structures on the leaves of  $\mathcal{H}^a$  and, if  $a \neq \pm 1$ , these are admissible with respect to natural hyperkähler structures. Furthermore, there exist orthogonal decompositions  $TM = \bigoplus_{a \in A} \mathcal{H}^a$  and  $\mathcal{V} = \bigoplus_{a \in A \setminus \{\pm 1\}} \mathcal{H}^a$ .

If the cardinal of  $A \setminus \{\pm 1\}$  is at least two then the leaves of  $\bigoplus_{a \in A \setminus \{\pm 1\}} \mathcal{H}^a$  are naturally endowed with two distinct hyperkähler structures with respect to which  $J_+$  and  $J_-$  define admissible complex structures, respectively.

Furthermore, if  $J_+ + J_-$  (or  $J_+ - J_-$ ) is invertible then as, locally,  $M$  is the product of a Kähler manifold and hyperkähler manifolds, its holomorphic Poisson structure is the pull-back of the product of the holomorphic symplectic structures of the hyperkähler factors.

Next, we prove the following.

**Theorem 2.3.** *Let  $(M, L_1, L_2)$  be a generalized Kähler manifold with  $J_+ + J_-$  (or  $J_+ - J_-$ ) invertible and for which the eigendistributions of  $(J_+ J_- + J_- J_+)|_{(\mathcal{H}^+ \oplus \mathcal{H}^-)^\perp}$  have dimensions at least eight.*

*Then the following assertions are equivalent:*

- (i)  $db = 0$ .
- (ii) *The eigendistributions of  $J_+ J_- + J_- J_+$  and their orthogonal complements are integrable.*

*Proof.* The implication (i)  $\implies$  (ii) is an immediate consequence of Remark 2.2.

Assume that (ii) holds. From [8, Corollary 6.3] it follows that we may suppose that, also,  $J_+ - J_-$  is invertible.

Then, locally, outside a set with empty interior there exists a finite set  $A$  of functions  $a : M \rightarrow (-1, 1)$  such that  $\mathcal{H}^a$  are distributions and  $TM = \bigoplus_{a \in A} \mathcal{H}^a$ .

Also,  $L_1 = L(T^{\mathbb{C}}M, \varepsilon_+)$  and  $L_2 = L(T^{\mathbb{C}}M, \varepsilon_-)$ , where  $\varepsilon_\pm$  are closed complex two-forms on  $M$ .

By Theorem 1.1, we have that (i) holds if and only if  $db(X, Y, Z) = 0$ , for any  $X \in \mathcal{H}^a$  and  $Y, Z \in \bigoplus_{a' \in A \setminus \{a\}} \mathcal{H}^{a'}$ , ( $a \in A$ ).

As  $\mathcal{H}^a$ , ( $a \in A$ ), are invariant under  $B$ -field transformations, we may assume  $\operatorname{Re} \varepsilon_- = 0$ ; equivalently,  $b = -g(J_+ - J_-)(J_+ + J_-)^{-1}$ . Together with the fact that  $\mathcal{H}^a$ , ( $a \in A$ ), and their orthogonal complements are holomorphic foliations, with respect to  $J_+$  and  $J_-$ , this gives that (i) holds if and only if  $\mathcal{H}^a$  are Riemannian foliations, ( $a \in A$ ).

Now, note that we, also, have

$$\operatorname{Re} \varepsilon_+ = b + g(J_+ + J_-)(J_+ - J_-)^{-1} = g[(J_+ + J_-)(J_+ - J_-)^{-1} - (J_+ - J_-)(J_+ + J_-)^{-1}].$$

As  $L_1$  is integrable,  $\operatorname{Re} \varepsilon_+$  is closed and, consequently,  $\mathcal{H}^a$  are Riemannian foliations, ( $a \in A$ ).

The proof is complete.  $\square$

We end with the following result.

**Corollary 2.4.** *Let  $(M, L_1, L_2)$  be a generalized Kähler manifold for which the eigendistributions of  $(J_+J_- + J_-J_+)|_{(\mathcal{H}^+ \oplus \mathcal{H}^-)^\perp}$  have dimensions at least eight.*

*Then the following assertions are equivalent:*

- (i)  $db = 0$ .
- (ii)  $\mathcal{H}^\pm$  and the sum of any two eigendistributions of  $J_+J_- + J_-J_+$  are integrable.

*Proof.* The implication (i)  $\implies$  (ii) is trivial.

If (ii) holds then  $\mathcal{H}^+ \oplus \mathcal{H}^-$  is integrable. Hence, by [8, Theorem 6.10], we may assume  $\mathcal{H}^+ = 0 = \mathcal{H}^-$ . The proof follows from Theorem 2.3.  $\square$

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