LOCAL ALMOST CONTACT METRIC 3-STRUCTURES

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Abstract. We study an odd-dimensional analogue of quaternion-Kähler geometry. We show that such manifolds are Einstein with positive scalar curvature, hence, if complete, they are compact with finite fundamental group. Moreover, under some regularity assumption, they fiber with 3-dimensional spherical space forms over Einstein orbifolds with positive scalar curvature. As a by-product, we derive a non-existence result for a certain type of real hypersurface of a quaternion-Kähler manifold.

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1. Introduction

One could see in recent years a renewed interest concerning (almost) contact 3-structures, especially motivated and stimulated by the work of Ch. Boyer, K. Galicki, B. Mann ([Bo-Ga-Ma]). They succeeded in throwing a new light on 3-Sasakian geometry, constructing a wide class of compact examples with different homotopy types. On the other hand, they made precise the relation between 3-Sasakian geometry, quaternionic geometry and complex (Kähler) geometry. One of their major novelty was to work in the orbifold category. 3-Sasakian structures are induced on totally umbilical real hypersurfaces of hyperkähler manifolds. Also, thecône over a 3-Sasakian manifold can be canonically endowed with a hyperkähler structure. In the same spirit G. Hernandez was interested in manifolds carrying three "nested" f-structures ([He]).

In the same time, the study of weaker (than the Sasakian ones) geometrical conditions imposed to a contact 3-structure was developed by other authors: Geiges and Thomas related hypercontact structures to gauge theory ([Ge-Th]) and A. Banyaga continued this line (cf. [Ba]).

All these structures have some common features: they are global, thus being odd-dimensional analogues of hyperhermitian or hyperkähler geometries and they rely on a specific, good definition of *normality* derived from the structure induced on a real hypersurface (usually totally umbilical) of an appropriate type of hyperhermitian manifold.

On the contrary, in this note we try to understand the local case. A previous attempt was made in [Or-Pi] where a notion of *local* 3-*Sasakian structure* was introduced, arising from the study of locally conformal quaternion-Kähler structures. We also want to propose a natural odd-dimensional analogue of quaternion-Kähler structures. A similar local structure was studied by A. Bejancu under the name of *generalized* 3-*Sasakian structures*, [Be], but his viewpoint is that of submanifold theory.

In the sequel all manifolds we deal with are connected. The manifolds and the geometric objects they carry on are of differentiability class $\mathcal{C}^\infty$.

2. The normality conditions

Let us first introduce

Definition 2.1. Let (M, g) be a Riemannian manifold of real dimension $4n+3$ endowed with: a rank 3 subbundle Ξ of TM, a rank 3 subbundle Φ of $End(TM)$ such that, on any trivializing open set U good for both Ξ and Φ, there exist a local orthonormal basis $\xi_1,\ \xi_2,\ \xi_3$ for Ξ and a local basis $\varphi_1,\ \varphi_2,\ \varphi_3$ for Φ satisfying the relations (here and in the sequel $\eta_i = \xi_i^\flat$):

(1)
$$\varphi_i\varphi_j - \eta_j \otimes \xi_i = -\varphi_j\varphi_i + \eta_i \otimes \xi_j = \epsilon_{ijk}\varphi_k,\ \ \eta_i\varphi_j = \epsilon_{ijk}\eta_k,\ \ \varphi_i\xi_j = \epsilon_{ijk}\xi_k$$

(2)
$$\varphi_i^2 = -Id + \eta_i \otimes \xi_i,\ \ \eta_i\varphi_i = 0,\ \ \varphi_i\xi_i = 0, g(\varphi_i\cdot, \varphi_i\cdot) = g(\cdot,\cdot) - \eta_i(\cdot)\eta_i(\cdot),$$

where (i, j, k) is any permutation of $(1, 2, 3)$. Then we say that (M, g) has *a local almost contact metric* 3-*structure.*

The global version of this definition was given in [Ku].

Following the same method as was already done for 3-Sasakian spaces (cf. [Bo-Ga-Ma]), we now consider the product $N = M \times \mathbf{R}^+$ endowed with the cone metric $G = t^2\pi_1^* g + \pi_2^* dt^2$ where π_1 (resp. π_2) is the canonical projection on M (resp.$\mathbf{R}$). One can define, locally, three almost complex structures as follows:

$$I_i(X, f\frac{d}{dt}) = (\varphi_i X + \frac{f}{t}\xi_i, -t\eta_i(X)\frac{d}{dt})$$

where $X \in \mathcal{X}(M)$ and f is a real function on N. It is easily seen that

$$I_iI_j + I_jI_i = -2\delta_{ij},\ \ G(I_i\cdot, I_i\cdot) = G(\cdot,\cdot)$$

thus, denoting with H the subbundle of $End(TN)$ generated by the I_i, (N, G, H) is a *quaternion Hermitian* manifold. We request that this structure be quaternion-Kähler. Hence, letting D (resp. ∇) be the metric connection of G (resp. g) we must have $DH = H$. Together with the relations between D and ∇ (see [On], p.206):

$$(3)\quad D_{\frac{d}{dt}}\frac{d}{dt} = 0, \quad D_{\frac{d}{dt}}X = D_X\frac{d}{dt} = \frac{1}{t}X, \quad D_XY = \nabla_XY - tg(X,Y)\frac{d}{dt}$$

this implies the existence of some local one-forms α_i satisfying

$$DI_i = \alpha_k \otimes I_j - \alpha_j \otimes I_k.$$

On the other hand, the well-known relations between D and ∇ imply that the pulled-back forms $\pi^*\alpha_i$ (for simplicity denoted equally α_i) satisfy the equation:

$$(4)\qquad \nabla\varphi_i = Id \otimes \eta_i - g \otimes \xi_i + \alpha_k \otimes \varphi_j - \alpha_j \otimes \varphi_k,$$

This motivates the following

Definition 2.2. A local almost contact metric 3-structure is called *normal* if on any trivializing open set good for both Φ and Ξ there exist three local one forms α_i satisfying (4).

Remark 2.1. 1) On a quaternionic Hermitian manifold (N, G, H) the two-forms $\omega_i(X,Y) = G(X, I_iY)$ are local, but the four-form $\omega = \sum_{i=1}^3 \omega_i \wedge \omega_i$ is global and the quaternion-Kähler condition is equivalent with the parallelism of ω. Similarly, on a manifold with a local almost contact metric 3-structure one defines the local two-forms $\psi_i(X,Y) = g(X, I_iY)$ and these produce the global four-form $\psi = \sum_{i=1}^3 \psi_i \wedge \psi_i$. One may directly compute the following expression for the covariant derivative of ψ: $2\nabla_T\psi = -\gamma \wedge \sum_{i=1}^3 \psi_i \wedge \eta_i$, where $\gamma = i_Tg$. Then, for $T \in \Xi^\perp$, $(\nabla_T\psi)(\xi_1, \xi_2, \xi_3, T) = -3$. Hence, the normality condition does not imply the parallelism of ψ.

2) Moreover, let us denote $\alpha_{ij} := \alpha_i \circ \varphi_j$ and, abusing a little, put $\alpha_i \wedge \varphi_i = \alpha_i \otimes \varphi_i - \varphi_i \otimes \alpha_i$. Now a straightforward computation proves that the Nijenhuis tensors N_{φ_i} of φ_i on a normal local almost contact metric 3-structure satisfy the following three conditions:

$$N_{\varphi_i} + d\eta_i \otimes \xi_i = \alpha_{ki} \wedge \varphi_j - \alpha_{ji} \wedge \varphi_k - \alpha_j \wedge \varphi_j - \alpha_k \wedge \varphi_k.$$

3) The following equations are, also, a consequence of normality:

$$(5)\qquad \nabla\xi_i = \varphi_i + \alpha_k \otimes \xi_j - \alpha_j \otimes \xi_k.$$

3. Embedding in quaternion-Kähler manifolds

Retracing our steps we immediately find that our manifolds can be embedded as real hypersurfaces of quaternion-Kähler manifolds.

Proposition 3.1. *If M has a normal local almost contact metric 3-structure, then $M \times \mathbf{R}^+$ with the cone metric has a natural quaternion-Kähler structure.*

As in [Bo-Ga], this allows us to give the following:

Equivalent definition 3.1. A $4n+3$-dimensional Riemannian manifold (M, g) admits a normal local almost contact metric 3-structure if and only if the holonomy of the cone metric G on $N = M \times \mathbf{R}^+$ is included in $Sp(n+1) \cdot Sp(1)$.

For the "only if part" one has only to note that the hypothesis on the holonomy is equivalent to (N, G) being quaternion Kähler . Then, identifying M with the slice $M \times 1$ of N, one defines a local almost contact 3-structure by:

$$\xi_i = I_i \frac{d}{dt} \ , \varphi_i X = tan(I_i X) \tag{6}$$

for any X tangent to M. The rest of the proof proceeds as in the previous paragraph. The "if part" is obvious.

¿From the third relation in (3) we see that the second fundamental form of the embedding of M in the cone is $h = -tg$, hence the embedding is totally umbilical, with mean curvature vector $-t$.

On the other hand, let M be an oriented real hypersurface of a quaternion-Kähler manifold N, with local unit normal vector field C. Then a local almost contact 3-structure is canonically induced on M by

$$\varphi_i X = tan(I_i X), \ \ \xi_i = -I_i C. \tag{7}$$

Note that the previous embedding of M in the cone $M \times \mathbf{R}^+$ is not of this type, because in (6) $\frac{d}{dt}$ is not a unit vector field for the cone metric.

Letting A be the shape operator of the hypersurface, the Gauss and Weingarten formulae imply:

$$(\nabla_X \varphi_i)Y = \eta_i(Y)AX - g(AX, Y)\xi_i + \alpha_k(X)\varphi_j(Y) - \alpha_j(X)\varphi_k(Y) \tag{8}$$

$$\nabla_X \xi_i = \varphi_i AX + \alpha_k(X)\xi_j - \alpha_j(X)\xi_k \tag{9}$$

If M is totally umbilical, with $A = Id$, it is trivially normal. Conversely, suppose the induced structure on M normal. Then there exist the local one-forms β_j such that the equations (4), (5) are satisfied with β instead of α. Comparing the expressions of $(\nabla_{\xi_i}\varphi_i)\xi_i$ obtained from (4) and (8) we derive

$$\alpha_k(\xi_i) = \beta_k(\xi_i),\ A\xi_i = a_i\xi_i$$

Hence the one-forms α_i, β_i coincide, respectively, on Ξ. We then let X be normal to Ξ and using the expressions of $(\nabla_X\varphi_i)\xi_i$ as well as the symmetry of the shape operator we conclude $\alpha_j(X) = \beta_j(X)$. Thus, the normality of the hypersurface implies the equality of the considered one-forms. This moreover yields $\varphi_i AY = \varphi_i Y$ for any $Y \in \mathcal{X}(M)$. Letting $Y = \xi_j$ we easily infer that A is the identity on Ξ. If Y is in $\Xi^\perp$, then $\varphi_i Y \in \Xi^\perp$, and the same holds for AY. Finally, from $\varphi_i^2 AY = \varphi_i^2 Y$ we obtain that $A = Id$ on $\Xi^\perp$ too. The two constructions are one-to-one. Summing up we have proved:

Theorem 3.1. *The local almost contact metric structure induced by (7) on a real orientable hypersurface of a quaternion-Kähler manifold is normal if and only if the hypersurface is totally umbilical with constant mean curvature* 1.

Remark 3.1. This result is to be compared with the similar one in [He], p. 320 and with Thm. 1 in [Be].

Recall that totally umbilical submanifolds with constant mean curvature are particular cases of *extrinsic spheres*, a type of submanifold studied by many authors, e.g. [Ch], p.69 and forward.

4. Main properties

From now on (M, g, Ξ, Φ) will denote a $4n+3$-dimensional Riemannian manifold endowed with a normal local almost contact metric 3-structure. The embedding procedure in the cone can be used to determine the curvature properties of M.

3-Sasakian manifolds are always Einstein. This is the case for our structure too. Precisely:

Proposition 4.1. *Local almost contact metric* 3*-structures are Einstein with positive scalar curvature* $4n+2$

Proposition 4.2. We embed M in $N = M \times \mathbf{R}$ as above. Then (N, G) is warped product with warp function $f = t$. The Ricci tensor of N is computed in [On], p. 211. Particularly, $Ric^N(X, \frac{d}{dt}) = 0$ for any projectable X on M. As G is known to be Einstein, this implies $Ric^N = 0$. Moreover,

$$Ric^M(X, Y) = Ric^N(X^*, Y^*) + G(X^*, Y^*)F,$$

where
$$F = \frac{\Delta}{f} + (\dim M - 1)\frac{G(\mathrm{grad} f, \mathrm{grad} f)}{f^2},$$
and X^* are projectable vector fields on X, Y. In our case $F = 4n + 2$, hence the desired result. □

From Myers' theorem we now deduce:

Corollary 4.1. *Complete manifolds carrying a normal almost contact metric 3-structure are compact, with finite fundamental group.*

Let us now recall the following result of N. Koiso:

Theorem 4.1. *(cf.* [Ko]*) Let (M, g) be a totally umbilical Einstein hypersurface in a complete Einstein manifold $(\overline{M}, \overline{g})$. Then the only possible cases are:*

(a) g has positive Ricci curvature. Then g and $\overline{g}$ have constant sectional curvature;

(b) $\overline{g}$ has negative Ricci curvature. If $\overline{M}$ is compact or $(\overline{M}, \overline{g})$ is homogeneous, then g and $\overline{g}$ have constant sectional curvature;

(c) g and $\overline{g}$ have zero Ricci curvature.

Note that this does not apply to the embedding in the cone, because the cone metric is not complete.

However, combining Koiso's theorem with our previous result and with Theorem 3.1, we obtain the following statement (which can also be viewed as a non-existence result):

Proposition 4.3. *A totally umbilical real hypersurface, with mean curvature 1, of a complete quaternion-Kähler manifold is necessarily a positive space form. Moreover, the ambient space is a space form too.*

Hence, more interesting examples can be looked for as hypersurfaces of non-complete quaternion-Kähler manifolds.

The above is to be compared to other non-existence results for hypersurfaces of quaternion-Kähler manifolds, *e.g.* [Ort-Pe].

We collect some computational facts in the next:

Lemma 4.1. *For any trivializing open set U good for both Ξ and Φ one has:*

$$\alpha_j(\xi_j) = \alpha_k(\xi_k) \text{ for any } j \neq k. \tag{10}$$

$$[\xi_i, \xi_j] = -2(1 - \alpha_j(\xi_j))\xi_k + \alpha_k(\xi_j)\xi_j + \alpha_k(\xi_i)\xi_i. \tag{11}$$

Moreover, ξ_i are Killing if and only if $\alpha_k(\xi_i) = 0$ for any $k \neq i$.

Proposition 4.4. We compute the Lie derivative of the metric on the direction of ξ_i. The result is:

$$L_{\xi_i} g = \alpha_k \otimes \eta_j - \alpha_j \otimes \eta_k + \eta_j \otimes \alpha_k - \eta_k \otimes \alpha_j. \tag{12}$$

Applying on (ξ_j, ξ_k) we get:

$$(L_{\xi_i} g)(\xi_j, \xi_k) = -\alpha_j(\xi_j) + \alpha_k(\xi_k).$$

Now the LHS of the above is symmetric and the RHS is antisymmetric. This proves (10) and the last assertion. As for (11), it is a direct consequence of (5). □

5. The leaves of Ξ

From (11) it is clear that Ξ determines a 3-dimensional foliation on M. Let F be any of its leaves. From (5) we see that F is totally geodesic in M. A direct computation of the curvature tensor of M shows that $R(X,Y)\xi_i$ is tangent to F for X, $Y \in \mathcal{X}(M)$ any i. On the other hand, for any totally geodesic submanifold P^p of a Riemannian manifold $\overline{P}^{p+k}$ one has the relation:

$$Ric^P(X,Y) = Ric^{\overline{P}}(X,Y) - \sum_{p+1}^{p+k} g(R^{\overline{P}}(X,E_i)Y, E_i)$$

where $\{E_i\}$ is a local orthonormal basis adapted to the submanifold. Hence, if one knows that $R^{\overline{P}}(X,E_i)Y$ is tangent to the submanifold, the last term vanishes and the two Ricci tensors are equal on the submanifold. It is the case for F in P (just put $X = \xi_i$, $Y = \xi_j$). As any 3-dimensional Einstein manifold has constant scalar curvature we may state:

Proposition 5.1. *The leaves of the foliation generated by Ξ are space forms of positive sectional curvature $2n+1$.*

Hence, if compact, the leaves are spherical space forms S^3/Γ.

A direct computation of the curvature tensor of M gives

$$\begin{aligned} Ric^M(Y,\xi_i) &= 2(2n+1)\eta_i(Y) + 2\left\{d\alpha_k + (\alpha_i \wedge \alpha_j)\right\}(\xi_j, Y) - \\ &- 2\left\{d\alpha_j - (\alpha_i \wedge \alpha_k)\right\}(\xi_k, Y). \end{aligned}$$

Together with the above proposition, this proves that the local one-forms α_i are subject to some conditions, their restrictions to F (denoted by the same letters) must satisfy the following equations:

$$(d\alpha_i + \alpha_k \wedge \alpha_j)(\xi_j, \xi_i) = 0$$
$$(d\alpha_i + \alpha_k \wedge \alpha_j)(\xi_k, \xi_i) = 0$$
$$(d\alpha_i + \alpha_k \wedge \alpha_j)(\xi_k, \xi_j) = \lambda$$

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$$(d\alpha_k + \alpha_i \wedge \alpha_j)(\xi_i, \xi_k) = 0$$
$$(d\alpha_k + \alpha_i \wedge \alpha_j)(\xi_j, \xi_k) = 0$$
$$(d\alpha_k + \alpha_i \wedge \alpha_j)(\xi_j, \xi_i) = \lambda$$

where $\lambda = \frac{1}{2}Scal^M - (2n+1)$. These formulae may be put into the more compact form:

$$d\alpha_i + \alpha_k \wedge \alpha_j = \lambda \eta_k \wedge \eta_j$$
$$d\alpha_j + \alpha_k \wedge \alpha_i = \lambda \eta_k \wedge \eta_i$$
$$d\alpha_k + \alpha_i \wedge \alpha_j = \lambda \eta_j \wedge \eta_i$$

From (12) we see that $L_\xi g(X,Y) = 0$ for any X, Y orthogonal to Ξ. Hence the metric is projectable on the leaf space, when this exists. However, we can say nothing about other projectable structures on the eventual leaf space. We can only state:

Proposition 5.2. *Let M be a Riemannian manifold carrying a normal almost contact metric 3-structure. If the leaves of the foliation Ξ are compact, then the projected metric on the orbifold $P = M/\Xi$ is Einstein with positive scalar curvature. The canonical projection is a Riemannian totally geodesic submersion whose leaves are 3-dimensional spherical space forms.*

Proof. Everything was already proven except for the Einstein property of P. We use the following formula connecting the Ricci tensors of the total space and of the base space in a Riemannian submersion with totally geodesic fibers, cf. [Bes], p.250:

$$Ric^M(X,Y) = Ric^P(X^*, Y^*) - 2g(A_{X^*}, A_{Y^*}).$$

Here * denotes horizontal lift of vector fields and A is the O'Neill tensor defined by $2A_E F = vertical part of [E,F]$. A local orthonormal basis for the vertical space is $\{\xi_1, \xi_2, \xi_3\}$, hence:

$$\begin{aligned} g(A_{X^*}, A_{Y^*}) &= \sum g(A_{X^*}\xi_i, A_{Y^*}\xi_i) = \sum g(\mathcal{H}\nabla_{X^*}\xi_i, \mathcal{H}\nabla_{Y^*}\xi_i) = \\ &= \sum g(\varphi_i X^*, \varphi_i Y^*) = 3g(X^*, Y^*). \end{aligned}$$

Finally this yields:

$$Ric^P = (Scal^M + 6)g = (4n+8)g$$

and the proof is complete. □

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