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Transformations of locally conformally Kähler manifolds

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Abstract. We consider several transformation groups of a locally conformally Kähler manifold and discuss their inter-relations. Among other results, we prove that all conformal vector fields on a compact Vaisman manifold which is neither locally conformally hyperkähler nor a diagonal Hopf manifold are Killing, holomorphic and that all affine vector fields with respect to the minimal Weyl connection of a locally conformally Kähler manifold which is neither Weyl-reducible nor locally conformally hyperkähler are holomorphic and conformal.

1. Introduction

We briefly present the necessary background for locally conformally Kähler (in short, LCK) geometry.

In the sequel, J will denote an integrable complex structure on a connected, smooth manifold M^{2n} of complex dimension $n \geq 2$. For a Hermitian metric g on (M, J), we denote by ∇^g its Levi-Civita connection and by ω its fundamental two-form $\omega(X, Y) := g(JX, Y)$.

Traditionally, a LCK metric g on the complex manifold (M, J) is defined by the following "integrability" condition satisfied by its fundamental two-form:

$$d\omega = \theta \wedge \omega, \quad d\theta = 0. \tag{1}$$

The closed one-form θ is called the *Lee form*. Note that, on manifolds of complex dimension at least 3, the first equation implies the closedness of θ, the second condition being relevant only on complex surfaces.

Any other Hermitian metric $e^f g$ which is conformal with g is a LCK metric too, its Lee form being $\theta + df$. Hence, a LCK metric determines a 1-cocycle associated merely to the conformal class $c = [g]$.

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Clearly, on local open sets on which θ is exact, g is conformal with some local Kähler metrics which, on overlaps, are homothetic. The above 1-cocycle is associated to this system of scale factors.

Note that on the universal cover of M, the pull-back of θ is exact (the pull-backs of the local Kähler metrics glue together to a global Kähler metric) and the fundamental group of M acts by biholomorphic homotheties with respect to this Kähler metric. This is usually taken for definition of a LCK manifold.

Returning on M, the Levi-Civita connection D^U of a local Kähler metric in the conformal class of g on some open set U is related to the Levi-Civita connection of g by the formula:

$$D^U = \nabla^g - \frac{1}{2}(\theta \otimes \mathrm{id} + \mathrm{id} \otimes \theta - g \otimes \theta^\sharp). \tag{2}$$

It is obvious that, in fact, these connections do not depend on the particular local Kähler metric, i.e. they glue together to a global connection, here denoted D, which has the following two properties:

$$DJ = 0, \quad Dg = \theta \otimes g.$$

The second equation implies that D preserves the conformal class: $Dc = 0$. Being torsion free (as, locally, any Levi-Civita connection), it is a Weyl connection. Notice that the Lee form of the Weyl connection D with respect to g defined by (2) coincides with the Lee form of the complex structure J with respect to g defined by (1).

Definition 1.1 A *Hermitian-Weyl structure* on M^{2n} is a triple (c, J, D) where c is a conformal structure, J is a complex structure compatible with c and D is a Weyl connection such that $DJ = 0$ and $Dc = 0$.

From the above, every LCK structure defines a Hermitian-Weyl structure on M, and conversely, every metric in the conformal class of a Hermitian-Weyl structure is LCK, provided $n \geq 3$. Notice that the Weyl connection D is uniquely defined by J (see e.g. [2, Lemma 5.1]). We call D the *minimal* Weyl connection.

In general, there is no way to choose a "canonical" metric in the conformal class of a Hermitian-Weyl manifold (M, c, J, D), unless M is compact, where there exists a unique, up to homothety, metric g_0 in c such that the Lee form θ_0 of D with respect to g_0 is co-closed (and hence harmonic), see [5, p. 502]. We call g_0 the *Gauduchon metric*.

A particular class of LCK manifolds are the Vaisman manifolds. These are defined by the condition $\nabla^g \theta = 0$. By the uniqueness up to homotheties, a Vaisman metric is, necessarily, the Gauduchon metric in its conformal class. The prototype of Vaisman manifolds are the Hopf manifolds $\mathbb{C}^n \backslash \{0\}/\mathbb{Z}$, with $\mathbb{Z}$ generated by a semi-simple endomorphism (see [10] for the structure of compact Vaisman manifolds). On the other hand, there exist examples of compact LCK manifolds which do not admit any Vaisman metric: such are the Inoue surfaces and the non-diagonal Hopf surfaces, see [1].

The hypercomplex version of LCK geometry is straightforward. One starts with a hypercomplex manifold (M, I, J, K) endowed with a conformal class $[c]$ and with a Weyl connection D satisfying:

$$Dc = 0, \quad DI = DJ = DK = 0.$$

One obtains the notion of hyperhermitian-Weyl structure. It can be seen that for any $g \in c$, the Lee forms associated to the three complex structures coincide. As above, the Lee form is closed in real dimension at least 8, and hence, with the exception of dimension 4, *a hyperhermitian-Weyl manifold is locally conformally hyperkähler (LCHK)*.

The local Kähler metrics of a LCHK manifold are now hyperkähler, hence Ricci-flat. It follows that LCHK manifolds are necessarily Einstein-Weyl. But on compact Einstein-Weyl manifolds, the Gauduchon metric is parallel, see [6]. Hence, a compact LCHK manifold bears three nested Vaisman structures.

For examples and properties of LCK and LCHK manifolds, we refer to [3], to more recent papers by Ornea and Verbitsky and to the references therein.

Remark 1.2 In the whole paper, we tacitly assume that the manifolds we consider are not globally conformally Kähler, i.e. the Lee form is never exact. This is especially important on compact manifolds, where LCK and Kähler structures impose completely different topologies.

We now consider the following transformation groups on a Hermitian-Weyl manifold (M, c, J, D):

- $\mathrm{Aff}(M, D)$, the group of affine transformations, i.e. preserving the Weyl connection.
- $\mathrm{H}(M, J)$, the group of biholomorphisms with respect to J.
- $\mathrm{Conf}(M, c)$, the group of conformal transformations.
- $\mathrm{Aut}(M) := \mathrm{Hol}(M, J) \cap \mathrm{Conf}(M, c)$, the group of automorphisms.

It is well known that these are Lie groups. Their Lie algebras will be denoted, respectively by: $\mathfrak{aff}(M, D)$, $\mathfrak{h}(M, J)$, $\mathfrak{conf}(M, c)$, $\mathfrak{aut}(M)$.

On Vaisman manifolds, the Lee field $\theta^\sharp$ is Killing and real-analytic but on non-Vaisman LCK manifolds, almost no information about these groups is available. It is the purpose of this note to clarify some relations among the above groups. Essentially, we prove that:

- $\mathfrak{aff}(M, D) = \mathfrak{aut}(M)$, provided that $\mathrm{Hol}_0(D)$ is irreducible and M is not LCHK (Corollary 2.3).
- $\mathfrak{conf}(M, c) = \mathfrak{aut}(M)$ on compact Vaisman manifolds which are neither LCHK nor diagonal Hopf manifolds (Theorem 3.2).

Note that on compact LCHK manifolds and on Hopf manifolds there exist examples of affine transformations which are not holomorphic, see Remark 2.4 (ii) below.

We believe that the second equality holds on all compact LCK manifolds which are neither LCHK nor diagonal Hopf manifolds. By Lemma 2.5 below, this amounts to prove that every conformal vector field is Killing for the Gauduchon metric, which is the LCK counterpart of the result saying that every conformal vector field on a compact Kähler manifold is Killing ([8, p. 148]).

2. Affine vector fields on LCK manifolds

Our first result is the LCK analogue of [8, Sect. 54], (see also [7]).

Lemma 2.1 *Let (M, c, J, D) be a LCK manifold which is not locally conformally hyperkähler and such that* $\mathrm{Hol}_0(D)$ *is irreducible. Then any $f \in \mathrm{Aff}(M, D)$ is holomorphic or anti-holomorphic.*

Proof. Let $f \in \mathrm{Aff}(M, D)$ and let $J'_x := (d_x f)^{-1} \circ J_{f(x)} \circ (d_x f)$ denote the image by f of the complex structure J. Then $J'^2 = -\mathrm{id}$ and, as J and df commute with the parallel transport induced by D, we easily derive that J' is D-parallel too.

Consider the decomposition of JJ' into symmetric and skew-symmetric parts: $JJ' = S + A$, where

$$\begin{cases} S := \frac{1}{2}(JJ' + J'J), \\ A := \frac{1}{2}(JJ' - J'J). \end{cases}$$

Since S is D-parallel, its eigenvalues are constant and the corresponding eigenbundles are D-parallel. The irreducibility assumption implies $S = k\mathrm{id}$ for some $k \in \mathbb{R}$. Similarly, the D-parallel symmetric endomorphism A^2 has to be a multiple of the identity: $A^2 = p\mathrm{id}$.

If A were non-zero, $A(X) \neq 0$ for some $X \in TM$, so

$$0 > -c(AX, AX) = c(A^2X, X) = pc(X, X),$$

whence $p < 0$. The endomorphism $K := A/\sqrt{-p}$ is then a D-parallel complex structure and satisfies $KJ = -JK$, so (J, K) defines a LCHK structure on (M, c), which is forbidden by the hypothesis. Thus $A = 0$ and $JJ' = k\mathrm{id}$, so $J' = -kJ$. Using $J'^2 = -\mathrm{id}$ we get $k = \pm 1$, so $J' = \pm J$, which just means that f is holomorphic or anti-holomorphic. □

In a similar manner one can prove the following

Lemma 2.2 *Let (M, c, J, D) be a LCK manifold such that* $\mathrm{Hol}_0(D)$ *is irreducible. Then any $f \in \mathrm{Aff}(M, D)$ is conformal.*

Proof. Since f is affine, the pull-back by f of the conformal structure c is a D-parallel conformal structure c'. The symmetric endomorphism B of TM defined by $c'(X, Y) = c(BX, Y)$ is D-parallel. The irreducibility of $\mathrm{Hol}_0(D)$ shows as before that B is a multiple of the identity. □

Corollary 2.3 *Let (M, c, J, D) be a LCK manifold which is not LCHK and such that* $\mathrm{Hol}_0(D)$ *is irreducible. Then any infinitesimal affine transformation of the Weyl connection is an infinitesimal automorphism, i.e.* $\mathfrak{aff}(M, D) = \mathfrak{aut}(M)$.

Remark 2.4

(i) We do not know whether the assumption of irreducibility of the Weyl connection can be relaxed or not, at least on compact M. But we can show that *a compact Vaisman manifold, which is not a (diagonal) Hopf manifold, is Weyl-irreducible*. Indeed, by the structure theorem in [10], M is a mapping torus of a Sasakian isometry and its universal cover is a Kählerian cone over the compact, hence complete, Sasakian fibre. Then, as D is, locally, the Levi-Civita connection of the local Kähler metrics, if D is reducible, the Kähler metric of the covering cone is reducible. Now, by Proposition 3.1 in [4], a cone over a complete manifold is reducible if and only if it is flat. But a flat cone is the cone over a sphere, hence M is a Hopf manifold.

(ii) Let W be a 3-Sasakian manifold. Then $M := W \times S^1$ is LCHK (for $W = S^{2n-1}$, M is a Hopf manifold) and each of the three Killing fields on W which generate the $\mathrm{SU}(2)$ action, induce Killing, hence affine with respect to the Weyl connection, fields on M and each of them is holomorphic only with respect to one of the three complex structures (see e.g. [3, Chapter 11]).

We apply the above to prove:

Lemma 2.5 *Let* (M, c, J, D) *be a compact LCK manifold which is not LCHK and such that* $\mathrm{Hol}_0(D)$ *is irreducible. Then every Killing vector field* ξ *of the Gauduchon metric is a holomorphic vector field.*

Proof. According to Corollary 2.3, it is enough to prove that ξ is affine. As the Weyl connection is uniquely defined by the Gauduchon metric and its Lee form θ_0, it will be enough to show that

$$\mathcal{L}_\xi \theta_0 = 0. \tag{3}$$

But, as ξ is g_0-Killing, the codifferential δ of g_0 commutes with $\mathcal{L}_\xi$, hence $\delta(\mathcal{L}_\xi \theta_0) = \mathcal{L}_\xi(\delta\theta_0) = 0$, because θ_0 is co-exact. Now, θ_0 is closed, so $d(\mathcal{L}_\xi \theta_0) = 0$, i.e. $\mathcal{L}_\xi \theta_0$ is harmonic. On the other hand, by Cartan's formula, $\mathcal{L}_\xi \theta_0 = d(\xi \lrcorner \theta_0)$. On a compact Riemannian manifold, a one-form which is both exact and harmonic necessarily vanishes, so (3) is proved. □

3. Conformal vector fields on Vaisman manifolds

As mentioned in the introduction, every conformal vector field on a compact Kähler manifold is Killing ([8, p. 148]). We consider the LCK analogue of this statement: *Is every conformal vector field on a compact LCK manifold Killing with respect to the Gauduchon metric?*

As no sphere S^{2n} can bear an LCK metric for $n \geq 2$ (S^{2n} being simply connected, such a structure would be automatically Kähler), by Obata's theorem we derive that *any conformal vector field on a compact LCK manifold is Killing for some metric in the given conformal class, but not necessarily the Gauduchon metric*. In fact, an argument entirely similar to the above, shows that a conformal vector field is Killing with respect to the Gauduchon metric if and only if it preserves the Lee form of the metric with respect to which it is Killing.

The answer to the above question is not known to us in general. Nevertheless, we can show that it holds on compact Vaisman manifolds, on which every conformal field is Killing with respect to the Gauduchon metric. This, actually, follows from a more general statement:

Theorem 3.1 *Let φ be an isometry of a compact Riemannian manifold (W, h) and let $(M, g) := (W, h) \times \mathbb{R}/_{\{(x,t)\sim(\varphi(x),t+1)\}}$ be the mapping torus of φ. Then every conformal vector field on (M, g) is Killing.*

Proof. In [9] it is shown that every twistor form on a Riemannian product is defined by Killing forms on the factors. We adapt the argument there to the situation of mapping tori.

Every conformal vector field on (M, g) induces a conformal vector field denoted ξ on the covering space $W\times\mathbb{R}$ of M, endowed with the product metric $\tilde{g} := h+dt^2$. We write ξ as

$$\xi = a\partial_t + \eta,$$

where $\partial_t = \partial/\partial t$, η is tangent to W and a is a function. Both a and η can be viewed as objects on W indexed upon the parameter t on $\mathbb{R}$. Since ξ is invariant under the isometry $(x, t) \mapsto (\varphi(x), t + 1)$ of $W \times \mathbb{R}$, we get in particular $a(w, t) = a(\varphi(w), t + 1)$, and thus, if we write $a_t(w)$ for $a(w, t)$,

$$a_t = \varphi^* a_{t+1}, \quad \dot{a}_t = \varphi^* \dot{a}_{t+1}, \qquad \forall\, (w, t) \in W \times \mathbb{R}. \tag{4}$$

Now, ξ being conformal, there exists a function f on $W \times \mathbb{R}$ such that $\mathcal{L}_\xi \tilde{g} = f\tilde{g}$. This can be written

$$\tilde{g}(\nabla_A \xi, B) + \tilde{g}(A, \nabla_B \xi) = f\tilde{g}(A, B) \tag{5}$$

for every tangent vectors to $W \times \mathbb{R}$, A and B. Taking $A = B = \partial_t$ in (5) gives

$$f = 2\tilde{g}(\nabla_{\partial_t}\xi, \partial_t),$$

hence, by the parallelism of ∂_t, one has

$$f = 2\partial_t(a) = 2\dot{a}. \tag{6}$$

Applying the same Eq. (5) on pairs (∂_t, X), (Y, Z), with X, Y, Z tangent to W and independent on t, we obtain, respectively:

$$\nabla_{\partial_t}\eta + da = 0,$$

and

$$h(\nabla_Y \eta, Z) + h(Y, \nabla_Z \eta) = fh(Y, Z).$$

Notice that here da denotes the exterior derivative of a on W and should be understood as a family of 1-forms on W indexed on t, identified by h with a family of vector fields on W. Taking (6) into account, the two equations above become:

$$\begin{cases} \dot\eta = -da, \\ h(\nabla^W_Y \eta, Z) + h(Y, \nabla^W_Z \eta) = 2\dot a h(Y, Z) \end{cases} \tag{7}$$

As the vector fields Y, Z are independent on t, we can take the derivative with respect to t in the second equation of (7):

$$h(\nabla^W_Y \dot\eta, Z) + h(Y, \nabla^W_Z \dot\eta) = 2\ddot a h(Y, Z).$$

Letting here $Y = Z = E_i$, where $\{E_i\}$ is a local orthonormal basis on W, we obtain

$$\delta^W \dot\eta = -m\ddot a, \qquad (m = \dim W).$$

Using the first equation in (7), we finally obtain:

$$\delta^W da = m\ddot a. \tag{8}$$

We use this equation to show that a is constant. To this end, we integrate $\|da\|^2$ on W:

$$\int_W h(da, da) = \int_W (\delta^W da)a = m\int_W a\ddot a = m\int_W (a\dot a)' - m\int_W (\dot a)^2.$$

If we let $b := ma\dot a$, then, from the above equation we have

$$\int_W \dot b = \int_W h(da, da) + m\int_W (\dot a)^2 \geq 0. \tag{9}$$

On the other hand, taking $t = 0$ in (4) yields

$$\int_W b_0 = \int_W \varphi^* b_1 = \int_W b_1,$$

because φ is a diffeomorphism. We can therefore compute

$$\int_{W\times[0,1]} \dot b = \int_0^1 \left(\int_W \dot b\right) dt = \int_W b_1 - \int_W b_0 = 0.$$

By (9), $\dot a = 0$ and $da = 0$. As M is connected, a is constant. It follows that $f = 0$ and thus ξ is Killing. □

We apply the previous result to compact Vaisman manifolds which, by the structure theorem in [10], are mapping tori with compact, Sasakian fibre. As a direct consequence of Remark 2.4(i), Lemma 2.5 and Theorem 3.1 we get:

Theorem 3.2 *Any conformal vector field on a compact Vaisman manifold which is neither LCHK nor a diagonal Hopf manifold is Killing and holomorphic.*

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