Journal of Geometry and Physics 59 (2009) 295–305

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Morse–Novikov cohomology of locally conformally Kähler manifolds

Liviu Ornea [a,b,*], Misha Verbitsky [c]

[a] University of Bucharest, Faculty of Mathematics, 14 Academiei str., 70109 Bucharest, Romania

[b] Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 21, Calea Grivitei Str. 010702-Bucharest, Romania

[c] Institute of Theoretical and Experimental Physics, B. Cheremushkinskaya, 25, Moscow, 117259, Russia

ARTICLE INFO

Article history:
Received 4 October 2008
Accepted 9 November 2008
Available online 14 November 2008

Keywords:
Locally conformally Kähler
Morse–Novikov cohomology
Bott–Chern cohomology
Vaisman manifold
Weight bundle

ABSTRACT

A locally conformally Kähler (LCK) manifold is a complex manifold admitting a Kähler covering, with the monodromy acting on this covering by holomorphic homotheties. We define three cohomology invariants, the Lee class, the Morse–Novikov class, and the Bott–Chern class, of an LCK-structure. These invariants play together the same role as the Kähler class in Kähler geometry. If these classes coincide for two LCK-structures, the difference between these structures can be expressed by a smooth potential, similar to the Kähler case. We show that the Morse–Novikov class and the Bott–Chern class of a Vaisman manifold vanish. Moreover, for any LCK-structure on a manifold, admitting a Vaisman structure, we prove that its Morse–Novikov class vanishes. We show that a compact LCK-manifold M with vanishing Bott–Chern class admits a holomorphic embedding into a Hopf manifold, if $\dim_{\mathbb{C}} M \geqslant 3$, a result which parallels the Kodaira embedding theorem.
© 2008 Elsevier B.V. All rights reserved.

1. Introduction

1.1. Cohomological invariants of LCK-structures

A locally conformally Kähler (LCK) manifold is a complex manifold M (in this paper we shall usually assume $\dim_{\mathbb{C}} M \geqslant 3$), equipped with a Hermitian metric ω in such a way that a certain covering $\tilde{M}$ of M is Kähler, and its Kähler metric is conformal to the pullback of ω by the covering map. In this case, we have $d\omega = \omega \wedge \theta$, and the 1-form θ is called **the Lee form** of M. It is easy to see that θ is closed: indeed,

$$0 = d^2\omega = d\theta \wedge \omega + \theta \wedge \theta \wedge \omega = d\theta \wedge \omega,$$

but the multiplication by ω induces an embedding $\Lambda^2(M) \longrightarrow \Lambda^4(M)$ if $\dim_{\mathbb{C}} M \geqslant 3$.[1] The LCK-manifolds are often considered up to conformal equivalence; indeed, a manifold which is conformally equivalent to an LCK-manifold is itself locally conformally Kähler.

If one performs a conformal change, $\omega_1 = e^f\omega$, the Lee form θ changes to $\theta_1 = \theta + df$. The cohomology class $[\theta] \in H^1(M)$ is an important invariant of an LCK-manifold. Clearly, $[\theta] \in H^1(M)$ vanishes if and only if ω is conformally equivalent to a Kähler structure. In this case (M, ω) is called **globally conformally Kähler**.

Another, more subtle, invariant of an LCK-manifold is called **the Morse–Novikov class** of $[\omega] \in H^2_\theta(M)$, defined as follows. Recall that the Morse–Novikov cohomology, also known as Lichnerowicz cohomology (defined independently by Novikov and Lichnerowicz in [13,15]) is the cohomology of the complex

$$\Lambda^0(M) \xrightarrow{d-\theta} \Lambda^1(M) \xrightarrow{d-\theta} \Lambda^2(M) \xrightarrow{d-\theta} \cdots \tag{1.1}$$

* Corresponding author at: University of Bucharest, Faculty of Mathematics, 14 Academiei str., 70109 Bucharest, Romania.
E-mail addresses: lornea@gta.math.unibuc.ro, Liviu.Ornea@imar.ro (L. Ornea), verbit@maths.gla.ac.uk, verbit@mccme.ru (M. Verbitsky).

1 When $\dim_{\mathbb{C}} M = 2$, one defines LCK-manifolds in the same way, but $d\theta = 0$ should be assumed as a part of the definition.

0393-0440/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2008.11.003

with θ a closed 1-form (see Section 3 for a more detailed exposition). It is well known that the cohomology $H^*_\theta(M)$ of (1.1) is naturally identified with the cohomology of the local system given by the character $\pi_1(M) \longrightarrow \mathbb{R}^{>0}$ associated with the cohomology class $[\theta] \in H^1(M)$.

An LCK-form ω on an LCK-manifold satisfies $d\omega = \omega \wedge \theta$, therefore it is $(d-\theta)$-closed. The cohomology class $[\omega] \in H^2_\theta(M)$ is called **the Morse–Novikov class of the LCK-manifold**. It is an invariant of the LCK-manifold, roughly analogous to the Kähler class on a Kähler manifold.

The third, even more subtle, cohomology invariant of an LCK-manifold has a complex-analytic nature; it is a Morse–Novikov version of the usual Bott–Chern class of a closed $(1, 1)$-form.

On a compact Kähler manifold, an exact (p, q)-form η satisfies

$$\eta = \partial\overline{\partial}\alpha, \quad \alpha \in \Lambda^{p-1,q-1}(M). \tag{1.2}$$

This statement (which is called global $\partial\overline{\partial}$-**lemma**) fails to be true on non-Kähler manifolds; however, the corresponding complex

$$\cdots \longrightarrow \Lambda^{p-1,q-1}(M) \xrightarrow{\partial\overline{\partial}} \Lambda^{p,q}(M) \xrightarrow{\partial\oplus\overline{\partial}} \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M) \longrightarrow \cdots \tag{1.3}$$

is still elliptic. Its cohomology groups $H^{p,q}_{\partial\overline{\partial}}(M)$ are called **the Bott–Chern cohomology groups of** M (see e.g. [23]). Explicitly, the Bott–Chern cohomology groups are:

$$H^{p,q}_{\partial\overline{\partial}}(M) = \frac{\ker\left(\Lambda^{p,q}(M) \xrightarrow{\partial} \Lambda^{p+1,q}(M)\right) \bigcap \ker\left(\Lambda^{p,q}(M) \xrightarrow{\overline{\partial}} \Lambda^{p,q+1}(M)\right)}{\operatorname{im}\left(\Lambda^{p-1,q-1}(M) \xrightarrow{\partial\overline{\partial}} \Lambda^{p,q}(M)\right)}.$$

For compact Kähler manifolds the Bott–Chern cohomology groups are isomorphic with the Dolbeault ones; this isomorphism is equivalent to the $\partial\overline{\partial}$-lemma. For non-Kähler manifolds, this isomorphism does not hold anymore.

For Morse–Novikov cohomology, a similar complex can be defined. Consider the Hodge components of the Morse–Novikov differential $d_\theta := d - \theta$: $d_\theta = \partial_\theta + \overline{\partial}_\theta$, with $\partial_\theta = \partial - \theta^{1,0}$ and $\overline{\partial}_\theta = \overline{\partial} - \theta^{0,1}$. Locally, the Morse–Novikov complex becomes the de Rham complex after a change $\eta \mapsto \psi\eta$, where $\psi = e^f$, with f a function which satisfies $df = \theta$. Therefore, the complex

$$\Lambda^{p-1,q-1}(M) \xrightarrow{\partial_\theta\overline{\partial}_\theta} \Lambda^{p,q}(M) \xrightarrow{\partial_\theta\oplus\overline{\partial}_\theta} \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M) \tag{1.4}$$

is also elliptic. Its cohomology groups $H^{p,q}_{\partial_\theta\overline{\partial}_\theta}(M)$ are called **the Bott–Chern cohomology** groups of the Morse–Novikov complex.

It is possible to express $H^{1,1}_{\partial_\theta\overline{\partial}_\theta}(M)$ in terms of Morse–Novikov cohomology of M and holomorphic cohomology of a flat line bundle L, with monodromy determined from θ (see Theorem 4.7).

The cohomology class $[\omega] \in H^{1,1}_{\partial_\theta\overline{\partial}_\theta}(M)$ of an LCK-form ω is called **the Bott–Chern class of** M.

To summarize: with every LCK-manifold, we associate three cohomological invariants: the Lee class $[\theta] \in H^1(M)$, the Morse–Novikov class $[\omega] \in H^2_\theta(M)$, and the Bott–Chern class $[\omega] \in H^{1,1}_{\partial_\theta\overline{\partial}_\theta}(M)$. Notice that the Morse–Novikov class can be reconstructed from the Bott–Chern class.

In different situations, these three cohomological invariants play the role of the Kähler class. However, unlike the Kähler class, the Morse–Novikov and Bott–Chern classes can be zero (the Morse–Novikov class vanishes for compact Vaisman manifolds, see Section 3). On the other hand, if M is compact and non-Kähler, the Lee class $[\theta]$ cannot vanish, and if M admits a Kähler structure, all LCK-structures on M are globally conformally Kähler and have $[\theta] = 0$ (a result proved by Vaisman, see [4, Theorem 2.1]).

It is not clear whether the global $\partial\overline{\partial}$-lemma is true for Morse–Novikov complex, associated with the Lee form of an LCK-structure. If it is true, then the tautological map $H^{1,1}_{\partial_\theta\overline{\partial}_\theta}(M) \longrightarrow H^{1,1}_\theta(M)$ is an isomorphism, and the Morse–Novikov class of an LCK-manifold is the same as its Bott–Chern class.

1.2. The space of LCK-structures

In Kähler geometry, the Kähler form ω determines the Kähler class $[\omega] \in H^{1,1}(M)$, and the difference of Kähler forms which have the same Kähler class is measured by a potential:

$$\omega_1 - \omega = \partial\overline{\partial}\varphi$$

(this follows from the $\partial\overline{\partial}$-lemma). The space of all Kähler metrics is locally modeled on $H^{1,1}(M, \mathbb{R}) \times (C^\infty(M)/\operatorname{const})$. A similar local description exists for the set of all LCK-structures on a given complex manifold, if we fix the cohomology class $[\theta]$ of a Lee form.

Let $[\omega] \in H^{1,1}_{\partial_\theta\overline{\partial}_\theta}(M)$ be the Bott–Chern class of an LCK-form ω. Given another LCK-form ω_1, with the same Bott–Chern class, we can write

$$\omega_1 = \omega + d_\theta d^c_\theta \varphi = \omega + \varphi(\theta \wedge I\theta + d^c\theta) - \theta \wedge d^c\varphi + I\theta \wedge d\varphi + dd^c\varphi, \tag{1.5}$$

where $d^c_\theta = -Id_\theta I = d^c - I\theta$. Here we use implicitly the equality

$$d_\theta d^c_\theta = -2\sqrt{-1}\partial_\theta\overline{\partial}_\theta$$

which is the Morse–Novikov version of $dd^c = -2\sqrt{-1}\partial\overline{\partial}$.

For any real-valued function $\varphi \in C^\infty(M)$, the form (1.5) satisfies $d\omega_1 = \omega_1 \wedge \theta$, as a simple calculation implies. If

$$\sup_M \left(|dd^c\varphi| + |d\varphi| + |\varphi|\right)$$

is sufficiently small, ω_1 is also positive. We obtained that the difference of two LCK-forms in the same Bott–Chern class is expressed by a potential, just like in Kähler case, and the set of LCK-structures is locally parametrized by

$$H^{1,1}_{\partial_\theta\overline{\partial}_\theta}(M) \times \left(C^\infty(M)/\ker d_\theta d^c_\theta\right).$$

With regard to the realization of cohomology classes by LCK-forms, one could ask the questions similar to those asked (and sometimes answered) in Kähler geometry.

Problem 1.1. Determine all 1-forms θ for which there exists a Hermitian 2-form ω having θ as its Lee form, and all the Morse–Novikov classes which can be realized by an LCK-form.

Problem 1.2. Let M be a compact complex manifold, admitting an LCK-metric, and $[\theta] \in H^1(M)$ its Lee class. Determine the set of all classes $[\omega] \in H^{1,1}_{\partial_\theta\overline{\partial}_\theta}(M)$ such that $[\omega]$ is the Bott–Chern class of an LCK-structure with the Lee class $[\theta]$.

If θ is fixed, a sum of two LCK-forms is again LCK; therefore, the set of possible Morse–Novikov and Bott–Chern classes of LCK-structures on a given manifold with a fixed Lee class $[\theta] \in H^1(M)$ is a convex cone, similar to a Kähler cone.

In algebraic geometry, one often finds all kind of geometric properties of a Kähler manifold encoded in the shape of its Kähler cone. One would expect that the LCK-cones defined above would be just as important.

1.3. Potentials on coverings of LCK-manifolds

An important special case of an LCK-manifold is a manifold with *vanishing* Bott–Chern class. In [20], we studied the LCK-manifolds admitting an embedding into a Hopf manifold. We have proven a theorem, which can be stated as follows (using the language developed in the present paper).

Recall that **a linear Hopf manifold** is a complex manifold of the form $H_A := (\mathbb{C}^n \setminus 0)/\Gamma_A$, where the group $\Gamma_A \cong \mathbb{Z}$ is generated by $x \longrightarrow A(x)$, where A is a linear operator with all eigenvalues $\{\alpha_i\}$ satisfying $|\alpha_i| < 1$. If, in addition, A can be diagonalized, H_A is called **a diagonal Hopf manifold**. It is easy to see that H_A is homeomorphic to $S^{2n-1} \times S^1$. Since this manifold has $b_1(H_A)$ odd, it cannot be Kähler. However, H_A is locally conformally Kähler (see [5]). In [20] we have studied LCK-manifolds which admit a holomorphic (but not necessarily isometric) embedding into a linear Hopf manifold.

The main result of [20] can now be stated in terms of Bott–Chern cohomology:

Theorem 1.3 (*[20]*). *Let M be a compact LCK-manifold,* $\dim_{\mathbb{C}} M \geqslant 3$. *Then the following statements are equivalent.*

(i) *M admits a holomorphic embedding into a linear Hopf manifold.*
(ii) *M admits an LCK-structure with rational Lee class and vanishing Bott–Chern class.*

In the present paper, we explore the geometry of LCK-structures with vanishing Bott–Chern class. We generalize Theorem 1.3 to all manifolds with vanishing Bott–Chern class.

Theorem 1.4. *Let M be a compact LCK-manifold with vanishing Bott–Chern class. Then M admits an LCK-structure with vanishing Bott–Chern class and rational cohomology class of its Lee form.*

Proof. See Corollary 5.3. ■

From Theorems 1.3 and 1.4, we obtain that an LCK-manifold of complex dimension at least 3 admits a holomorphic embedding into a Hopf manifold if and only if it admits an LCK-structure with vanishing Bott–Chern class. We conjecture that the Bott–Chern class of an LCK-manifold admitting a holomorphic embedding into a Hopf manifold always vanishes.

For an important class of LCK-manifolds a weaker form of this conjecture can be proven. One could define a compact Vaisman manifold as an LCK-manifold admitting a holomorphic flow acting by conformalities (Section 2.2). It is known [5,28] that all diagonal Hopf manifolds are Vaisman. In [19], we have shown that any Vaisman manifold admits a holomorphic immersion into a diagonal Hopf manifold, and in [20] we proved that for $\dim_{\mathbb{C}} M \geqslant 3$ there exists an embedding into a diagonal Hopf manifold. In fact, the assumption $\dim_{\mathbb{C}} M \geqslant 3$ is not needed, because in [3], all two-dimensional Vaisman manifolds were classified, and embeddability of those into H_A can be easily checked using the same arguments as in [20].

In the present paper, we study the LCK-structures which can be defined on a given Vaisman manifold. We show that the Morse–Novikov class of any such structure vanishes (Theorem 6.1), and the Lee class is rational.

2. Locally conformally Kähler geometry

In this section we give the necessary definitions and properties of locally conformally Kähler (LCK) manifolds.

In what follows, M will denote a connected, smooth manifold of real dimension $2n$; I will be an integrable complex structure. For a Hermitian metric g, we denote with ∇^g its Levi-Civita connection and with ω its fundamental 2-form defined as $\omega(X, Y) = g(IX, Y)$.

2.1. LCK manifolds

Definition 2.1. A complex manifold (M, I) is LCK if it admits a Kähler covering $(\widetilde{M}, \widetilde{\omega})$, such that the covering group acts by holomorphic homotheties.

Equivalently, there exists on M a *closed* 1-form θ, called *the Lee form*, such that ω satisfies the integrability condition:

$$d\omega = \theta \wedge \omega.$$

Clearly, the metric $g := \omega(\cdot, I\cdot)$ on M is locally conformal to some Kähler metrics and its lift to the Kähler cover in the definition is globally conformal to h.

To an LCK manifold one associates the *weight bundle* $L_{\mathbb{R}} \longrightarrow M$. It is a real line bundle associated to the representation[2]

$$\mathrm{GL}(2n, \mathbb{R}) \ni A \mapsto |\det A|^{\frac{1}{n}}.$$

The Lee form induces a connection in $L_{\mathbb{R}}$ by the formula $\nabla = d - \theta$. ∇ is associated to the Weyl covariant derivative (also denoted ∇) determined on M by the LCK metric and the Lee form (the Weyl covariant derivative is uniquely defined by the properties $\nabla I = 0$, $\nabla g = \theta \otimes g$; in this context, θ is called the Higgs field). As $d\theta = 0$, then $\nabla^2 = d\theta = 0$, and hence $L_{\mathbb{R}}$ is flat.

The complexification of the weight bundle will be denoted by L. The Weyl connection extends naturally to L and its $(0, 1)$-part endows L with a holomorphic structure. On the other hand, as L is flat, one can pick a nowhere degenerate section λ satisfying

$$\nabla(\lambda) = \lambda \otimes (-\theta).$$

Accordingly, one chooses a Hermitian structure on L such that $|\lambda| = 1$ and considers the associated Chern connection. Then one proves:

Theorem 2.2 (*[27, Theorem 6.7]*). *The curvature of the Chern connection on L with respect to the above holomorphic and Hermitian structure is $-2\sqrt{-1}d^c\theta$.*

2.2. Vaisman manifolds

Definition 2.3. A Vaisman manifold is an LCK manifold with ∇^g-parallel Lee form.

On a Vaisman manifold, the Lee field $\theta^\sharp$ is real holomorphic and Killing (see [4]). On compact manifolds, this statement can be taken as a definition:

Theorem 2.4 (*[12]*). *A compact LCK manifold admits a LCK metric with parallel Lee form if and only if its Lie group of holomorphic conformalities has a complex one-dimensional Lie subgroup, acting non-isometrically on its Kähler covering.*

The structure of compact Vaisman manifolds is now fully understood:

Theorem 2.5 (*[18]*). *Let (M, I, g) be a compact Vaisman manifold. Then:*

(1) *The monodromy of $L_{\mathbb{R}}$ is $\mathbb{Z}$.*
(2) *(M, g) is a suspension over S^1 with fibre a compact Sasakian manifold (W, g_W). Moreover, M admits a conic Kähler covering $(W \times \mathbb{R}_+, t^2 g_W + dt^2)$ such that the covering group is an infinite cyclic group, generated by the transformation $(w, t) \mapsto (\varphi(w), qt)$ for some Sasakian automorphism φ and $q \in \mathbb{Z}$.*

The typical example of a compact Vaisman manifold is the diagonal Hopf manifold $H_A := \mathbb{C}^n/\langle A\rangle$ with $A = \mathrm{diag}(\alpha_i)$, with $|\alpha_i| < 1$. The complex structure is the projection of the standard one of $\mathbb{C}^n$ and the LCK metric is constructed as follows.

---

[2] In conformal geometry, the weight bundle usually corresponds to $|\det A|^{\frac{1}{2n}}$. For LCK-geometry, $|\det A|^{\frac{1}{n}}$ is much more convenient.

Let $C > 1$ be a constant. Then one constructs on $\mathbb{C}^n$ the potential

$$\varphi(z_1, \ldots, z_n) = \sum |z_i|^{\beta_i}, \quad \beta_i = \log_{|\alpha_i|^{-1}} C$$

which is acted on by A as follows: $A^*\varphi = C^{-1}\varphi$.

Hence:

$\Omega = \sqrt{-1}\partial\overline{\partial}\varphi$ is Kähler and $\Gamma \cong \mathbb{Z}$ acts by holomorphic homotheties with respect to it.

The Lee field is : $\theta^\sharp = -\sum z_i \log|\alpha_i|\partial z_i$ and one can see it is parallel.

Note than diagonal Hopf manifolds are generalizations of the rank 1 Hopf surfaces.

Other examples (in fact, the whole list) of compact complex surfaces with Vaisman structure are given in [3].

Non-Vaisman LCK manifolds are some of the Inoue surfaces (cf. [25,3]) and their generalizations to higher dimensions [16], the rank 0 Hopf surfaces [5].

2.3. LCK manifolds with potential

Not only the Vaisman metric of the Hopf manifold can be constructed out of a potential on the Kähler covering, but the Kähler metric on the universal cover of any Vaisman manifold has a global potential given by $|\theta^\sharp|^2$.

A wider class of LCK manifolds, strictly containing the Vaisman ones, is the following:

Definition 2.6 (*[20]*). A complex manifold (M, I) is *LCK with potential* if it admits a Kähler cover $(\widetilde{M}, \Omega)$ with global potential $\varphi : \widetilde{M} \to \mathbb{R}_+$ satisfying the following conditions:

(i) φ is proper (i.e. it has compact level sets).
(ii) The monodromy map τ acts on φ by multiplication with a constant: $\tau(\varphi) = const \cdot \varphi$.

On compact manifolds, the properness of the potential is equivalent (cf. [20]) to the deck group being isomorphic to $\mathbb{Z}$ (a condition satisfied by compact Vaisman manifolds).

In [19] we showed that there exist deformations which preserve the Vaisman class on compact manifolds. Moreover, on compact manifolds, one can always deform a Vaisman structure into a quasi-regular one. Using a similar argument, one proves:

Theorem 2.7 (*[20]*). *The class of compact LCK manifolds with potential is stable under small deformations.*

As a consequence, one sees that the Hopf manifold $(\mathbb{C}^N\backslash 0)/\Gamma$, where now Γ is a cyclic group generated by a non-diagonal linear operator, is LCK with potential. This is a generalization of the (non-Vaisman) rank 0 Hopf surface.

The main property of LCK manifolds with potential is that they satisfy an embedding theorem similar to the Kodaira–Nakano one in Kähler geometry:

Theorem 2.8 (*[20]*). *Any compact LCK manifold with potential of complex dimension at least 3 can be holomorphically embedded in a Hopf manifold. Moreover, a compact Vaisman manifold of complex dimension at least 3 can be holomorphically embedded in a diagonal Hopf manifold.*

3. Morse–Novikov cohomology

3.1. Morse–Novikov complex and cohomology of local systems

Let M be a smooth manifold, and θ a closed 1-form on M. Denote by $d_\theta : \Lambda^i(M) \longrightarrow \Lambda^{i+1}(M)$ the map $d - \theta$. Since $d\theta = 0$, $d_\theta^2 = 0$.

Consider the complex

$$\Lambda^0(M) \xrightarrow{d_\theta} \Lambda^1(M) \xrightarrow{d_\theta} \Lambda^2(M) \xrightarrow{d_\theta} \cdots$$

This complex is called **the Morse–Novikov complex**, (see e.g. [21,22,14]) and its cohomology **the Morse–Novikov cohomology**. In Jacobi and locally conformal symplectic geometry, this object is called **Lichnerowicz–Jacobi, or Lichnerowicz cohomology**, motivated by Lichnerowicz's work [13] on Jacobi manifolds (see e.g. [9,2]).

For an early introduction to this subject, we refer to [21].

A closed 1-form θ defines a flat connection $d - \theta$ on the trivial bundle $M \times C^\infty(M)$. A sheaf of $d - \theta$-closed functions on M is obviously locally trivial, and hence it defines a local system. Its monodromy is associated with the character $\chi : \pi_1(M) \longrightarrow \mathbb{R}^{>0}$ given by the exponent $e^\theta \in H^1(M, \mathbb{R}^{>0})$, considered as an element of $\mathbb{R}^{>0}$-valued cohomology.

The cohomology of this local system is equal to the Morse–Novikov cohomology, as we shall see presently.

Let L be a real local system on M, that is, a locally trivial sheaf on M locally modeled on $M \times \mathbb{R}$. Assume also that L is oriented, that is, its monodromy lies in $\mathbb{R}^{>0} \subset \mathbb{R}^*$. Then the vector bundle $L_\mathbb{R} = L \otimes_\mathbb{R} C^\infty(M)$ is trivial. The structure of a local system induces on $L_{\mathbb{R}}$ a flat connection ∇ (see [7]). Choose a trivialization, that is, a nowhere degenerate section $\xi \in L_{\mathbb{R}}$, and let

$$\nabla_\xi : L_{\mathbb{R}} \longrightarrow L_{\mathbb{R}} \otimes C^\infty(M)$$

be the trivial connection on $L_{\mathbb{R}}$ mapping ξ to 0. Then $\theta = \nabla - \nabla_\xi$ is a 1-form with values in $\operatorname{End}_{C^\infty(M)}(L_{\mathbb{R}}) = C^\infty(M)$. Since $\nabla^2 = 0$, θ is closed. The following elementary claim is well known.

Proposition 3.1 ([15]). *The cohomology of the local system L is naturally identified with the cohomology of the Morse–Novikov complex* $(\Lambda^*(M), d_\theta)$. □

3.2. Morse–Novikov cohomology of LCK-manifolds

It is well known that the LCK form represents a class in the Morse–Novikov cohomology. Indeed, let (M, ω) be an LCK-manifold, with θ its Lee form. Since $d\omega = \theta \wedge \omega$, we have $d_\theta \omega = 0$. Therefore, ω represents a cohomology class in the complex $(\Lambda^*(M), d_\theta)$

Definition 3.2. The Morse–Novikov cohomology class $[\omega]$ of ω is called **the Morse–Novikov class of** M.

This notion is similar to the notion of a Kähler class of a Kähler manifold.

Morse–Novikov cohomology for locally conformally symplectic manifolds was first considered in [6] where it was proven to vanish in the top dimension. Then Vaisman, [26], studied it under the name of "adapted cohomology" on LCK manifolds and identified it with the cohomology with values in the sheaf of germs of smooth d_θ-closed functions. Later, it was proven in [9] that it vanishes in all dimensions for compact locally conformally symplectic manifolds which admit a compatible Riemannian metric for which the Lee form is parallel, hence, in particular, for compact Vaisman manifolds (see also [17]). But for compact Vaisman manifolds, the vanishing of the Morse–Novikov cohomology follows almost directly from the Structure Theorem 2.5.

Indeed, according to Proposition 3.1, this is the cohomology of the local system L. But, by Theorem 2.5, M is $W \times S^1$ topologically and the monodromy of L is $\mathbb{Z}$, hence L is the pull-back p^*L' of a local system L' on S^1.

Now, the cohomology of the local system L is the derived direct image $R^iP_*(L)$, where P is a projection onto a point. By the above remark and changing the base, $R^iP_*(L) = R^iP_*p^*(L') = R^i(\mathbb{C} \otimes L')$, where $\mathbb{C}$ is viewed as a trivial local system. From the Künneth formula it follows that $R^i(\mathbb{C})$ is a locally constant sheaf on S^1, with fiber $H^*(W)$. By the Leray spectral sequence of composition, the hypercohomology of the complex of sheaves $R^*(\mathbb{C} \otimes L')$ converges to $R^iP_*(L)$. Finally, each $R^i(\mathbb{C} \otimes L')$ has zero cohomology, being a locally trivial sheaf on S^1 with non-trivial constant monodromy. Therefore this spectral sequence vanishes in E_2. It then converges to zero.

Remark 3.3. There exist compact LCK manifolds with non-vanishing Morse–Novikov class. Indeed, it is proven in [2] that the compact four-dimensional LCK solvmanifold constructed in [1] has non-vanishing Morse–Novikov class. On the other hand, it is shown in [11] that this solvmanifold is biholomorphic with an Inoue surface [8] which is known, [3], to not admit any Vaisman metric.

4. Bott–Chern class of an LCK-form

4.1. LCK structures up to potential

Definition 4.1. Let (M, ω_1, θ) and (M, ω_2, θ) be two LCK-structures on the same compact manifold M, having the same Lee form θ. These structures are called **equivalent up to a potential** if the following conditions hold.

(i) Consider a covering $\widetilde{M}$ where θ becomes exact, $\theta = d\!\!\!\underset{\sim}{f}$, and let $\widetilde{\omega}_i = e^{-f}\omega_i$ be the corresponding Kähler forms on $\widetilde{M}$. Then $\widetilde{\omega}_1 - \widetilde{\omega}_2 = \partial\bar{\partial}\varphi$, for some smooth function $\varphi : \widetilde{M} \longrightarrow \mathbb{R}$.

(ii) Let Γ be the deck transformation (also known as monodromy) group of the covering $\widetilde{M} \longrightarrow M$, and $\chi : \Gamma \longrightarrow \mathbb{R}^{>0}$ the character corresponding to e^f,

$$\gamma^* e^f = \chi(\gamma) e^f, \quad \forall \gamma \in \Gamma.$$

Then φ has the same automorphy:

$$\gamma^* \varphi = \chi(\gamma)\varphi, \quad \forall \gamma \in \Gamma.$$

A Kähler potential φ satisfying these automorphy conditions is called **the automorphic potential** for the LCK metric.

Given an LCK-structure on a compact complex manifold, it is very easy to construct other LCK-structures, equivalent up to potential. Let (M, ω) be an LCK-manifold, $\widetilde{\omega}$ the natural Kähler form on its covering $\widetilde{M}$. Consider a function $v : \widetilde{M} \longrightarrow \mathbb{R}$ which satisfies $|\nabla^2 v|_{L^\infty} < \varepsilon$, and has the same automorphy condition

$$\gamma^* v = \chi(\gamma) v, \quad \forall \gamma \in \Gamma. \tag{4.1}$$

If $\varepsilon < 1$, the form $\widetilde{\omega}_1 := \widetilde{\omega} + \partial\overline{\partial} v$ is Kähler, with the same automorphy properties as $\widetilde{\gamma}$. Therefore, $\widetilde{\omega}_1$ induces an LCK-structure on $\widetilde{M}$, obviously, equivalent to (M, ω, θ). All LCK-structures equivalent to a given one are obtained this way.

Remark 4.2. Denote by $C^\infty(\widetilde{M})_\chi$ the space of smooth functions satisfying (4.1). We have shown that the set of LCK-structures equivalent to a given one is an open convex cone in an affine space modeled on a quotient of $C^\infty(\widetilde{M})_\chi$ by the kernel of $\partial\overline{\partial}$.

4.2. Bott–Chern cohomology

The set of equivalence classes of LCK-structures with a given monodromy can be described in terms of cohomology.

Definition 4.3. Let M be a complex manifold, $\dim_{\mathbb{C}} M = n$, $0 < p, q < n$ integer numbers, and L a complex line bundle with flat connection. Consider the complex

$$\longrightarrow \Lambda^{p-1,q-1}(M, L) \xrightarrow{\partial\overline{\partial}} \Lambda^{p,q}(M, L) \xrightarrow{\partial \oplus \overline{\partial}} \Lambda^{p+1,q}(M, L) \oplus \Lambda^{p,q+1}(M, L) \longrightarrow \tag{4.2}$$

where ∂, $\overline{\partial}$ denote the (1,0) and (0,1)-parts of the connection operator $\nabla : \Lambda^i(M, L) \longrightarrow \Lambda^{i+1}(M, L)$. The cohomology of (4.2) is called **the Bott–Chern cohomology of** M **with coefficients in** L, denoted by $H^{p,q}_{\partial\overline{\partial}}(M, L)$. It is well known that (4.2) is elliptic, hence $H^{p,q}_{\partial\overline{\partial}}(M, L)$ is finite-dimensional.

Definition 4.4. Let (M, ω, θ) be an LCK-manifold, and L its weight bundle, that is, a trivial complex line bundle with the flat connection $d - \theta$. Consider ω as a closed L-valued (1,1)-form on M. Its Bott–Chern class $[\omega] \in H^{1,1}_{\partial\overline{\partial}}(M, L)$ is called **the Bott–Chern class of the LCK-manifold**.

Clearly, the vanishing of $[\omega] \in H^{1,1}_{\partial\overline{\partial}}(M, L)$ implies the existence of an automorphic potential for M. Hence:

Proposition 4.5. *If the Bott–Chern class of an LCK-manifold M vanishes and the monodromy of L is $\mathbb{Z}$, then M is LCK with potential.*

Directly from the definition we have:

Proposition 4.6. *Let M be a complex manifold, and ω_1 and ω_2 LCK-metrics having the same Lee form θ. Then the following conditions are equivalent:*

(i) *The Bott–Chern classes of ω_1, ω_2 are equal.*
(ii) *The LCK-structures ω_1 and ω_2 are equivalent up to a potential.*

We obtain a remarkable analogy between the Kähler manifolds and LCK-manifolds. A Kähler structure on a complex manifold is determined by a Kähler class in $H^{1,1}(M)$ and a choice of a Kähler metric in this Kähler class. The latter is obtained by choosing an element in a cone locally modeled on $C^\infty(M)/\,\mathrm{const}$.

An LCK-structure on a given complex manifold with a prescribed conformal structure is determined by a Bott–Chern class and a choice of an LCK-metric with a prescribed Bott–Chern class. A metric with prescribed Bott–Chern class metric is obtained by choosing an element in a cone locally modeled on $C^\infty(\widetilde{M})_\chi/\ker(\partial\overline{\partial})$ (see Remark 4.2).

4.3. Bott–Chern classes and Morse–Novikov cohomology

The holomorphic cohomology of a bundle can be realized as cohomology of a complex

$$C^\infty(L) \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M, L) \xrightarrow{\overline{\partial}} \Lambda^{0,2}(M, L) \xrightarrow{\overline{\partial}} \cdots \tag{4.3}$$

If L is equipped with a flat connection, $\partial : \Lambda^{0,1}(M, L) \longrightarrow \Lambda^{1,1}(M, L)$ induces a map

$$H^1(\mathcal{L}) \xrightarrow{\partial} H^{1,1}_{\partial\overline{\partial}}(M, L) \tag{4.4}$$

from the holomorphic cohomology of the underlying holomorphic bundle (denoted as $\mathcal{L}$) to the Bott–Chern cohomology. The complex

$$C^\infty(L) \xrightarrow{\nabla^{1,0}} \Lambda^{1,0}(M, L) \xrightarrow{\nabla^{1,0}} \Lambda^{2,0}(M, L) \xrightarrow{\nabla^{1,0}} \cdots \tag{4.5}$$

computes the holomorphic cohomology of a bundle $\mathcal{L}'$ with a holomorphic structure defined by the complex conjugate of the $\nabla^{1,0}$-part of the connection. When the bundle L is real, we have $\mathcal{L} \cong \mathcal{L}'$. Then the cohomology of the complex (4.5) is naturally identified with $\overline{H^*(\mathcal{L})}$. The map $\overline{\partial} : \Lambda^{1,0}(M, L) \longrightarrow \Lambda^{1,1}(M, L)$ defines a homomorphism

$$\overline{H^1(\mathcal{L})} \xrightarrow{\overline{\partial}} H^{1,1}_{\partial\overline{\partial}}(M, L) \tag{4.6}$$

which is entirely similar to (4.4).

The following result allows one to compute the Bott–Chern classes in terms of holomorphic cohomology and Morse–Novikov cohomology.

Theorem 4.7. *Let M be a complex manifold, $L_{\mathbb{R}}$ a trivial real line bundle with flat connection $d - \theta$, where θ is a real closed 1-form. Denote by L its complexification, and let $\mathcal{L}$ be the underlying holomorphic bundle. Then there is an exact sequence*

$$H^1(\mathcal{L}) \oplus \overline{H^1(\mathcal{L})} \xrightarrow{\partial+\overline{\partial}} H^{1,1}_{\partial\overline{\partial}}(M, L) \xrightarrow{\nu} H^2_\theta(M) \tag{4.7}$$

where $H^2_\theta(M)$ is the Morse–Novikov cohomology, ν a tautological map, and the first arrow is obtained as a direct sum of (4.4) and (4.6).

Proof. If a $(1, 1)$-form η with coefficients in L is Morse–Novikov cohomologous to zero, we have $\eta = d_\theta\alpha$, where $d_\theta = d - \theta$ is the corresponding differential. Taking $(1, 0)$ and $(0, 1)$-parts, we obtain that $\eta = \overline{\partial}_\theta\alpha^{1,0} + \partial_\theta\alpha^{0,1}$, where $\partial_\theta = \partial - \theta^{1,0}$ and $\overline{\partial}_\theta = \partial - \theta^{0,1}$ are the Hodge components of d_θ. However, these operators are precisely those that are used to define the first arrow in (4.7). Moreover, since η has type $(1, 1)$,

$$\partial_\theta\alpha^{1,0} = 0, \quad \text{and} \quad \overline{\partial}_\theta\alpha^{0,1} = 0.$$

Since $\alpha^{0,1}$ and $\alpha^{1,0}$ are closed under the respective differentials, they represent classes in the cohomology: $[\alpha^{0,1}] \in H^1(\mathcal{L})$ and $[\alpha^{1,0}] \in \overline{H^1(\mathcal{L})}$ Now the Bott–Chern class of of η is obtained as $\partial[\alpha^{0,1}] + \overline{\partial}[\alpha^{1,0}]$, hence the sequence (4.7) is exact. We proved Theorem 4.7. ■

Proposition 4.8. *Let (M, ω, θ) be a compact LCK-manifold, L the corresponding flat line bundle, and $\mathcal{L}$ the underlying holomorphic bundle. Assume that $H^1(M, \mathcal{L}) = 0$, and $H^2_\theta(M) = 0$. Then $H^{1,1}_{\partial\overline{\partial}}(M, L) = 0$, and the Kähler form $\widetilde{\omega}$ on the covering of M admits an automorphic potential.*

Proof. It follows immediately from Theorem 4.7. ■

Remark 4.9. The vanishing of the Bott–Chern class is hard to control. Hence the utility of Proposition 4.8 which reduces the analysis to the more manageable Morse–Novikov cohomology of M and holomorphic cohomology of L.

5. The Bott–Chern class and deformations

We now want to determine the influence of the vanishing of the Bott–Chern class of a compact LCK manifold. To this end, we need some preliminaries about the automorphic potentials on the covering.

Let φ be an automorphic potential function on $\widetilde{M}$. As such, it can be thought of as a section of L. This means that φ can be viewed as a function defined on M, which becomes important in problems of approximations, as M is compact and $\widetilde{M}$ generally not. Moreover, as on $\widetilde{M}$ one has $\theta = d \log \varphi$, the potential is uniquely determined, up to a constant, by θ.

Before stating the main result of this section, we prove a simple technical result which was known to be true for Vaisman manifolds:

Lemma 5.1. *Let M be a LCK manifold and $\widetilde{M}$ a Kähler covering on which the Lee form is exact. Suppose the Kähler form of $\widetilde{M}$ admits an automorphic potential. Then the LCK-form of M is conformally equivalent to an LCK-form*

$$\omega = -d^c\theta + \theta \wedge I(\theta), \tag{5.1}$$

where $-\theta$ is the Lee form of ω, $d\omega = \omega \wedge \theta$.

Proof. If φ is the potential on the Kähler covering $\widetilde{M}$, the Kähler form $\widetilde{\omega}$ on $\widetilde{M}$ satisfies $\widetilde{\omega} = dd^c\varphi$. Then $\omega := \frac{dd^c\varphi}{\varphi}$ is an LCK-form on M conformally equivalent to $\widetilde{\omega}$, and the corresponding Lee form can be found from $d\omega = \omega \wedge d \log \varphi$, giving $\theta = d \log \varphi$. Therefore,

$$\begin{aligned} \omega &= \frac{dd^c\varphi}{\varphi} = -\frac{d^c d\varphi}{\varphi} \\ &= -d^c\left(\frac{d\varphi}{\varphi}\right) + \frac{d\varphi \wedge d^c\varphi}{\varphi^2} \end{aligned}$$

$$= -d^c d\log\varphi + d\log\varphi \wedge \frac{d^c\varphi}{\varphi}$$
$$= -d^c\theta + \theta \wedge I(\theta). \quad \blacksquare$$

Proposition 5.2. *Let M be a compact LCK manifold such that the Kähler form on a Kähler covering admits an automorphic potential, and let θ be its Lee form. Let θ' be any closed 1-form sufficiently close to θ in the metric*

$$|\theta - \theta'|_{L_1^\infty} = \sup_M |\theta - \theta'| + \sup_M |\nabla\theta - \nabla\theta'|,$$

and let φ' be defined from $\theta' = d\log\varphi'$. Then $\omega' := \sqrt{-1}\partial_{\theta'}\overline{\partial}_{\theta'}\varphi'$ is also positive.

Proof. Note that in this statement the monodromy of the cover is not assumed to be $\mathbb{Z}$.

Deforming θ to θ' changes the monodromy of the cover. Let the new monodromy be Γ'. By definition, $\theta' = d\log\varphi'$ is Γ'-invariant, hence $a^*\log\varphi' = \log\varphi' + c$ for any $a \in \Gamma'$ (here c is a real constant). This implies $a^*\varphi' = \exp(c)\varphi'$, and hence φ' is Γ'–automorphic. As such, φ' is a trivialization of L.

Now $\sqrt{-1}\partial_{\theta'}\overline{\partial}_{\theta'}\varphi' > 0$ is equivalent to $d_{\theta'}d^c_{\theta'}\varphi' > 0$. As a trivialization, φ' is constant, hence the above inequality is equivalent on M with $dd^c\varphi' > 0$. Now $dd^c\varphi' = \varphi'\omega'$ and from Lemma 5.1 this is equal to $-d^c\theta' + \theta' \wedge I(\theta')$, which is positive because $-d^c\theta + \theta \wedge I(\theta) > 0$ and θ' is close to θ. $\blacksquare$

Corollary 5.3. *Any compact LCK manifold with vanishing Bott–Chern class admits an LCK metric with potential, in the sense of Definition 2.6 (hence, if $\dim_{\mathbb{C}} M \geqslant 3$, it is embeddable in a Hopf manifold).*

Proof. The vanishing of the Bott–Chern class of M assures the existence of a potential on a Kähler covering $\widetilde{M} \longrightarrow M$. The weight bundle L is associated to the monodromy of this covering and the monodromy can be a priori a subgroup of $(\mathbb{R}^{>0}, \cdot) \cong (\mathbb{R}, +)$, which is not necessarily discrete. Considering L as a trivial line bundle with connection $\nabla_{\mathrm{triv}} - \theta$, we deform L by adding a small term to θ to obtain a bundle L' with monodromy $\mathbb{Z}$.

This is possible to do as follows. A local system on M is defined by a group homomorphism $H_1(M, \mathbb{Z}) \longrightarrow \mathbb{R}$, and its monodromy is $\mathbb{Z}$ if this map is rational. However, each real homomorphism from $H_1(M, \mathbb{Z})$ can be approximated by a rational one. This allows one to deform L into L' with integer monodromy.

But deforming the monodromy amounts at deforming the 1-form θ and this, as follows from the formula $\theta = d\log\varphi$, gives a corresponding deformation of the potential φ. Hence we deform the pair (L, φ) to a pair (L', φ') in which φ' is automorphic function on $\widetilde{M}$, with monodromy determined by L'. By Proposition 5.2, φ' is plurisubharmonic, if θ' is sufficiently close to θ. By construction, L' has monodromy $\mathbb{Z}$, hence φ' defines an LCK-metric with potential, in the sense of Definition 2.6. $\blacksquare$

6. LCK-structures on compact Vaisman manifolds

Given a closed form η on a manifold M with an action of a connected compact group G, we can average this form over the group action, obtaining a closed form η_G. If η is exact, η_G is also exact. Since G is connected, it acts trivially on cohomology. Therefore, the form η is cohomology equivalent to η_G:

$$[\eta] = [\eta_G]. \tag{6.1}$$

The same is true for d_θ-closed forms, if the form θ is G-invariant. Indeed, the action of G maps d_θ-closed and d_θ-exact forms to d_θ-closed and d_θ-exact forms, and acts trivially on the cohomology of the local system defined by θ. Therefore, averaging a d_θ-closed form η, we obtain a d_θ-closed form η_G, which is cohomology equivalent to η.

If η is symplectic, the form η_G is not necessarily symplectic (this form can be degenerate). However, if (M, η) is a Hermitian manifold, and G preserves the complex structure on M, the form η_G is positive definite, because it is an average of positive definite forms over a compact group. In particular, if (M, η) is Kähler, then the form η_G is a Kähler form too.

By this argument, it follows that the average of an LCK-form ω over the action of a compact group G is again an LCK-form, assuming that the Lee form of ω is G-invariant.

Theorem 6.1. *Let (M, J) be a compact complex manifold endowed with a Vaisman structure with 2-form ω and Lee form θ, $\dim_{\mathbb{C}} M \geqslant 3$. Let ω_1 be another LCK-form (not necessarily Vaisman) on (M, J), and θ_1 its Lee form. Then θ_1 is cohomologous with the Lee form of a Vaisman metric, and the Morse–Novikov class of ω_1 vanishes.*

Proof. Denote by ρ the Lee flow corresponding to the Vaisman structure ω on M. It is well known that ρ can be chosen with compact leaves, giving an action $\rho : S^1 \longrightarrow \mathrm{Aut}(M)$ (see [19]). Let θ_1 be the Lee form of ω_1.

For any closed form θ_1' in the same cohomology class as θ, we can find an LCK-form conformally equivalent to ω_1, with the Lee form equal to θ_1'. Averaging θ_1 over ρ, we find a 1-form which is ρ-invariant and has the same cohomology class, by (6.1). Hence, replacing ω_1 by a conformally equivalent form, we may assume from the beginning that θ_1 is ρ-invariant.

For any $t \in S^1$, $\rho(t)^*\omega_1$ satisfies

$$d(\rho(t)^*\omega_1) = \rho(t)^*\omega_1 \wedge \rho(t)^*\theta_1 = \rho(t)^*\omega_1 \wedge \theta_1. \tag{6.2}$$

Averaging ω_1 over S^1 and applying (6.2), we find an S^1-invariant Hermitian form ω_1' which satisfies

$$d\omega_1' = \omega_1' \wedge \theta_1.$$

Since the Morse–Novikov class takes values in a cohomology group which is S^1-invariant, it does not change under averaging, and ω_1' has the same Morse–Novikov class as ω_1. Therefore, we may also assume that ω_1 is ρ-invariant. Denote by G_0 the closure of the group of holomorphic and conformal automorphisms of M generated by $I(\theta^\sharp)$. G_0 is compact, as a closed subgroup in the compact group of conformalities of $(M, [g])$ (see also [12]) and commutative because the group generated by $I(\theta^\sharp)$ is such and taking limits preserves commutativity. Repeating the same procedure as above, we may assume that θ_1 and ω_1 are G_0-invariant.

Let now $\widetilde{M} \xrightarrow{\pi} M$ be a connected covering of M on which the pullback of θ_1 is exact, hence $\widetilde{M}$ is globally conformal Kähler. Let also $\widetilde{\rho} : \mathbb{R} \longrightarrow \mathrm{Aut}(\widetilde{M})$ be the holomorphic flow obtained by lifting ρ to $\widetilde{M}$.

Denote by $\widetilde{\omega}_1$ a Kähler form on $\widetilde{M}$ globally conformal to the lift of ω_1. For all $t \in \mathbb{R}$, the form $\widetilde{\rho}(t)^*\widetilde{\omega}_1$ is a Kähler form, conformally equivalent to ω_1. Since $\dim_{\mathbb{C}} M > 2$ and $\widetilde{M}$ is connected, the conformal factor χ is a constant (indeed, in general, if α, α' are closed conformal 2-forms, α non-degenerate, and $\alpha' = f\alpha$, then $df \wedge \alpha = 0$ implies $df = 0$). In [12] it was shown that if $\theta^\sharp$ and $I(\theta^\sharp)$ act conformally and holomorphically on an LCK-manifold, and $\theta^\sharp$ cannot be lifted to an isometry of $\widetilde{M}$, then M is Vaisman. Unless the conformal factor χ is 1, we may apply this result and obtain that (M, ω_1) is Vaisman. It remains to prove Theorem 6.1 assuming that $\widetilde{\omega}_1$ is $\widetilde{\rho}$-invariant.

Let $\varphi : \widetilde{M} \longrightarrow \mathbb{R}$ be a function defined by $\widetilde{\omega}_1 = \varphi\pi^*\omega_1$. Clearly, $\pi^*\theta_1 = d\log\varphi$. Since φ is $\widetilde{\rho}$-invariant, we have

$$\mathrm{Lie}_{\widetilde{v}} \log\varphi = \langle d\log\varphi, \widetilde{v}\rangle = 0, \tag{6.3}$$

where $\widetilde{v}$ is the vector field generating the flow $\widetilde{\rho}$.

From (6.3) it follows that $\langle\theta_1, v\rangle = 0$, where v is the Lee field of the Vaisman structure ω. As θ_1 is invariant, this means that θ_1 is a *basic form* associated with the foliation ρ (see [24] for a definition and fundamental properties of basic forms with respect to foliations). Now, a basic 1-form on a Vaisman manifold is cohomologous to a sum of a holomorphic and antiholomorphic basic forms: this follows by applying the Hodge decomposition to basic forms with respect to a transversally Kähler foliation. Then $d^c\theta_1 = 0$. This implies

$$dd^c\omega_1^{n-1} = (n-1)^2\theta_1 \wedge I(\theta_1) \wedge \omega_1^{n-1} \tag{6.4}$$

where $n = \dim_{\mathbb{C}} M$. The form (6.4) is manifestly positive, and strictly positive unless θ_1 is identically zero. Since

$$0 = \int_M dd^c\omega_1^{n-1} = \int_M (n-1)^2\theta_1 \wedge I(\theta_1) \wedge \omega_1^{n-1},$$

the form $\theta_1 \wedge I(\theta_1) \wedge \omega_1^{n-1}$ vanishes everywhere, hence θ_1 is zero. Then (M, ω_1) is Kähler. This is a contradiction, since a compact Kähler manifold cannot support a Vaisman structure as compact Vaisman manifolds have odd first Betti number [10,4]. We proved Theorem 6.1. ■

Remark 6.2. Notice that Theorem 6.1 implies, in particular, that the weight bundle of any LCK-structure ω_1 on a Vaisman manifold has monodromy $\mathbb{Z}$. Indeed, M has the same monodromy as a Vaisman manifold, and for Vaisman manifold the weight bundle has monodromy $\mathbb{Z}$, as follows from [18].

It is interesting to determine all Bott–Chern classes realized by an LCK-form on a Vaisman manifold. In this regard, we state:

Conjecture 6.3. *Let M be a Vaisman manifold, equipped with an additional LCK-form ω_1 (not necessarily Vaisman). Then the Bott–Chern class of ω_1 vanishes; equivalently, ω_1 is an LCK-structure with potential.*

Acknowledgements

Liviu Ornea thanks the University of Glasgow, the Independent University and the Steklov Institute in Moscow for hospitality during part of the work at this paper.

Liviu Ornea is partially supported by grant 2-CEx-06-11-22/25.07.2006.

References

[1] L.C. de Andrés, L.A. Cordero, M. Fernández, J.J. Mencía, Examples of four-dimensional compact locally conformal Kähler solvmanifolds, Geom. Dedicata 29 (1989) 227–232.
[2] A. Banyaga, Examples of non d_ω-exact locally conformal symplectic forms, J. Geom. 87 (2007) 1–13. arxiv:math/0308167.
[3] F.A. Belgun, On the metric structure of non-Kähler complex surfaces, Math. Ann. 317 (2000) 1–40.
[4] S. Dragomir, L. Ornea, Locally conformal Kähler geometry, in: Progress in Math., vol. 155, Birkhäuser, Boston, Basel, 1998.
[5] P. Gauduchon, L. Ornea, Locally conformally Kähler metrics on Hopf surfaces, Ann. Inst. Fourier 48 (1998) 1107–1127.
[6] F. Guédira, A. Lichnerowicz, Géométrie des algèbres locales de Kirillov, J. Math. Pures Appl. 63 (1984) 407–483.
[7] R. Hartshorne, Algebraic Geometry, GTM, vol. 52, Springer, 1977.

[8] M. Inoue, On surfaces of class VII_0, Invent Math. 24 (1974) 269–310.
[9] M. de León, B. López, J.C. Marrero, E. Padrón, On the computation of the Lichnerowicz–Jacobi cohomology, J. Geom. Phys. 44 (2003) 507–522.
[10] T. Kashiwada, S. Sato, On harmonic forms on compact locally conformal Kähler manifolds with parallel Lee form, Ann. Fac. Sci. Kinshasa, Zaire 6 (1980) 17–29.
[11] Y. Kamishima, Note on locally conformal Kähler surfaces, Geom. Dedicata 84 (2001) 115–124.
[12] Y. Kamishima, L. Ornea, Geometric flow on compact locally conformally Kahler manifolds, Tohoku Math. J. 57 (2005) 201–221. arxiv:math/0105040.
[13] A. Lichnerowicz, Les variétés de Poisson et leurs algèbres de Lie associées, J. Differential Geom. 12 (2) (1977) 253–300.
[14] D.V. Millionshchikov, Cohomology of solvmanifolds with local coefficients and problems in the Morse–Novikov theory, Russian Math. Surveys 57 (2002) 813–814. arXiv:math/0203067.
[15] S.P. Novikov, The Hamiltonian formalism and a multivalued analogue of Morse theory (Russian), Uspekhi Mat. Nauk 37 (1982) 3–49.
[16] K. Oeljeklaus, M. Toma, Non-Kähler compact complex manifolds associated to number fields, Ann. Inst. Fourier 55 (2005) 1291–1300.
[17] L. Ornea, Locally conformal Kähler manifolds, A Selection of Results, in: Lect. Notes Semin. Interdiscip. Mat., vol. IV, S.I.M. Dep. Mat. Univ. Basilicata, Potenza, 2005. arxiv:math/0411503.
[18] L. Ornea, M. Verbitsky, Structure theorem for compact Vaisman manifolds, Math. Res. Lett. 10 (2003) 799–805. arxiv:math/0305259.
[19] L. Ornea, M. Verbitsky, Immersion theorem for compact Vaisman manifolds, Math. Ann. 332 (2005) 121–143. arXiv:math/0306077.
[20] L. Ornea, M. Verbitsky, Locally conformal Kähler manifolds with potential, Math. Ann. (in press). arXiv:math/0407231.
[21] A.V Pazhitnov, Exactness of Novikov-type inequalities for the case $\pi_1(M) = \mathbb{Z}^m$ and for Morse forms whose cohomology classes are in general position, Soviet Math. Dokl. 39 (3) (1989) 528–532.
[22] A. Ranicki, Circle valued Morse theory and Novikov homology, in: Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001), pp. 539–569, in: ICTP Lect. Notes, 9, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002.
[23] A. Teleman, The pseudo-effective cone of a non-Kählerian surface and applications, Math. Ann. 335 (4) (2006) 965–989.
[24] P. Tondeur, Foliations on Riemannian manifolds, Universitext, Springer-Verlag, 1988.
[25] F. Tricerri, Some examples of locally conformal Kähler manifolds, Rend. Sem. Mat. Univ. Politec. Torino 40 (1982) 81–92.
[26] I. Vaisman, Remarkable operators and commutation formulas on locally conformal Kähler manifolds, Compos. Math. 40 (1980) 227–259.
[27] M. Verbitsky, Theorems on the vanishing of cohomology for locally conformally hyper-Kähler manifolds, Proc. Steklov Inst. Math. 246 (2004) 54–78. arXiv:math/0302219.
[28] M. Verbitsky, Stable bundles on positive principal elliptic fibrations, Math. Res. Lett. 12 (2005) 251–264. math.AG/0403430.