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Potential 1-forms for hyper-Kähler structures with torsion

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Abstract

It is shown that an HKT space with closed parallel potential 1-form has $D(2, 1; -1)$ symmetry. Every locally conformally hyper-Kähler manifold generates this type of geometry. The HKT spaces with closed parallel potential 1-form arising in this way are characterized by their symmetries and an inhomogeneous cubic condition on their torsion.

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Introduction

HKT geometry is a metric geometry with multiple complex structures which arises in various physical theories, including supersymmetric nonlinear sigma models, type IIA string theory, and black hole moduli. Good references for the physical background are [5] and [8] and the citations therein. For a mathematical approach, we refer the reader to [4]. Since the geometry is typically hyper-Hermitian and non-Kählerian, it is of great interest and challenging to find potential functions [8].

In the context of multi-particle quantum mechanics, Michelson and Strominger studied the phenomenon of superconformal symmetry. Motivated by application to dynamics of black holes [9], they demonstrated in [8] a relation between a $D(2, 1; \alpha)$ superconformal symmetry and classical differential geometry on HKT manifolds. Given supersymmetry such as this, potential functions are already found [12, 13].

On the other hand, a maximum principle argument shows that potential functions could not exist on compact manifolds [4]. We therefore replace locally defined potential functions by a globally defined closed 1-form in our consideration (see definition 4). We focus on the case when the potential 1-form is parallel with respect to the HKT connection in this investigation. Combining corollary 9 and proposition 11, we obtain the following result in this direction.

If V is the dual vector field of a closed parallel potential 1-form θ of the HKT space with metric $\hat{g}$ and hypercomplex structures I_1, I_2, I_3, then

$$d\theta = 0, \qquad \mathcal{L}_V \hat{g} = 0, \qquad \mathcal{L}_{I_r V} \hat{g} = 0, \qquad \mathcal{L}_{I_r V} I_s = \epsilon^{rst} I_t.$$

Conversely, if there is such a vector field on an HKT space, then the dual 1-form is a parallel potential function.

Due to the theorem of Michelson and Strominger [8], this type of symmetry is a degenerate version of $D(2, 1; \alpha)$ symmetry, namely $D(2, 1; -1)$. Since the above symmetry makes sense on the HKT space, we shall refer to it as $D(2, 1; -1)$ symmetry in this paper despite an apparent singularity that occurs in the structural equations [8, (3.44)]. Kac [7, proposition 2.5.4] shows that for $\alpha \neq -1, 0, \infty$, the superalgebras $D(2, 1; \alpha)$ are simple. On the other hand the algebras $D(2, 1; -1)$, $D(2, 1; 0)$ and $D(2, 1; \infty)$ have a decoupled SU(2) and share many features. In this paper, we interpret $D(2, 1; -1)$ symmetry after Michelson and Strominger's theorem [8, (3.56)]. A precise description is given in definition 10. Through a construction, we shall prove the following observation.

If (M, g, I_1, I_2, I_3) is a locally conformally hyper-Kähler manifold whose Lee form is parallel with respect to the Levi-Civita connection, then there exists an HKT metric $\hat{g}$ such that the Lee form of g is a potential 1-form of $\hat{g}$, and is parallel with respect to the HKT connection of $\hat{g}$.

As a result, in potential theory, the above observation supplements what is already known for HKT spaces with $D(2, 1; \alpha)$ symmetry when $\alpha \neq -1, 0, \infty$. From a geometric perspective, it implicitly links HKT geometry to Weyl geometry, quarternionic geometry and Sasakian geometry through the theory of locally conformally hyper-Kähler manifolds.

We conclude with a discussion on how to distinguish the class of HKT spaces associated with locally conformally hyper-Kähler manifolds.

Throughout this paper we adopt the conventions in [1, 3]. Here we warn casual readers that the metrics concerned for locally conformally hyper-Kähler structure and its associated HKT structure are in different conformal classes.

1. HKT-manifolds

A Hermitian structure on a smooth manifold M consists of a Riemannian metric $\hat{g}$ and an integrable complex structure J such that for any tangent vectors X and Y on the manifold M,

$$\hat{g}(JX, JY) = \hat{g}(X, Y).$$

A triple of integrable complex structure I_r, $r = 1, 2, 3$, forms a hypercomplex structure on the manifold M if they satisfy the quaternion relations:

$$I_1^2 = I_2^2 = I_3^2 = -I, \qquad I_1 I_2 = I_3 = -I_2 I_1.$$

If each complex structure I_r with the metric $\hat{g}$ forms a Hermitian structure, then $(M, \hat{g}, I_1, I_2, I_3)$ is said to be a hyper-Hermitian manifold.

We denote by $\hat{F}_r$ the fundamental 2-form associated with the complex structure I_r and we observe the convention

$$\hat{F}_r(X, Y) = \hat{g}(I_r X, Y).$$

For a k-form ω let

$$(I_r\omega)(X_1, \ldots, X_k) = (-1)^k \omega(I_r X_1, \ldots, I_r X_k). \tag{1}$$

The complex operators d_r, ∂_r and $\bar{\partial}_r$ are respectively defined as

$$d_r\omega = (-1)^k I_r\, dI_r\omega \quad \text{for a } k\text{-form } \omega, \qquad \partial_r = \tfrac{1}{2}(d + \mathrm{i}d_r), \quad \bar{\partial}_r = \tfrac{1}{2}(d - \mathrm{i}d_r).$$

Definition 1. *A linear connection D with torsion tensor T^D on M is called hyper-Kähler with torsion if*

(i) it is hyper-Hermitian: $DI_1 = DI_2 = DI_3 = 0$, $Dg = 0$ and
(ii) the tensor field c defined by $c(X, Y, Z) = \hat{g}(T^D(X, Y), Z)$ is a 3-form.

Such a connection is denoted as HKT by physicists [5, 8] and we shall preserve this name. Among mathematicians, the HKT connection is also known as a Bismut connection for each of the complex structures I_r [3]. Using the characterization of the Bismut connection and the fact that it is uniquely associated with a Hermitian structure, one obtains the following equivalent observation [4, 5]:

Proposition 2. *On any hyper-Hermitian manifold $(M, \hat{g}, I_1, I_2, I_3)$, the following two conditions are equivalent:*

(i) $d_1\hat{F}_1 = d_2\hat{F}_2 = d_3\hat{F}_3$.
(ii) $\partial_1(\hat{F}_2 + \mathrm{i}\hat{F}_3) = 0$.

An HKT-connection exists if and only if one of the above two conditions is satisfied. When it exists, it is unique.

As demonstrated in [8], an efficient way for constructing examples of HKT structures is the use of HKT potentials. These are generalizations of hyper-Kähler potentials [4].

Definition 3. *Let $(M, \hat{g}, I_1, I_2, I_3)$ be an HKT manifold. A (possibly locally defined) function $\mu : U \subseteq M \to \mathbb{R}$ is a* potential function *for the HKT structure if*

$$\hat{F}_1 = \tfrac{1}{2}(dd_1 + d_2d_3)\mu, \qquad \hat{F}_2 = \tfrac{1}{2}(dd_2 + d_3d_1)\mu, \qquad \hat{F}_3 = \tfrac{1}{2}(dd_3 + d_1d_2)\mu. \tag{2}$$

Alternatively, the potential function μ is characterized by

$$\hat{F}_2 + \mathrm{i}\hat{F}_3 = 2\partial_1 I_2 \bar{\partial}_1 \mu. \tag{3}$$

Potential functions do not always exist. When one exists, the torsion form of an HKT structure deriving from a potential μ is

$$c = -\tfrac{1}{2}d_1d_2d_3\mu = -d_1\hat{F}_1 = -d_2\hat{F}_2 = -d_3\hat{F}_3.$$

As an example, the function $\log \sum_i |z_i|^2$ is an HKT potential on $\mathbb{C}^{2n}\backslash\{0\}$. Moreover, it descends locally to the Hopf manifold $S^1 \times S^{4n-1}$.

This should be noted that like Kähler potentials, HKT potentials could not exist globally on compact manifolds due to a typical maximum principle argument [4]. Moreover, a generic HKT manifold is non-Kählerian and the $\partial\bar{\partial}$-lemma is not applicable. Therefore, we propose to develop a global version of potential theory through the Poincaré lemma for 1-forms.

Definition 4. *A 1-form ω is a potential 1-form for an HKT manifold $(M, \hat{g}, I_1, I_2, I_3)$ if the fundamental 2-forms are given by*

$$\hat{F}_1 = \tfrac{1}{2}(d\omega_1 + d_2\omega_3), \qquad \hat{F}_2 = \tfrac{1}{2}(d\omega_2 + d_3\omega_1), \qquad \hat{F}_3 = \tfrac{1}{2}(d\omega_3 + d_1\omega_2), \tag{4}$$

where $\omega_r := I_r\omega$. A potential 1-form is closed if $d\omega = 0$.

In such terminology, the HKT structure on Hopf manifolds has a globally defined potential 1-form. Implicitly, Poincaré lemma provides the locally defined potential functions whenever a potential 1-form exists and is closed. Moreover, the torsion 3-form is now given by

$$c = -\tfrac{1}{2}d_1d_2\omega_3 = -\tfrac{1}{2}d_2d_3\omega_1 = -\tfrac{1}{2}d_3d_1\omega_2. \tag{5}$$

2. Parallel potential forms

In this section, we analyse the structure of HKT spaces with parallel potential 1-forms. Since HKT connections are Riemannian connections, vector fields dual to parallel potential forms are parallel. Therefore, briefly, we extend our investigation to parallel vector fields in general before focusing again on potential 1-forms and their dual vector fields.

Lemma 5. *Let V be a vector field on an HKT space. The following statements are equivalent:*

(i) *V is parallel with respect to the HKT connection D.*
(ii) *V, I_1V, I_2V, I_3V are parallel with respect to the HKT connection D.*
(iii) *V, I_1V, I_2V, I_3V are Killing vector fields with respect to the HKT metric.*

Proof. Since the HKT connection preserves the hypercomplex structure, the equivalence between the first two statements is obvious.

For any vector fields W, Y, Z, as D is a metric connection, we have the identity

$$\begin{aligned}\mathcal{L}_W\hat{g}(Y,Z) &= \hat{g}(D_YW,Z)+\hat{g}(Y,D_ZW)+\hat{g}(T^D(W,Y),Z)+\hat{g}(Y,T^D(W,Z))\\ &= \hat{g}(D_YW,Z)+\hat{g}(Y,D_ZW)+c(W,Y,Z)+c(Y,W,Z).\end{aligned}$$

Since c is totally skew, we have

$$\mathcal{L}_W\hat{g}(Y,Z) = \hat{g}(D_YW,Z)+\hat{g}(Y,D_ZW). \tag{6}$$

Applying this identity to the vector fields V, I_1V, I_2V, I_3V and using the fact that the HKT connection preserves the hypercomplex structure, we derive the implication from the second statement to the third.

Conversely, if the vector fields V, I_1V, I_2V, I_3V are Killing, we apply the above identity to V to conclude that the symmetric part of DV is equal to zero. Let β be the skew-symmetric part of DV, i.e., $DV=\beta$. Since the connection preserves the complex structures, the above identity is equivalent to

$$\hat{g}(D_Y(I_rV),Z)=\hat{g}(I_rD_YV,Z)=-\beta(Y,I_rZ). \tag{7}$$

On the other hand, as the vector fields are Killing,

$$\hat{g}(D_Y(I_rV),Z)+\hat{g}(D_Z(I_rV),Y)=\left(\mathcal{L}_{(I_rV)}\hat{g}\right)(Y,Z)=0. \tag{8}$$

Therefore, $\beta(Y,I_rZ)+\beta(Z,I_rY)=0$. Then

$$\begin{aligned}\beta(Y,I_1Z) &= -\beta(Z,I_1Y)=\beta(I_1Y,Z)=\beta(I_2I_3Y,Z)\\ &= \beta(I_3Y,I_2Z)=\beta(Y,I_3I_2Z)=-\beta(Y,I_1Z).\end{aligned}$$

Therefore, $\beta=0$. This implies that $DV=0$. □

Lemma 6. *Suppose that V is a parallel vector field with respect to the HKT connection D. Let $\hat{\theta}$ be its dual 1-form with respect to $\hat{g}$. Then*

$$d\hat{\theta}=\iota_Vc, \qquad d\hat{\theta}_r=\iota_{I_rV}c. \tag{9}$$

Proof. Let $0\leqslant m\leqslant 3$. Let I_0 denote the identity endomorphism on tangent space. For any vector fields X and Y,

$$\begin{aligned}d\hat{\theta}_m(X,Y) &= X(\hat{\theta}_m(Y))-Y(\hat{\theta}_m(X))-\hat{\theta}_m([X,Y])\\ &= X(\hat{g}(I_mV,Y))-Y(\hat{g}(I_mV,X))-g(I_mV,[X,Y])\\ &= \hat{g}(I_mV,D_XY)-\hat{g}(I_mV,D_YX)-\hat{g}(I_mV,[X,Y])\\ &= \hat{g}(I_mV,T^D(X,Y))=c(I_mV,X,Y).\end{aligned}$$

□

Lemma 7. *Suppose that V is a parallel vector field with respect to the HKT connection D. It is parallel with respect to the Levi-Civita connection $\hat{\nabla}$ of the metric $\hat{g}$ if and only if $\iota_V c = 0$.*

Proof. This is due to the identity $\hat{g}(\hat{\nabla}_X V, Y) = \hat{g}(D_X V, Y) + c(X, V, Y) = c(X, V, Y)$. □

Next we investigate the behaviour of the vector fields V, I_1V, I_2V, I_3V with respect to the hypercomplex structure $\{I_1, I_2, I_3\}$.

Lemma 8. *If $-2\hat{\theta}$ is a closed potential 1-form and is parallel with respect to the HKT connection, then $\mathcal{L}_V I_r = 0$ and $\mathcal{L}_{I_r V} I_s = \epsilon^{rst} I_t$.*

Proof. Since the vector fields V, I_1V, I_2V, I_3V are Killing vector fields, it suffices to show that $\mathcal{L}_V \hat{F}_r = 0$, and $\mathcal{L}_{I_r V} \hat{F}_s = \epsilon^{rst}\hat{F}_t$.

In the following computation, we use the results in lemma 6 extensively. For any tangent vectors X and Y,

$$\iota_V\, d\hat{F}_r(X, Y) = (\iota_V I_r c)(X, Y) = -c(I_rV, I_rX, I_rY) = -d\hat{\theta}_r(I_rX, I_rY) = -I_r d\hat{\theta}_r(X, Y).$$

On the other hand, $\iota_V \hat{F}_r(X) = \hat{g}(I_rV, X) = \hat{\theta}_r(X)$. Therefore,

$$\mathcal{L}_V \hat{F}_r = \iota_V\, d\hat{F}_r + d\iota_V \hat{F}_r = -I_r d\hat{\theta}_r + d\hat{\theta}_r.$$

As the torsion form is of type (1, 2) + (2, 1) with respect to all I_r,

$$c(Z, X, Y) = c(Z, I_rX, I_rY) + c(I_rZ, X, I_rY) + c(I_rZ, I_rX, Y). \tag{10}$$

Substituting Z by I_rV and applying lemma 6, we have

$$d\hat{\theta}_r(X, Y) = I_r d\hat{\theta}_r(X, Y) - d\hat{\theta}(X, I_rY) - d\hat{\theta}(I_rX, Y).$$

Therefore, $\mathcal{L}_V \hat{F}_r(X, Y) = -d\hat{\theta}(X, I_rY) - d\hat{\theta}(I_rX, Y)$. As $\hat{\theta}$ is closed, $\mathcal{L}_V \hat{F}_r = 0$. Next,

$$\iota_{I_rV} \hat{F}_r(X) = \hat{F}_r(I_rV, X) = \hat{g}\big(I_r^2V, X\big) = -\hat{\theta}(X). \tag{11}$$

With lemma 6, we have

$$\iota_{I_rV}\, d\hat{F}_r(X, Y) = \iota_{I_rV} I_r c(X, Y) = -c\big(I_r^2V, I_rX, I_rY\big) = c(V, I_rX, I_rY) = I_r\, d\hat{\theta}(X, Y). \tag{12}$$

Therefore,

$$\mathcal{L}_{I_rV} \hat{F}_r = \iota_{I_rV}\, d\hat{F}_r + d\iota_{I_rV} \hat{F}_r = I_r\, d\hat{\theta} - d\hat{\theta}. \tag{13}$$

Since $d\hat{\theta} = 0$, $\mathcal{L}_{I_rV} \hat{F}_r = 0$. Finally,

$$\iota_{I_1V} \hat{F}_2(X) = \hat{F}_2(I_1V, X) = \hat{g}(I_2I_1V, X) = -\hat{\theta}_3(X). \tag{14}$$

By lemma 6 and (10),

$$\begin{aligned}
\iota_{I_1V}\, d\hat{F}_2(X, Y) &= \iota_{I_1V} I_2 c(X, Y) = I_2 c(I_1V, X, Y) = c(I_3V, I_2X, I_2Y) \\
&= c(I_3V, I_3I_2X, I_3I_2Y) + c\big(I_3^2V, I_2X, I_3I_2Y\big) + c\big(I_3^2V, I_3I_2X, I_2Y\big) \\
&= c(I_3V, I_1X, I_1Y) + c(V, I_2X, I_1Y) + c(V, I_1X, I_2Y) \\
&= I_1 d\hat{\theta}_3(X, Y) + d\hat{\theta}(I_2X, I_1Y) + d\hat{\theta}(I_1X, I_2Y).
\end{aligned} \tag{15}$$

Therefore,

$$\mathcal{L}_{I_1V} \hat{F}_2(X, Y) = -d\hat{\theta}_3(X, Y) + I_1 d\hat{\theta}_3(X, Y) + d\hat{\theta}(I_2X, I_1Y) + d\hat{\theta}(I_1X, I_2Y). \tag{16}$$

On the other hand, if $-2\hat{\theta}$ is a potential 1-form, then $d\hat{\theta} = 0$. It follows that

$$\mathcal{L}_{I_1V} \hat{F}_2 = -d\hat{\theta}_3 + I_1\, d\hat{\theta}_3.$$

In addition,

$$\hat{F}_3 = \tfrac{1}{2}(d(-2\hat{\theta}_3) + d_1(-2\hat{\theta}_2)) = -d\hat{\theta}_3 + I_1\, dI_1I_2\hat{\theta} = -d\hat{\theta}_3 + I_1\, d\hat{\theta}_3.$$

Therefore, $\mathcal{L}_{I_1V} \hat{F}_2 = \hat{F}_3$. □

Summarizing the results in lemmas 5 and 8 in the context of parallel potential 1-forms, we have the next result.

Corollary 9. *Suppose that $-2\hat{\theta}$ is a closed potential 1-form and parallel with respect to the HKT connection. If V is the dual of $\hat{\theta}$ with respect to the HKT metric $\hat{g}$, then*

$$\mathcal{L}_V \hat{g} = 0, \qquad \mathcal{L}_{I_r V} \hat{g} = 0, \qquad \mathcal{L}_{I_r V} I_s = \epsilon^{rst} I_t. \tag{17}$$

Comparing with [8, (3.56)] and keeping in mind that the dual 1-form $\hat{\theta}$ is closed, we conclude that the HKT space in question is induced by the $D(2, 1; -1)$ supersymmetry. Although such supersymmetry is singular as seen in [8, (3.44)], we retain the notion of $D(2, 1; -1)$ symmetry. To be precise, we make a definition.

Definition 10. *A $D(2, 1; -1)$ symmetry on an HKT space is a vector field V satisfying the conditions in (17) and whose dual 1-form $\hat{\theta}$ is closed.*

In a previous investigation on potential functions [12, 13], such symmetry was not extensively studied due to the degeneracy of supersymmetry. Below is a remedy.

Proposition 11. *Suppose that a vector field V generates a $D(2, 1; -1)$ symmetry on an HKT space. Let $\hat{\theta}$ be the dual vector field. Then $-2\hat{\theta}$ is a parallel potential 1-form. In particular, local potential function exists.*

Proof. By definition, V, I_1V, I_2V, I_3V are Killing vector fields. By lemma 5, V is parallel with respect to the HKT connection. In particular, lemma 6 is applicable. With it, we obtain equation (15). With identity (14), we obtain equation (16). Since $\hat{\theta}$ is closed, $\mathcal{L}_{I_1 V}\hat{F}_2 = -d\hat{\theta}_3 + I_1 d\hat{\theta}_3$. On the other hand, as I_1V is a Killing vector field and $\mathcal{L}_{I_1V} I_2 = I_3$, it follows that $\mathcal{L}_{I_1V}\hat{F}_2 = \hat{F}_3$. Therefore,

$$\hat{F}_3 = -d\hat{\theta}_3 + I_1\, d\hat{\theta}_3 = \tfrac{1}{2}(d(-2\hat{\theta}_3) + d_1(-2\hat{\theta}_2)).$$

The above calculation is repeated with the indices permuted to conclude that $-2\hat{\theta}$ is a potential 1-form. □

Remark. By lemmas 6 and 7, the closedness of $\hat{\theta}$ along with the parallelism of the dual vector field V together implies that the vector field of symmetry is parallel with respect to the Levi-Civita connection of the HKT metric $\hat{g}$. In view of lemma 8, it implies that $\mathcal{L}_V I_r = 0$.

3. Locally conformally hyper-Kähler manifolds

Locally conformally hyper-Kähler manifolds have been studied in relation to Weyl geometry, quaternionic geometry as well as Sasakian geometry [10, 11]. In this section, we demonstrate a way to generate HKT structures with $D(2, 1; -1)$ symmetry and parallel potential 1-form from a locally conformally hyper-Kähler structure. We begin our investigation with a review of definitions.

Definition 12. *(i) A hyper-Hermitian manifold (M, g, I_1, I_2, I_3) is called hyper-Kähler if the Levi-Civita connection of g parallelizes each complex structure I_r: $\nabla I_r = 0$.*

(ii) A hyper-Hermitian manifold (M, g, I_1, I_2, I_3) is called locally conformally hyper-Kähler if there exists an open cover $\{U_i\}$ such that the restriction of the metric to each U_i is conformal to a local hyper-Kähler metric g_i:

$$g|_{U_i} = \mathrm{e}^{f_i} g_i, \qquad f_i \in \mathcal{C}^\infty U_i. \tag{18}$$

We shall focus on the second notion. Taking $\theta|_{U_i} = df_i$, condition (18) is equivalent to the existence of a globally defined 1-form θ satisfying the integrability conditions:

$$dF_r = \theta \wedge F_r, \qquad r = 1, 2, 3. \tag{19}$$

The standard example of locally conformally hyper-Kähler manifold is the Hopf manifold $H^n_{\mathbb{H}} = (\mathbb{H}\backslash\{0\})/\Gamma_2$, where Γ_2 is the cyclic group generated by the quaternionic automorphism $(q_1, \ldots, q_n) \mapsto (2q_1, \ldots, 2q_n)$. The hypercomplex structure of $\mathbb{H}^n$ is easily seen to descend to $H^n_{\mathbb{H}}$. Moreover, the globally conformal hyper-Kähler metric $\left(\sum_i q_i\bar{q}_i\right)^{-1} \sum_i dq_i \otimes d\bar{q}_i$ on $\mathbb{H}^n\backslash\{0\}$ is invariant to the action of Γ_2, hence induces a locally conformally hyper-Kähler metric on the Hopf manifold with Lee form

$$\theta = -\frac{\sum_i (q_i d\bar{q}_i + \bar{q}_i dq_i)}{\sum_i q_i\bar{q}_i}.$$

Note that, as in the complex case, $H^n_{\mathbb{H}}$ is diffeomorphic with a product of spheres $S^1 \times S^{4n-1}$. Consequently, its first Betti number is 1 and it cannot admit any hyper-Kähler metric. Other examples are presented in [10] where also a complete classification of compact homogeneous locally conformally hyper-Kähler manifolds is given.

One should note that locally conformally hyper-Kähler manifolds are hyper-Hermitian Weyl and as such, Einstein–Weyl Ricci-flat (here, the conformal class is that of g and the Weyl connection is constructed out of the Levi-Civita connection of g and the Lee form). Hence, if compact, one applies a well-known result of Gauduchon [2] to obtain the existence of a metric g_0, conformal with g and having the Lee form parallel with respect to the Levi-Civita connection of g_0. The metric we just wrote on the Hopf manifold has this property. Therefore, when working with compact locally conformally hyper-Kähler manifolds, one can always assume the metric with parallel Lee form. We shall need the following computational result [10]:

Lemma 13. *Let (M, g, I_1, I_2, I_3) be a locally conformally hyper-Kähler manifold with parallel Lee form θ. Let $\theta_r = I_r\theta$. Assume that θ has unit length. Then*

$$d\theta_r = \theta \wedge \theta_r - F_r. \tag{20}$$

It should be noted that the unit length condition may be achieved by rescaling g by a homothety and that

$$I_r\, d\theta_r = I_r\theta \wedge I_r\theta_r - I_r F_r = -\theta_r \wedge \theta - F_r = d\theta_r. \tag{21}$$

Also,

$$I_r\, dF_r = I_r\theta \wedge I_r F_r = \theta_r \wedge F_r. \tag{22}$$

That the Hopf manifolds admit HKT structures is not by chance. We can state

Theorem 14. *Let (M, g, I_1, I_2, I_3) be a locally conformally hyper-Kähler manifold with parallel Lee form θ. Assume that θ has unit length. Then the metric*

$$\hat{g} = g - \tfrac{1}{2}\{\theta \otimes \theta + \theta_1 \otimes \theta_1 + \theta_2 \otimes \theta_2 + \theta_3 \otimes \theta_3\} \tag{23}$$

is HKT. Moreover, θ is a closed potential 1-form for $\hat{g}$.

Proof. Let $g_2 = \theta \otimes \theta + \theta_1 \otimes \theta_1 + \theta_2 \otimes \theta_2 + \theta_3 \otimes \theta_3$ be the restriction of the metric g on the quaternionic span of the vector field V. Let g_1 be the restriction of the metric g on the orthogonal complement of the quaternionic span of V. Then the metric g pointwisely and smoothly splits into two parts $g = g_1 + g_2$. Since the norm of θ and its dual vector field V

have unit length with respect to g, the bilinear form $\hat{g}$ is equal to $g_1 + \frac{1}{2}g_2$. In particular, this is a Riemannian metric.

Note first that, due to (1) we have

$$I_r F_r = F_r, \qquad I_r F_s = -F_s \qquad \text{for} \quad r \neq s, \quad I_r\theta_s = \epsilon^{rst}\theta_t. \tag{24}$$

As a matter of convention, for exterior products we use that

$$\alpha_1 \wedge \cdots \wedge \alpha_n(X_1, \ldots, X_n) := \det(\alpha_i(X_j)). \tag{25}$$

In particular, $\theta \wedge \theta_1 = \theta \otimes \theta_1 - \theta_1 \otimes \theta$. From the definitions and (23),

$$\hat{F}_1 = F_1 - \tfrac{1}{2}\{\theta \wedge \theta_1 + \theta_2 \wedge \theta_3\}. \tag{26}$$

Now we have successively, using $d\theta = 0$, $dF_r = \theta \wedge F_r$ and formula (20),

$$\begin{aligned} d\hat{F}_1 &= dF_1 - \tfrac{1}{2}\{d\theta \wedge \theta_1 - \theta \wedge d\theta_1 + d\theta_2 \wedge \theta_3 - \theta_2 \wedge d\theta_3\} \\ &= dF_1 - \tfrac{1}{2}\{-\theta \wedge (\theta \wedge \theta_1 - F_1) + (\theta \wedge \theta_2 - F_2) \wedge \theta_3 - \theta_2 \wedge (\theta \wedge \theta_3 - F_3)\} \\ &= \tfrac{1}{2}\{\theta \wedge F_1 - 2\theta \wedge \theta_2 \wedge \theta_3 + \theta_3 \wedge F_2 - \theta_2 \wedge F_3\}. \end{aligned} \tag{27}$$

$$\begin{aligned} I_1\, d\hat{F}_1 &= \tfrac{1}{2}\{\theta_1 \wedge I_1F_1 - 2\theta_1 \wedge I_1\theta_2 \wedge I_1\theta_3 + I_1\theta_3 \wedge I_1F_2 - I_1\theta_2 \wedge I_1F_3\} \\ &= \tfrac{1}{2}\{\theta_1 \wedge F_1 + \theta_2 \wedge F_2 + \theta_3 \wedge F_3 - 2\theta_1 \wedge \theta_2 \wedge \theta_3\}. \end{aligned} \tag{28}$$

The above formula is symmetric in the indices 1, 2, 3. Due to proposition 2, $\hat{g}$ is an HKT metric.

We prove the assertion on the potential 1-form by demonstrating that any locally defined function f with $df = \theta$ is a potential function:

$$\begin{aligned} &\bar{\partial}_1 f = \tfrac{1}{2}(df - \mathrm{i}I_1\, df) = \tfrac{1}{2}(\theta - \mathrm{i}I_1\theta) = \tfrac{1}{2}(\theta - \mathrm{i}\theta_1), \\ &I_2\bar{\partial}_1 f = \tfrac{1}{2}(I_2\theta - \mathrm{i}I_2\theta_1) = \tfrac{1}{2}(\theta_2 + \mathrm{i}\theta_3), \\ &\partial_1 I_2\bar{\partial}_1 f = \tfrac{1}{4}(d\theta_2 + \mathrm{i}d\theta_3 - \mathrm{i}I_1\, d(I_1\theta_2 + \mathrm{i}I_1\theta_3)) = \tfrac{1}{4}(d\theta_2 + \mathrm{i}d\theta_3 - \mathrm{i}I_1\, d(\theta_3 - \mathrm{i}\theta_2)) \\ &\qquad = \tfrac{1}{4}(\theta \wedge \theta_2 - F_2 + \mathrm{i}(\theta \wedge \theta_3 - F_3) - \mathrm{i}I_1(\theta \wedge \theta_3 - F_3) - I_1(\theta \wedge \theta_2 - F_2)) \\ &\qquad = -\tfrac{1}{2}(F_2 + \mathrm{i}F_3) + \tfrac{1}{4}(\theta + \mathrm{i}\theta_1) \wedge (\theta_2 + \mathrm{i}\theta_3). \end{aligned}$$

On the other hand, $\hat{F}_r = F_r - \frac{1}{2}\{\theta \wedge \theta_r + \theta_s \wedge \theta_t\}$ implies that

$$\hat{F}_2 + \mathrm{i}\hat{F}_3 = F_2 + \mathrm{i}F_3 - \tfrac{1}{2}(\theta + \mathrm{i}\theta_1) \wedge (\theta_2 + \mathrm{i}\theta_3). \tag{29}$$

It shows that the function f_i satisfies the condition in (3). □

Next, we investigate the geometry of the Lee field with respect to the geometry of the HKT metric $\hat{g}$ and its associated HKT connection D. The following result can be found in [11].

Proposition 15. *Let V be the vector field dual to the parallel Lee form with respect to the locally conformally hyper-Kähler metric g, then the algebra $\{V\} \oplus \{I_1V, I_2V, I_3V\}$ is isomorphic to $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$. Moreover,*

$$\mathcal{L}_V I_r = 0, \qquad \mathcal{L}_V g = 0, \qquad \mathcal{L}_{I_rV} g = 0, \qquad \mathcal{L}_{I_rV} I_s = \epsilon^{rst} I_t. \tag{30}$$

To understand the relation between HKT geometry and the Lee field V, we need to describe the behaviour of the Lee field with respect to the forms θ and θ_r.

Lemma 16. *Let V be the Lee field, $\theta_r = I_r\theta$ for $1 \leqslant r \leqslant 3$. Then*

$$\mathcal{L}_V\theta = 0, \qquad \mathcal{L}_V\theta_r = 0, \qquad \mathcal{L}_{I_rV}\theta = 0, \qquad \mathcal{L}_{I_rV}\theta_s = \epsilon^{rst}\theta_t. \tag{31}$$

Proof. The Lee form θ is invariant along its dual vector field because it is parallel with respect to the Levi-Civita connection of the locally conformally hyper-Kähler metric g. The forms θ_r are invariant with respect to the Lee field because the Lee form is invariant and the Lee field is hypercomplex.

Next, for any vector field Y,

$$\begin{aligned}(\mathcal{L}_{I_rV}\theta)Y &= I_rV(\theta(V)) - \theta(\mathcal{L}_{I_rV}Y) = I_rVg(V,Y) - g(V,[I_rV,Y]) \\ &= g(\nabla_{I_rV}V,Y) + g(V,\nabla_{I_rV}Y) - g(V,[I_rV,Y]) = g(V,\nabla_{I_rV}Y - [I_rV,Y]) \\ &= g(V,\nabla_Y(I_rV)) = Yg(V,I_rV) - g(\nabla_YV,I_rV) = 0.\end{aligned}$$

It follows that $\mathcal{L}_{I_rV}\theta = 0$. This equality is combined with $\mathcal{L}_{I_rV}I_s = \epsilon^{rst}I_t$ to yield the last one in this lemma. □

Due to lemma 5, we learn the following.

Theorem 17. *The potential 1-form for the HKT metric $\hat{g}$ is parallel.*

Proof. The tensor $\theta^2+\theta_1^2+\theta_2^2+\theta_3^2$ is invariant with respect to the given vector fields due to the last lemma. As $\mathcal{L}_Vg = 0$ and $\mathcal{L}_{I_rV}g = 0$, the vector fields V, I_1V, I_2V, I_3V are Killing vector fields of the HKT metric $\hat{g}$. By lemma 5, the vector field V is parallel with respect to the HKT connection D. Since D is a Riemannian connection, the dual 1-form $\hat{\theta}$ is parallel. □

3.1. Additional examples of HKT spaces with parallel potential 1-form

Once we construct HKT spaces with $D(2,1;-1)$ symmetry, we can generate new examples through direct products. Indeed let $(M_1,g_1,I_r^{(1)})$, $(M_2,g_2,I_r^{(2)})$ be two locally conformally hyper-Kähler manifolds with parallel Lee forms. Then $\hat{g}_i$ are HKT metrics with special homotheties V_i, $i=1,2$. On $M = M_1\times M_2$ consider the product metric

$$\hat{g} = \tfrac{1}{2}(\pi_1^*\hat{g}_1 + \pi_2^*\hat{g}_2) \tag{32}$$

and complex structures $I_r = (I_r^{(1)}, I_r^{(2)})$. This geometry on M is HKT, since

$$F_r = \tfrac{1}{2}\left(\pi_1^*F_r^{(1)} + \pi_2^*F_r^{(2)}\right)$$

and $c = -d_rF_r = -I_r\,dF_r = \frac{1}{2}(\pi_1^*c_1 + \pi_2^*c_2)$ is independent of $r=1,2,3$. Let

$$V = (V_1,V_2), \qquad \hat{\theta} = \tfrac{1}{2}(\pi_1^*\hat{\theta}^{(1)} + \pi_2^*\hat{\theta}^{(2)}). \tag{33}$$

Then V generates a $D(2,1;-1)$ symmetry, since this is true of V_1 and V_2. Moreover, $\hat{\theta}$ is a potential 1-form. Note that the normalization of $\hat{g}$ has been chosen to fit with conventions of the following section.

4. Relating torsion 3-forms and potential 1-forms

The previous section demonstrates that locally conformally hyper-Kähler manifolds with parallel Lee form generate HKT spaces with $D(2,1;-1)$ symmetry. In this section, we demonstrate that the latter type of geometry is more general than the former. This is achieved through an analysis of the torsion 3-form.

Consider now an HKT structure obtained from a locally conformally hyper-Kähler metric with parallel Lee form. The torsion 3-form is given by the following lemma.

Lemma 18. *The torsion 3-form is determined by* $\hat\theta$ *as*

$$c = -(\hat\theta_1 \wedge \hat F_1 + \hat\theta_2 \wedge \hat F_2 + \hat\theta_3 \wedge \hat F_3 - 2\hat\theta_1 \wedge \hat\theta_2 \wedge \hat\theta_3). \tag{34}$$

Proof. To calculate the torsion 3-form when the HKT structure is generated by a locally conformally hyper-Kähler structure, we recall $\hat\theta = \frac{1}{2}\theta$. Next, we write equation (26) as

$$F_1 = \hat F_1 + 2(\hat\theta \wedge \hat\theta_1 + \hat\theta_2 \wedge \hat\theta_3). \tag{35}$$

Then from equation (28), we have

$$\begin{aligned} c &= -I_1\, d\hat F_1 = -\tfrac{1}{2}(\theta_1 \wedge F_1 + \theta_2 \wedge F_2 + \theta_3 \wedge F_3 - 2\theta_1 \wedge \theta_2 \wedge \theta_3) \\ &= -(\hat\theta_1 \wedge \hat F_1 + \hat\theta_2 \wedge \hat F_2 + \hat\theta_3 \wedge \hat F_3 - 2\hat\theta_1 \wedge \hat\theta_2 \wedge \hat\theta_3), \end{aligned}$$

as claimed. Thus the torsion is an inhomogeneous cubic function of the 1-form $\hat\theta$. □

The torsion 3-form c determines a torsion 1-form τ by

$$\tau(X) = \frac{1}{2}\sum_{i=1}^{4m} c(I_r X, e_i, I_r e_i), \tag{36}$$

where $\{e_i, 1 \leqslant i \leqslant 4m\}$ is an orthogonal frame. The HKT condition ensures that τ is independent of the choice of I_r, $r = 1, 2, 3$ [6]. Under the current constraints,

$$\tau(X) = (2m - 1 + \|\hat\theta\|^2)\hat\theta(X). \tag{37}$$

Thus $\hat\theta = \lambda\tau$, where λ is the unique real (and positive) solution to the cubic equation

$$\lambda(2m - 1 + \lambda^2) = 1. \tag{38}$$

On an arbitrary HKT manifold, whose torsion 1-form is non-zero, one may always find a 1-form $\hat\theta$ satisfying (37). By rescaling $\hat g$ by a homothety, we may ensure that $\|\hat\theta\|^2 = 1/2$ at some base point. With these conventions we call $\hat\theta$ a *normalized torsion 1-form* of M. We say that an HKT manifold M is of *cubic type* if its torsion 3-form c is related to the normalized torsion 1-form $\hat\theta$ by equation (34).

Let V be the vector field dual to $\hat\theta$ via $\hat g$ in this normalization. Then $\hat\theta = \hat g(V, \cdot)$ and

$$\hat g(V, V) = \tfrac{1}{2}, \quad \text{or equivalently} \quad \hat\theta(V) = \tfrac{1}{2}. \tag{39}$$

Theorem 19. *Suppose* $(M, \hat g, I_1, I_2, I_3)$ *is an HKT manifold with a normalized torsion 1-form* $\hat\theta$. *If the torsion* c *is given by*

$$c = -\{\hat\theta_1 \wedge \hat F_1 + \hat\theta_2 \wedge \hat F_2 + \hat\theta_3 \wedge \hat F_3 - 2\hat\theta_1 \wedge \hat\theta_2 \wedge \hat\theta_3\} \tag{40}$$

and the dual vector field of the torsion 1-form generates a $D(2, 1; -1)$ *symmetry, then*

$$g = \hat g + 2\{\hat\theta \otimes \hat\theta + \hat\theta_1 \otimes \hat\theta_1 + \hat\theta_2 \otimes \hat\theta_2 + \hat\theta_3 \otimes \hat\theta_3\} \tag{41}$$

is locally conformally hyper-Kähler with parallel Lee form.

Proof. We first compute the derivatives of $\hat\theta$ and $\hat\theta_r$. Let V be the dual vector field of the 1-form $\hat\theta$. By definition of symmetry and lemma 5, V is parallel. By lemma 6, we have

$$d\hat\theta(X, Y) = c(V, X, Y), \qquad d\hat\theta_1(X, Y) = c(I_1 V, X, Y). \tag{42}$$

The form of c now gives

$$\begin{aligned} d\hat\theta_1 &= -\left(\tfrac{1}{2}\hat F_1 - \hat\theta_1 \wedge F_1(I_1 V, \cdot) - \hat\theta_2 \wedge F_2(I_1 V, \cdot) - \hat\theta_3 \wedge F_3(I_1 V, \cdot) - \hat\theta_2 \wedge \hat\theta_3\right) \\ &= -\tfrac{1}{2}\hat F_1 + \hat\theta \wedge \hat\theta_1 - \hat\theta_2 \wedge \hat\theta_3 = -\tfrac{1}{2}F_1 + \tfrac{1}{4}\{\theta \wedge \theta_1 + \theta_2 \wedge \theta_3\} - \tfrac{1}{4}\theta \wedge \theta_1 + \tfrac{1}{4}\theta_2 \wedge \theta_3 \\ &= -\tfrac{1}{2}F_1 + \tfrac{1}{2}\theta \wedge \theta_1, \end{aligned}$$

where $\theta = 2\hat{\theta}$ and $F_1 = g(I_1\cdot, \cdot)$ is given by (26). Thus $F_1 = \theta \wedge \theta_1 - d\theta_1$ and this has derivative

$$dF_1 = d(\theta \wedge \theta_1 - d\theta_1) = -\theta \wedge d\theta_1 = \theta \wedge F_1. \tag{43}$$

As similar equations hold for F_2 and F_3, we conclude that g is locally conformally hyper-Kähler. The Lee form is a constant multiple of θ, which is closed and hence parallel. $\square$

The condition on the structure of the torsion 3-form is rather strong. However, this is a necessary condition. The example in section 3.1 demonstrates that the existence of $D(2, 1; -1)$ symmetry itself does not necessarily come from a locally conformally hyper-Kähler manifold. This is consistent with the fact that in general the product of locally conformally Kähler manifolds is not necessarily locally conformally Kähler. In fact, the torsion of the example given in section 3.1 is not of cubic type. If we consider the case where each factor is locally conformally hyper-Kähler, put $g = \hat{g} + 2\{\hat{\theta} \otimes \hat{\theta} + \hat{\theta}_1 \otimes \hat{\theta}_1 + \hat{\theta}_2 \otimes \hat{\theta}_2 + \hat{\theta}_3 \otimes \hat{\theta}_3\}$ and $\theta = 2\hat{\theta}$, the Kähler form F_1 is equal to

$$\tfrac{1}{2}\big(\pi_1^* F_1^{(1)} + \pi_2^* F_2^{(2)} + \pi_1^*\theta^{(1)} \wedge \pi_2^*\theta_1^{(2)} + \pi_1^*\theta_2^{(1)} \wedge \pi_2^*\theta_3^{(2)} + \pi_2^*\theta^{(2)} \wedge \pi_1^*\theta_1^{(1)} + \pi_2^*\theta_2^{(2)} \wedge \pi_1^*\theta_3^{(1)}\big),$$

so

$$\begin{aligned}
2\,dF_1 = {} & \pi_1^*\theta^{(1)} \wedge F_1^{(1)} + \pi_2^*\theta^{(2)} \wedge F_1^{(2)} \\
& - \pi_1^*\theta^{(1)} \wedge \pi_2^*\big(\theta^{(2)} \wedge \theta_1^{(2)} - F_1^{(2)}\big) + \pi_1^*\big(\theta^{(1)} \wedge \theta_2^{(1)} - F_2^{(1)}\big) \wedge \pi_2^*\theta_3^{(2)} \\
& - \pi_1^*\theta_2^{(1)} \wedge \pi_2^*\big(\theta^{(2)} \wedge \theta_3^{(2)} - F_3^{(2)}\big) - \pi_2^*\theta^{(2)} \wedge \pi_1^*\big(\theta^{(1)} \wedge \theta_1^{(1)} - F_1^{(1)}\big) \\
& + \pi_2^*\big(\theta^{(2)} \wedge \theta_2^{(2)} - F_2^{(2)}\big) \wedge \pi_1^*\theta_3^{(1)} - \pi_2^*\theta_2^{(2)} \wedge \pi_1^*\big(\theta^{(1)} \wedge \theta_3^{(1)} - F_3^{(1)}\big) \\
\neq {} & 2\theta \wedge F_1,
\end{aligned}$$

since the expression contains non-zero terms involving for example $\pi_1^* F_2^{(1)}$ and terms such as $\theta^{(1)} \wedge F_1^{(2)}$ occur with the wrong coefficients . Thus g is *not* locally conformally hyper-Kähler.

Remark. There is an alternative way to see when an HKT space with $D(2, 1; -1)$ symmetry will generate a locally conformally hyper-Kähler metric using the transformation of the last theorem. Suppose that the dual vector field of a closed 1-form $\hat{\theta}$ is a $D(2, 1, ; -1)$ symmetry on an HKT space. Now we do not assume that the torsion of the HKT space is of cubic type. Define $\theta = 2\hat{\theta}$. By proposition 11, $-\theta$ is a potential 1-form for the HKT metric $\hat{g}$. Again, consider the Riemannian metric (41). Due to the choice of V, θ is the dual of the vector field V with respect to the metric g. Define $g_0 = \theta \otimes \theta + \theta_1 \otimes \theta_1 + \theta_2 \otimes \theta_2 + \theta_3 \otimes \theta_3$. Then for any vector fields X and Y, when rst is a cyclic permutation of 123,

$$g_0(I_r X, Y) = (\theta \wedge \theta_r + \theta_s \wedge \theta_t)(X, Y).$$

Therefore, $F_r = \hat{F}_r + \frac{1}{2}(\theta \wedge \theta_r + \theta_s \wedge \theta_t) = \hat{F}_r + 2(\hat{\theta} \wedge \hat{\theta}_r + \hat{\theta}_s \wedge \hat{\theta}_t)$. Since $-\theta$ is a potential 1-form,

$$\hat{F}_r = -\tfrac{1}{2}(d\theta_r + d_s\theta_t) = -\tfrac{1}{2}(d\theta_r - I_s\, d\theta_r) = -\tfrac{1}{2}(d\theta_r - I_t\, d\theta_r). \tag{44}$$

It follows that

$$\begin{aligned}
F_r &= -\tfrac{1}{2}(d\theta_r - I_s\, d\theta_r) + \tfrac{1}{2}(\theta \wedge \theta_r + \theta_s \wedge \theta_t) = -\tfrac{1}{2}\{(d\theta_r - \theta \wedge \theta_r) - I_s(d\theta_r - \theta \wedge \theta_r)\} \\
&= -\tfrac{1}{2}\{(d\theta_r - \theta \wedge \theta_r) - I_t(d\theta_r - \theta \wedge \theta_r)\}.
\end{aligned}$$

Therefore, $F_r = -(d\theta_r - \theta \wedge \theta_r)$ if and only if for $s \neq r$, $I_s(d\theta_r - \theta \wedge \theta_r) = -(d\theta_r - \theta \wedge \theta_r)$. On the other hand, we check that $I_a(d\theta_a - \theta \wedge \theta_a) = d\theta_a - \theta \wedge \theta_a$. The conclusion is the following observation.

Proposition 20. *The metric g is a locally conformal hyper-Kähler metric with parallel Lee form θ if and only if for all $s \neq r$, $I_s(d\theta_r - \theta \wedge \theta_r) = -(d\theta_r - \theta \wedge \theta_r)$.*

Remark. An HKT structure is said to be strong if the torsion 3-form c is closed [5, 8]. We calculate the exterior differential of the torsion 3-form when the HKT structure is generated by a locally conformally hyper-Kähler structure. We continue to use the notation in lemma 18. With the aid of (19) and (20),

$$\begin{aligned} dc = -\tfrac{1}{2}(d\theta_1 \wedge F_1 + d\theta_2 \wedge F_2 + d\theta_3 \wedge F_3 - \theta_1 \wedge dF_1 - \theta_2 \wedge dF_2 - \theta_3 \wedge dF_3 \\ - 2\, d\theta_1 \wedge \theta_2 \wedge \theta_3 + 2\theta_1 \wedge d\theta_2 \wedge \theta_3 - 2\theta_1 \wedge \theta_2 \wedge d\theta_3) = \tfrac{1}{2}((F_1 - \theta \wedge \theta_1 \\ - \theta_2 \wedge \theta_3)^2 + (F_2 - \theta \wedge \theta_2 - \theta_3 \wedge \theta_1)^2 + (F_3 - \theta \wedge \theta_3 - \theta_1 \wedge \theta_2)^2). \end{aligned}$$

This formula demonstrates that the restriction of dc on the quaternionic span of V is equal to zero. On the quaternionic complement it is equal to

$$\tfrac{1}{2}(F_1 \wedge F_1 + F_2 \wedge F_2 + F_3 \wedge F_3). \tag{45}$$

In particular, it shows the following observation.

Proposition 21. *If M is a locally conformally hyper-Kähler space with real dimension at least 8, then the associated HKT structure $\hat{g}$ is never strong.*

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