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The Fundamental Equations of Conformal Submersions

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Abstract. The purpose of this note is to establish the fundamental equations of a conformal submersion; they are the analogue of the fundamental equations of Riemannian submersions as found by B. O'Neill. Our method consists in attaching to each class of conformal submersion a unique Riemannian submersion, then, by means of appropriate projectors, taking the conformal invariant components of its fundamental equations.

1. Introduction

For Riemannian submersions B. O'Neill in [5] and A. Gray in [2] introduced two tensor fields A and T, playing the role of the second fundamental form of an isometric immersion and gave a set of five equations which are the analogue of the Gauss, Codazzi and Ricci equations.

The aim of this paper is to achieve a similar task for conformal submersions. Their precise definition is

Definition. *Let (N,G) and (B,g) be Riemannian manifolds. A surjective differentiable mapping $\Pi : (N,G) \longrightarrow (B,g)$ is a conformal submersion if:*
i) Π *has maximal rank*
ii) Π_* *preserves the angles between horizontal vectors.*

We shall firstly associate to a conformal submersion two tensor fields analogue to A and T; they will determine the conformal submersion. We then use a technique developed in [6] and [7] for conformal immersions to derive the equations corresponding to those of O'Neill and Gray.

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Our method consists in attaching to a conformal submersion the unique Riemannian submersion with the same total space, base space and projection, writing down its fundamental equations, then taking their conformal invariant components by means of appropriate projections.

All manifolds and geometric objects on them are differentiable and of class $\mathcal{C}^\infty$.

2. Riemannian submersions

For a submersion we shall denote by $\mathcal{H}$ and $\mathcal{V}$ the orthogonal projectors of the tangent space of the total space on its horizontal and, respectively, vertical subspaces.

We recall that, for a Riemannian submersion $P:(N,G)\longrightarrow(B,g)$ the fundamental tensors T and A are defined for $E,F\in\Gamma(TN)$ by (see also [3]):

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}(\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{V}E}(\mathcal{H}F)$$
$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}(\mathcal{H}F) + \mathcal{H}\nabla_{\mathcal{H}E}(\mathcal{V}F)$$

where ∇ is the Levi-Civita connection of G.

The purpose of this section is to reformulate the fundamental equations of a Riemannian submersion in a form suitable for our further considerations.

Let R be the curvature tensor field of (N,G) and $R_{\mathcal{V}}$ its restriction to the vertical subspaces. Let R_f be the curvature tensor field of the fibre f of the submersion.

For a two-form h on the vertical space with values in the horizontal space we associate a $(0,4)$ tensor field $\tilde{h}$ on the vertical space by letting:

$$\tilde{h}(U,V,E,F) = G(h(U,V),h(E,F)) - G(h(V,E),h(U,F))$$

where U,V,E,F are vertical vector fields. The following equation holds for a Riemannian submersion

$$R_{\mathcal{V}} = R_f + \tilde{T} \tag{2.1}$$

We call (2.1) the vertical Gauss equation (note that here T denotes its restriction to the vertical subspace). Let now $R_{\mathcal{H}}$ be the restriction of R to the horizontal spaces. Denote by $\hat{R}$ the horizontal lift of the curvature R_B of the base space (B,g). More precisely, for horizontal vector fields X,Y,Z,H define:

$$\hat{R}(X,Y,Z,H) = R_B(P_*X,P_*Y,P_*Z,P_*H)$$

For a two-form k on the horizontal space with values in the vertical space we define the following (0,4) tensor field on the horizontal space:

$$\tilde{\tilde{k}}(X,Y,Z,H) = 2G(k(Z,H),k(Y,X)) - G(k(H,Y),k(Z,X)) + G(k(Z,Y),k(H,X))$$

If we denote also by A the restriction of A to the horizontal spaces, then the second fundamental equation for Riemannian submersions can be written:

$$R_{\mathcal{H}} = \hat{R} + \tilde{\tilde{A}} \tag{2.2}$$

We call (2.2) the horizontal Gauss equation. Let us denote by $\mathcal{H}$R the horizontal component of the restriction of the (1,3) curvature tensor field of (N, G) to the vertical subspaces.

For a field of bilinear forms l on the vertical (resp. horizontal) space with values in the horizontal (resp. vertical) space we define the following field of trilinear forms on the vertical (resp. horizontal) space with values in the horizontal (resp. vertical) space:

$$(\tilde{\nabla} l)(X, Y, Z) = (\nabla_X l)(Y, Z) - (\nabla_Y l)(X, Z)$$

where X, Y, Z are vertical (resp. horizontal) vector fields and where:

$$(\nabla_X l)(Y, Z) = \mathcal{H}(\nabla_X l(Y, Z)) - l(\mathcal{V}(\nabla_X Y), Z) - l(Y, \mathcal{V}(\nabla_X Z))$$

$$(\text{resp. } (\nabla_X l)(Y, Z) = \mathcal{V}(\nabla_X l(Y, Z)) - l(\mathcal{H}(\nabla_X Y), Z) - l(Y, \mathcal{H}(\nabla_X Z)))$$

Thinking, as before, T as the restriction to the vertical space, the third fundamental equation of a Riemannian submersion assumes the form:

$$\mathcal{H}R = \tilde{\nabla} T \tag{2.3}$$

We call (2.3) the vertical Codazzi equation.

Let now $\mathcal{V}R$ be the vertical component of the restriction to horizontal spaces of the (1,3) curvature tensor field of (N, G). We define a trilinear form L on the horizontal spaces with values in the vertical spaces by the formula:

$$G(L(X, Y, Z), V) = G(T_V Z, A_X Y)$$

where X, Y, Z (resp. V) are horizontal (resp. vertical) vector fields.

An equivalent form for the fourth fundamental equation will now be:

$$\mathcal{V}R = \tilde{\nabla} A - 2L \tag{2.4}$$

We shall call it the horizontal Codazzi equation. Finally we write the last fundamental equation of a Riemannian submersion in its usual form:

$$\begin{aligned} G(R(X, V)Y, W) = {} & G((\nabla_V A)_X Y, W) - G((\nabla_X T)_V W, Y) + \\ & + G(A_X V, A_Y W) - G(T_V X, T_W Y) \end{aligned} \tag{2.5}$$

where X, Y (resp. V, W) are horizontal (resp. vertical) vector fields.

Remark 2.1 - The tensors $\tilde{h}$ and $\tilde{\tilde{k}}$ above were obtained from the given tensors h and k by means of the metric G. How do they behave at a conformal change of metric on the total spaceΓ To answer the question let $G^\pi = e^{2\pi} G$, $\pi \in \mathcal{C}^\infty(N)$ and define for X, Y, Z, U (vertical or horizontal)

$$\tilde{h}^\pi(X, Y, Z, U) = G^\pi(h(X, Y), h(Z, U)) - G^\pi(h(Y, Z), h(X, U))$$

$$\tilde{k}^\pi(X,Y,Z,U) = 2G^\pi(k(Z,U),k(Y,X)) - G^\pi(k(U,Y),k(Z,X)) + G^\pi(k(Z,Y),k(U,X))$$

We obtain:

$$\tilde{h}^\pi = e^{2\pi}\tilde{h}, \quad \tilde{\tilde{k}}^\pi = e^{2\pi}\tilde{\tilde{k}}$$

3. Algebraic preparation

In this section we briefly recall some linear algebraic facts to be used in the following sections.

Let (V^n, G) be an Euclidian vector space and $\{E_1, ..., E_n\}$ an orthonormal frame on V^n $(n \geq 3)$.

We denote by $\mathcal{E}$ the vector space of (0,4) tensors on V^n satisfying the following properties

1) $T(X,Y,Z,W) + T(Y,X,Z,W) = 0$
2) $T(X,Y,Z,W) + T(Y,X,W,Z) = 0$
3) $T(X,Y,Z,W) + T(Z,X,Y,W) + T(Y,Z,X,W) = 0$

An element of $\mathcal{E}$ is called an algebraic curvature tensor (a.c.t.). Starting with an a.c.t. T one associates a symmetric (0,2) tensor $c(T)$ and a scalar function $r(T)$ defined respectively by:

$$c(T)(X,Y) = tr_G T(X,\cdot,Y,\cdot) = \sum_\alpha T(X,E_\alpha,Y,E_\alpha)$$

$$r(T) = tr_G c(T) = \sum_{\alpha,\beta} T(E_\beta,E_\alpha,E_\beta,E_\alpha)$$

The Weyl part of an a.c.t. is defined as

$$\mathcal{C}(T) = T + \frac{1}{n-2} c(T) \owedge G - \frac{r(T)}{2(n-1)(n-2)} G \owedge G$$

where $\owedge$ is the Kulkarni-Nomizu product defined, in general, for two symmetric (0,2) tensors h and k on V^n as follows:

$$(h \owedge k)(X,Y,Z,W) = h(X,Z)k(Y,W) + h(Y,W)k(X,Z) - \\ - h(X,W)k(Y,Z) - h(Y,Z)k(X,W)$$

It is well-known that $\mathcal{C} : \mathcal{E} \longrightarrow \mathcal{E}$ is a projector: $\mathcal{C}^2 = \mathcal{C}$ and that $\mathcal{C}(\mathcal{E})$ is an invariant irreducible subspace of $\mathcal{E}$ with respect to the natural action of $\mathcal{O}(n)$ on $\mathcal{E}$ (cf [1]).

As an example, the Riemannian curvature tensor R of a Riemannian manifold is an a.c.t.; then $c(R)$ is the Ricci tensor and $r(R)$ is the scalar curvature; finally $\mathcal{C}(R)$ is the Weyl tensor.

Let now $V^{\bar{n}}$ be another vector space and denote by T a $V^{\bar{n}}$-valued trilinear form on V^n. For $X, Y, Z \in V^n$ we define

$$C(T)(X,Y,Z) = T(X,Y,Z) - \frac{1}{n-1}\sum_\alpha \{T(E_\alpha, Y, E_\alpha)G(X,Z) - T(E_\alpha, X, E_\alpha)G(Y,Z)\}$$

We proved in [7] that C is a projector ($C^2 = C$) and, if $\mathcal{F}$ is the subspace of tensors T satisfying:

$$T(X,Y,Z) + T(Y,X,Z) = 0$$

$$T(X,Y,Z) + T(Z,X,Y) + T(Y,Z,X) = 0$$

then $C(\mathcal{F})$ is an irreducible invariant subspace of $\mathcal{F}$ with respect to the natural action of $\mathcal{O}(n)$ on $\mathcal{F}$.

Obviously the definition of $\mathcal{C}$ (resp. C) depends on the metric G on V^n. If $G^\pi = e^{2\pi}G$ is a conformal metric with G on $V^n, \pi \in \mathbf{R}$, denote with $\mathcal{C}^\pi$ (resp. C^π) the same operator constructed out of G^π. It is easy to check that $\mathcal{C}^\pi(T) = \mathcal{C}(T)$ (resp. $C^\pi(T) = C(T)$). We shall use in the following only metrics belonging to a fixed conformal class, thus we shall not specify in which metric $\mathcal{C}$ (resp. C) is computed.

Remark 3.1 In the notations of the previous section, let k be a field of bilinear forms on the vertical (resp. horizontal) spaces with values in the horizontal (resp. vertical) spaces in such a way that one may consider $\tilde{\nabla}k$. For a metric $G^\pi = e^{2\pi}G$ denote with ∇^π its Levi-Civita connection and, for vertical (resp. horizontal) vector fields X, Y, Z consider:

$$(\tilde{\nabla}^\pi k)(X,Y,Z) = (\nabla^\pi_X k)(Y,Z) - (\nabla^\pi_Y k)(X,Z)$$

Then a straightforward computation based on the well-known formula:

$$\nabla^\pi_X Y = \nabla_X Y + X(\pi)Y + Y(\pi)X - g(X,Y)grad_G\pi \tag{3.1}$$

shows that, if k is symmetric,

$$C(\tilde{\nabla}^\pi k) = C(\tilde{\nabla}k)$$

4. Conformal submersions

Let $\Pi : (N, G^\pi) \longrightarrow (B, g)$ be a conformal submersion as in the Introduction (note that here G^π is only a notation). The following equivalent definition is natural:

Lemma 4.1 Π *is a conformal submersion if and only if there exists a function* $\pi \in \mathcal{C}^\infty(N)$, *uniquely associated to* Π, *such that for every* $x \in M$ *and for every horizontal vectors* $X, Y \in T_xN$ *one has:*

$$G^\pi_x(X,Y) = e^{2\pi(x)}g_x(\Pi_*X, \Pi_*Y)$$

.

We shall call π the associated function of the conformal submersion π.

One easily deduces:

Lemma 4.2 *If π is the associated function of Π, then the metric G*

$$G = e^{-2\pi}G^{\pi}$$

is the unique metric on N, conformal with G, with the property that $p : (N, G) \longrightarrow (B, g)$, where p is defined by $p(x) = \Pi(x),\ x \in N$, is a Riemannian submersion.

We shall call $p : (N, G) \longrightarrow (B, g)$ the associated Riemannian submersion.
The following result is also immediate and its proof is left to the reader.

Lemma 4.3 *All the submersions obtained from a given conformal submersion $\Pi : (N, G^{\pi}) \longrightarrow (B, g)$ by conformal changes on the total space are conformal submersions and have the same associated Riemannian submersion $p : (N, G) \longrightarrow (B, g)$.*

From formula (3.1), the Levi-Civita connection ∇ of G is related to the Levi-Civita ∇^{π} of G^{π} by:

$$\nabla_X Y = \nabla^{\pi}_X Y - X(\pi)Y - Y(\pi)X + G^{\pi}(X, Y)grad_{\Pi}\pi$$

where $grad_{\Pi}\pi$ is the gradient of π with respect to G^{π}.

Let now $\Pi : (N, G^{\pi}) \longrightarrow (B, g)$ be the generic conformal submersion with associated Riemannian submersion a given (fixed) one $p : (N, G) \longrightarrow (B, g)$.

From Lemma 4.3 and (3.1) we deduce:

Lemma 4.4 *The equality:*

$$D_X Y = \nabla^{\pi}_X Y - X(\pi)Y - Y(\pi)X + G^{\pi}(X, Y)grad_{\Pi}\pi$$

defines a connection on N that is independent from the conformal submersion Π: it coincides in fact, with the Levi-Civita connection on the total space of the associated Riemannian submersion $p : (N, G) \longrightarrow (B, g)$.

Remark 4.5 The above construction is closely related to the one of a G-connection in [4].

In the following we consider a fixed class of conformal submersions, associated to a fixed Riemannian submersion, thus we shall no more specify the submersion by means of which we compute the connection D.

Let us now define, for the given conformal submersion Π, the (1,2) tensors $\mathcal{T}$ and $\mathcal{A}$:

$$\mathcal{T}_E F = \mathcal{H}D_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}D_{\mathcal{V}E}\mathcal{H}F, \quad E, F \in \Gamma(TN)$$

$$\mathcal{A}_E F = \mathcal{V}D_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}D_{\mathcal{H}E}\mathcal{V}F, \quad E, F \in \Gamma(TN)$$

Remark 4.6 If T and A are the fundamental tensors of the associated Riemannian submersion, then $\mathcal{T} = T, \ \mathcal{A} = A$.

Remark 4.7 If Π' is a conformal submersion obtained from Π by a conformal change of G^π, then $\mathcal{T}' = \mathcal{T}, \ \mathcal{A}' = \mathcal{A}$ so that we have not to specify in what conformal submersion $\mathcal{T}$ and $\mathcal{A}$ are computed when, as we mentioned above, the conformal class is fixed.

We now recall that a totally geodesic submanifold of a Riemannian manifold is changed into a totally umbilical submanifold by a conformal change of the metric of the ambient space. We also recall that $T = 0$ for a Riemannian submersion is a condition equivalent with: all fibres are totally geodesic. Then from Remark 4.6 we derive:

Lemma 4.8 *The fibres of a conformal submersion are totally umbilical if and only if* $\mathcal{T} = 0$.

In view of Lemma 4.2 the tensors $\mathcal{T}$ and $\mathcal{A}$ are uniquely associated to the conformal submersion Π. Moreover, they determine it, namely we have:

Theorem 4.9 *Let* $\Pi_i : (N, G^{\pi_i}) \longrightarrow (B, g), \ i = 1, 2$, *be two conformal submersions with* $\mathcal{T}^1 = \mathcal{T}^2, \mathcal{A}^1 = \mathcal{A}^2$. *If* N *is connected and* Π_1 *and* Π_2 *have equal derivatives in a certain point* $x_0 \in N$, *then* $\Pi_1 = \Pi_2$.

Proof. We only have to observe that in our hypothesis the Riemannian submersions associated to Π_1 and Π_2 have equal fundamental tensors; the result thus follows from the corresponding theorem of O'Neill [5].

5. The three first fundamental equations

In this section we establish the analogues for conformal submersions of the vertical Gauss equation (2.1), the horizontal Gauss equation (2.2) and of the vertical Codazzi equation (2.3).

Let $\Pi : (N, G^\pi) \longrightarrow (B, g)$ be a generic conformal submersion, as above, with associated Riemannian submersion a fixed Riemannian submersion $p : (N, G) \longrightarrow (B, g)$, i.e. $G^\pi = e^{2\pi} G, \ \pi \in \mathcal{C}^\infty(N)$. Let

$$R_{\mathcal{V}} = R_f + \tilde{T}$$

be the vertical Gauss equation of p. From Remark 4.6 we can write:

$$R_{\mathcal{V}} = R_f + \tilde{\mathcal{T}}$$

and applying to both members the projector $\mathcal{C}$ we obtain:

$$\mathcal{C}(R_{\mathcal{V}}) = \mathcal{C}(R_f) + \mathcal{C}(\tilde{\mathcal{T}})$$

But $\mathcal{C}(R_{\mathcal{V}})$ is the Weyl tensor, $W_{\mathcal{V}}$, of the restriction of R to the vertical space and $\mathcal{C}(R_f)$ is the Weyl tensor, W_f, of the fibre f through the point of N where the equation is considered. Then we have:

$$W_{\mathcal{V}} = W_f + \mathcal{C}(\tilde{\mathcal{T}}).$$

We now recall that changing the metric G with the metric $G^\pi = e^{2\pi}G$ changes the Weyl tensor W_f in:

$$W_f^\pi = e^{2\pi} W_f$$

and the Weyl tensor $W_\mathcal{V}$ (cf. [6]) in:

$$W_\mathcal{V}^\pi = e^{2\pi} W_\mathcal{V}$$

Moreover, from Remark 2.1, we have:

$$\mathcal{C}(\tilde{\mathcal{T}}^\pi) = e^{2\pi}\mathcal{C}(\tilde{\mathcal{T}})$$

thus the equation

$$W_\mathcal{V}^\pi = W_f^\pi + \mathcal{C}(\tilde{\mathcal{T}}^\pi) \tag{5.1}$$

holds good. Lemma 4.3 insures us that equality (5.1) is invariant at conformal changes of the metric G^π. We call (5.1) the vertical conformal Gauss equation.

Let now

$$R_\mathcal{H} = \hat{R} + \tilde{\tilde{A}}$$

be the horizontal Gauss equation of p. Recalling Remark 4.6 we apply $\mathcal{C}$ to both members and get:

$$\mathcal{C}(R_\mathcal{H}) = \mathcal{C}(R) + \mathcal{C}(\tilde{\tilde{\mathcal{A}}}\ )$$

We now define

$$\hat{W}^\pi = e^{2\pi}\mathcal{C}(\hat{R})$$

and observe that $\mathcal{C}(R_\mathcal{H})$ is the Weyl tensor, $W_\mathcal{H}$, of the restriction of R to the horizontal spaces. Changing G into G^π alters $W_\mathcal{H}$ by multiplication with $e^{2\pi}$ (cf. [7]):

$$W_\mathcal{H}^\pi = e^{2\pi} W_\mathcal{H}$$

From Remark 2.1 we also have:

$$\mathcal{C}(\tilde{\tilde{\mathcal{A}}}^{\ \pi}) = e^{2\pi}\mathcal{C}(\tilde{\tilde{\mathcal{A}}}\ )$$

and we can write

$$W_\mathcal{H}^\pi = \hat{W}^\pi + \mathcal{C}(\tilde{\tilde{\mathcal{A}}}^{\ \pi}) \tag{5.2}$$

The conformal invariance of (5.2) follows, as before, from Lemma 4.3. We call (5.2) horizontal conformal Gauss equation.

We now consider the vertical Codazzi equation of the Riemannian submersion p.

$$\mathcal{H}R = \tilde{\nabla} T$$

From Remark 4.6 we can also write:

$$\mathcal{H}R = \tilde{\nabla}\mathcal{T}$$

Applying to both members the projector C yields:

$$C(\mathcal{H}R) = C(\tilde{\nabla}\mathcal{T})$$

The conformal invariance of $C(\mathcal{H}R)$ (cf. [7]) and of $C(\tilde{\nabla}\mathcal{T})$ (cf. Remark 3.1) allows us to assert:

$$C(\mathcal{H}R^{\pi}) = C(\tilde{\nabla}\mathcal{T}^{\pi}) \tag{5.3}$$

where $\mathcal{H}R^{\pi}$ is the horizontal component of the restriction to the vertical subspace of the (1,3) curvature tensor of (N, G). The conformal invariance of equation (5.3) is now a consequence of Lemma 4.3. We call it the vertical conformal Codazzi equation.

Summing up we may state:

Theorem 5.1. *Let $\Pi : (N, G^{\pi}) \longrightarrow (B, g)$ be a conformal submersion. The following equations hold good and are invariant at the conformal changes of the metric G^{π}.*

$$W^{\pi}_{\mathcal{V}} = W^{\pi}_{f} + \mathcal{C}(\tilde{\mathcal{T}}^{\pi}) \qquad \textit{(vertical conformal Gauss equation)}$$

$$W^{\pi}_{\mathcal{H}} = \hat{W}^{\pi} + \mathcal{C}(\tilde{\tilde{\mathcal{A}}}^{\pi}) \qquad \textit{(horizontal conformal Gauss equation)}$$

$$C(\mathcal{H}R^{\pi}) = C(\tilde{\nabla}^{\pi}\mathcal{T}) \qquad \textit{(vertical conformal Codazzi equation)}$$

6. The last two equations

Let

$$\mathcal{V}R = \tilde{\nabla}A - 2L$$

be the horizontal Codazzi equation of the Riemannian submersion p. Using Remark 4.6 we substitute $\mathcal{A}$ to A and apply the projector C:

$$C(\mathcal{V}R) = C(\tilde{\nabla}\mathcal{A}) - 2C(L)$$

As $C(\mathcal{V}R)$ is conformally invariant (cf. [7]) we obtain:

$$C(\mathcal{V}R) = C(\mathcal{V}R^{\pi})$$

where $\mathcal{V}R^{\pi}$ is the vertical component of the restriction to the horizontal spaces of the (1,3) curvature tensor of (N, G^{π}).

We define a vertical valued trilinear form on the horizontal vectors by the equality:

$$G^{\pi}(L^{\pi}(X, Y, Z), V) = G^{\pi}(\mathcal{T}_V Z, \mathcal{A}_X Y),$$

for each vertical vector V. Clearly $L^\pi = L$ and also

$$C(L^\pi) = C(L)$$

The conformal class of the metric G being fixed we do not specify by means of what metric is L computed.

Unfortunately $C(\tilde{\nabla}\mathcal{A})$ is not conformally invariant because $\mathcal{A}$ is not symmetric on horizontal vectors, but antisymmetric. To obtain a conformally invariant equation we consider $C(\tilde{D}\mathcal{A})$, where $\tilde{D}\mathcal{A}$ is defined as:

$$(\tilde{D}\mathcal{A})(X,Y,Z) = (D_X\mathcal{A})(Y,Z) - (D_Y\mathcal{A})(X,Z)$$

From Lemma 4.4 we know that $D = \nabla$ and then:

$$C(\tilde{\nabla}\mathcal{A}) = C(\tilde{D}\mathcal{A})$$

The conformal invariance of D and $\mathcal{A}$ implies:

$$C(\mathcal{V}R^\pi) = C(\tilde{D}\mathcal{A}) - 2C(L) \tag{6.1}$$

We call (6.1) the horizontal Codazzi equation. It clearly doesn't change at conformal changes on the total space.

Finally we consider the analogue of the Ricci equation for the Riemannian submersion p:

$$\begin{aligned} G(R(X,V)Y,W) =& G((\nabla_V A)_X Y, W) - G((\nabla_X T)_V W, Y) + \\ &+ G(A_X V, A_Y W) - G(T_V X, T_W Y) \end{aligned}$$

From the Bianchi identity:

$$G(R(X,Y)V,W) = G(R(X,V)Y,W) - G(R(Y,V)X,W)$$

and from the antisymmetry of $(\nabla_V A)_X Y$ in X and Y we derive:

$$\begin{aligned} G(R(X,Y)V,W) = & \, 2G((\nabla_V A)_X Y, W) + G((\nabla_X T)_V W, Y) - \\ & - G((\nabla_Y T)_V W, X) + G(A_X V, A_Y W) - G(A_Y V, A_X W) + \\ & + G(T_V Y, T_W X) - G(T_V X, T_W Y) \end{aligned}$$

It was proved in [7] that $G^\pi((R^\pi(X,Y)V,W) = e^{2\pi}G(R(X,Y)V,W)$. Multiplying both members of the above equation by $e^{2\pi}$ and applying Lemmas 4.4 and 4.6 we obtain a conformally invariant equation.

$$\begin{aligned} G^\pi(R^\pi(X,Y)V,W) = & \, 2G^\pi((D_V\mathcal{A})_X Y, W) + G^\pi((D_X\mathcal{T})_V W, Y) - \\ & - G^\pi((D_Y\mathcal{T})_V W, X) + G^\pi(\mathcal{A}_X V, \mathcal{A}_Y W) - G^\pi(\mathcal{A}_Y V, \mathcal{A}_X W) + \\ & + G^\pi(\mathcal{T}_V Y, \mathcal{T}_W X) - G^\pi(\mathcal{T}_V X, \mathcal{T}_W Y) \end{aligned} \tag{6.2}$$

This leads to:

Theorem 6.1 *For a conformal submersion $\Pi : (N, G^\pi) \longrightarrow (B, g)$ equation (6.1), called the horizontal conformal Codazzi equation, and equation (6.2) hold good and are conformally invariant.*

Remark 6.2 The same method we used to prove the above theorem may be used to derive another form of the vertical conformal Codazzi equation, namely:

$$C(\mathcal{H}R^\pi) = C(\tilde{D}\mathcal{T})$$

Remark 6.3 One may consider equations (5.1), (5.2), (5.3), (6.1) and (6.2) for Riemannian submersions too; in this case $\mathcal{T} \equiv T$, $\mathcal{A} \equiv A$, $D = \nabla$ still the equations do not coincide with (2.1), (2.2), (2.3), (2.4), (2.5) respectively.

Remark 6.4 Changing the metric g of the base space of a conformal submersion $\Pi : (N, G^\pi) \longrightarrow (B, g)$ with a conformal one $g^\star = e^{2\sigma} g$, $\sigma \in \mathcal{C}^\infty(B)$, produces another conformal submersion $\Pi^\star : (N, G^\pi) \longrightarrow (B, g^\star)$ whose associated function is $\pi^\star = \pi - \sigma \circ \Pi^\star$. But, as a direct check shows, the connection D, defined by means of this associated function, doesn't change when we substitute $\pi^\star$ to π; thus the tensors $\mathcal{T}$ and $\mathcal{A}$ will not change. Taking into account that the metric G^π remains the same one may easily conclude that all the equations we found, except the horizontal conformal Gauss equation, are also invariant at conformal changes of the metric on the base space.

References

[1] Besse, A.: *Einstein manifolds.* Springer Verlag, 1987.

[2] Gray, A.: *Pseudo-Riemannian almost product manifolds and submersions.* J. Math. Mech. 16 (1967), 715-738.

[3] Ianus, S.: *Differential geometry with applications in relativity theory* (in Romanian). Ed. Academiei, Bucharest 1983.

[4] Okubo, K.: *Invariant connections for conformal and projective changes.* J. Math. Kyoto Univ. 31 (1991), 1087-1094.

[5] O'Neill, B.: *The fundamental equations of a Riemannian submersion.* Mich. Math. J. 34 (1966), 459-469.

[6] Ornea, L.; Romani, G.: *Conformal geometry of Riemannian submanifolds I. Gauss equation.* Preprint, Univ. degli Studi di Roma 41/92.

[7] Ornea, L.; Romani, G.: *Conformal geometry of Riemannian submanifolds II. Codazzi and Ricci equations.* (to appear).