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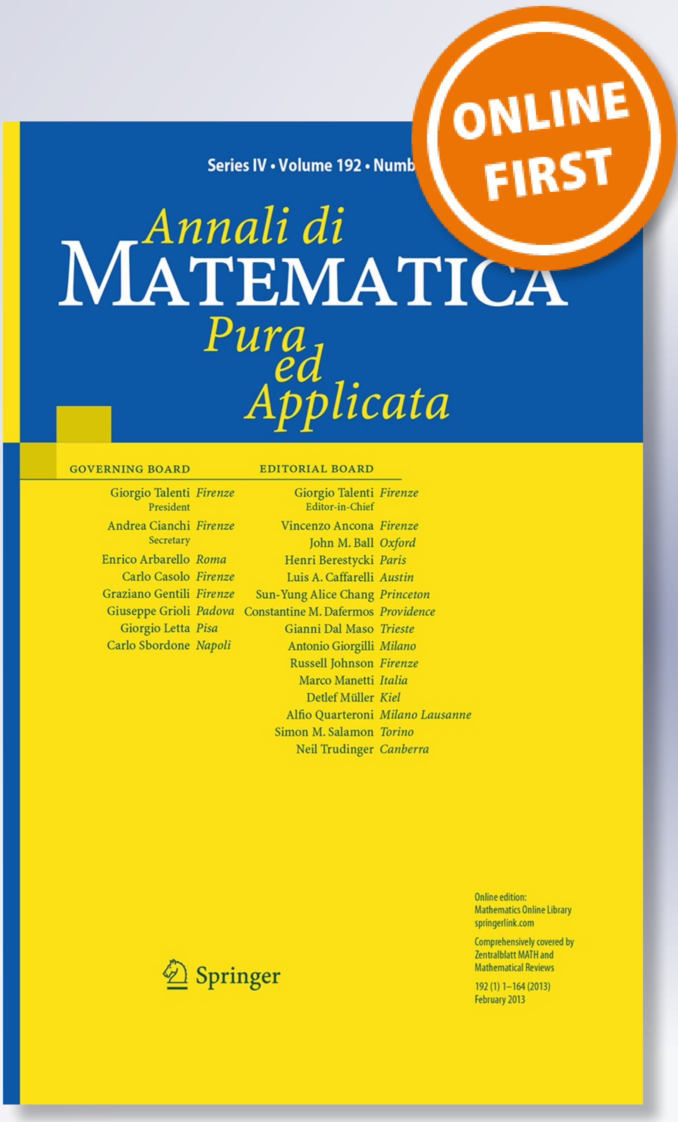

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Holomorphic submersions of locally conformally Kähler manifolds

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Abstract A locally conformally Kähler (LCK) manifold is a complex manifold covered by a Kähler manifold, with the covering group acting by homotheties. We show that if such a compact manifold X admits a holomorphic submersion with positive-dimensional fibers at least one of which is of Kähler type, then X is globally conformally Kähler or biholomorphic, up to finite covers, to a small deformation of a Vaisman manifold (i.e., a mapping torus over a circle, with Sasakian fiber). As a consequence, we show that the product of a compact non-Kähler LCK and a compact Kähler manifold cannot carry a LCK metric.

Keywords Locally conformally Kähler manifold · Holomorphic submersion · Vaisman manifold

Mathematics Subject Classification (2010) 53C55

Dedicated to the memory of Stere Ianuş.

L. Ornea (✉) · V. Vuletescu
Faculty of Mathematics, University of Bucharest,
14 Academiei str., 010014 Bucharest, Romania
e-mail: lornea@gta.math.unibuc.ro

V. Vuletescu
e-mail: vuli@gta.math.unibuc.ro

L. Ornea
Institute of Mathematics "Simion Stoilow" of the Romanian Academy,
21, Calea Grivitei Street, 010702 Bucharest, Romania
e-mail: liviu.ornea@imar.ro

M. Parton
Dipartimento di Scienze, Universita di Chieti—Pescara, Pescara, Italy
e-mail: parton@sci.unich.it

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1 Introduction and statement of results

Locally conformally Kähler (LCK) manifolds are Hermitian manifolds (X, g, J), such that the fundamental two-form $\omega = g \circ J$ satisfies the integrability condition

$$d\omega = \theta \wedge \omega, \quad \text{for a closed one-form } \theta,$$

where θ is called the Lee form.

This definition is known to be equivalent with a covering space $\tilde{X}$ of (X, J) to carry a global Kähler metric Ω with respect to which the covering group Γ acts by holomorphic homotheties (see [4,6,12] for a recent survey). As such, the LCK structure defines a character associating with each covering transformation its scale factor:

$$\chi : \Gamma \longrightarrow \mathbb{R}^+, \quad \chi(\gamma) = \frac{\gamma^* \Omega}{\Omega}. \tag{1.1}$$

If θ is exact, the metric ω is globally conformal to a Kähler metric (we say that the Hermitian manifold (X, ω) is GCK for short). In particular, X admits a Kähler metric (i.e., it is *of Kähler type*).

In a LCK manifold, if θ is moreover parallel with respect to the Levi-Civita connection of the LCK metric, the manifold is called Vaisman. Compact Vaisman manifolds are mapping tori over the circle with fibers isometric with a Sasakian manifold (see [11]). The topology of compact Vaisman manifolds is very different from the topology of Kähler manifolds, *e.g.*, their first Betti number is always odd.

Almost all compact complex surfaces in class VII are LCK, and many of them (e.g., diagonal Hopf, Kodaira) are Vaisman (see [2,3]). In higher dimensions, main examples are diagonal Hopf manifolds (which are Vaisman), non-diagonal Hopf manifolds (non-Vaisman, see [15]), Oeljeklaus-Toma manifolds (see [10,17]).

On a Vaisman manifold X, the Lee field $\theta^\sharp$ (the g-dual of θ) is analytic and Killing and hence generates a complex, totally geodesic foliation $\mathcal{F} = \{\theta^\sharp, J\theta^\sharp\}$. If $\mathcal{F}$ is regular (and in this case, the manifold X itself is called *regular*), then X admits a holomorphic submersion (which is moreover a principal bundle map) over a projective orbifold. But, in general, very little is known about the existence of holomorphic submersions from compact LCK manifolds (papers like [8,9] assume the existence of the submersion and are mainly concerned with the structure it imposes on the total space or on the base and by the geometry of the fibers).

In this note, we partially solve the existence problem. Our principal result is the following:

Theorem *Let X be a compact complex manifold which admits a homolorphic submersion $\pi : X \longrightarrow B$ with positive-dimensional fibers. Assume one of the fibers of π is of Kähler type. If X has an LCK metric g, then g is GCK or X is biholomorphic to a finite quotient of a small deformation of a Vaisman manifold.*

This result is rather general, as it does not assume that the submersion relates in any way the Riemannian geometries of the total and base spaces (we do not restrict to Riemannian or conformal submersions, for example).

There is no natural product construction in the category of LCK manifolds, because $\mathrm{CO}(m) \times \mathrm{CO}(n) \not\subset \mathrm{CO}(m+n)$. The following by-product of the Theorem (already proven differently in [18]) is, therefore, an useful information:

Corollary 1 *Let X_1, X_2 be compact regular Vaisman manifolds. Then, $X_1 \times X_2$ carries no LCK metric.*

But more can be said. Applying the above Theorem to the projection of the first factor of a product $X \times Y$ where X is a compact LCK (non-Kähler) manifold and Y is compact Kähler, both positive-dimensional, one obtains:

Corollary 2 *The product of a compact LCK non-Kähler manifold with a compact Kähler manifold admits no LCK metric.*

2 Proof of the Theorem

The main ingredient is the following "lemma on fibrations."

Lemma *Let X be a compact complex manifold which admits a homolorphic submersion $\pi : X \longrightarrow B$ with positive-dimensional fibers. If X has an LCK metric g whose Lee form θ is (cohomologically) a pull-back, $[\theta] = \pi^*([\eta])$, $[\eta] \in H^1(B)$, then g is GCK.*

Proof The proof is basically the same as in [16], but since the statement is a little bit different, we include the details here.

First, let us fix some notations. If M is any manifold and $\alpha \in H^1(M)$ is arbitrary, we will denote by $\alpha_* : H_1(M, \mathbb{Z}) \longrightarrow \mathbb{R}$ the morphism given by

$$\alpha_*([\gamma]) = \int_\gamma \alpha.$$

Notice that in our setup, we have

$$\eta_* \circ \pi_* = \theta_*$$

where $\pi_* : H_1(X, \mathbb{Z}) \longrightarrow H_1(B, \mathbb{Z})$ is the map induced at homology by π.

Moreover, we will denote by M^{ab} its maximal abelian cover, whose fundamental group is just $[\pi_1(M), \pi_1(M)]$. Observe that the deck group of M^{ab} over M is $H_1(M, \mathbb{Z})$.

Now, let $K = \ker(\eta_*)$; it is a subgroup of $H_1(B, \mathbb{Z})$ so letting $\overline{B} = B^{ab}/K$ we see $H_1(\overline{B}, \mathbb{Z}) \cong H_1(B^{ab}, \mathbb{Z})/K$. In particular, the pullback $\overline{\eta}$ of η to $\overline{B}$ is exact, since $\overline{\eta}_* \equiv 0$.

Now let $\overline{X} = \overline{B} \times_B X$, i.e.,

$$\begin{array}{ccc} \overline{X} & \longrightarrow & X \\ {\scriptstyle \overline{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \overline{B} & \longrightarrow & B \end{array}$$

Then, $\overline{X}$ is a cover of X, and the fibers of the induced map $\overline{\pi} : \overline{X} \longrightarrow \overline{B}$ are the same as the fibers of π, thus $\overline{\pi}$ is proper as well. Let $\overline{\theta}$ be the pullback of θ to $\overline{X}$. Since $\overline{[\theta]} = \overline{\pi}^*([\overline{\eta}])$, we see that $\overline{\theta}$ is also exact, as $\overline{\eta}$ is exact. This implies that the pullback $\overline{g}$ of g to $\overline{X}$ is globally conformal to a Kähler metric ω.

Assume now that g is not GCK. Then, there exists a deck transformation $\overline{\varphi}$ of $\overline{X}$ acting on ω by a non-isometric homothety:

$$\overline{\varphi}^*(\omega) = \varrho \cdot \omega, \quad \rho \neq 1. \tag{2.1}$$

Let F be any fiber of $\overline{\pi}$. Since F is also a fiber of π, it is compact, hence its volume $\mathrm{Vol}_\omega(F)$ is finite. Let $F' = \varphi(F)$. Then, F' is also a fiber of $\overline{\pi}$, and since ω is Kähler, we have

$$\mathrm{Vol}_\omega(F) = \mathrm{Vol}_\omega(F').$$

But from (2.1), we get

$$\mathrm{Vol}_\omega(F) = \varrho^{\dim_{\mathbb{C}}(F)} \mathrm{Vol}_\omega(F'),$$

a contradiction. □

Proof of the Theorem We shall prove the following facts:

1) If the fibers are at least 2-dimensional, then g is GCK.
2) If the fibers are 1-dimensional and their genus is not 1, g is again GCK. And finally,
3) If the fibers are 1-dimensional and their genus is 1, then X is biholomorphic to a GCK manifold or to a finite quotient of a small deformation of a Vaisman manifold.

To prove 1), let F_0 be a fiber of Kähler type and let F be any fiber of π. Also, let i_0 and i be the respective immersions of the fibers in X. As B is arcwise connected, from Ehresmann's theorem, it follows that F and F_0 have the same homotopy type, which implies the exact sequence

$$0 \longrightarrow H^1(B) \xrightarrow{\pi^*} H^1(X) \xrightarrow{i^*} H^1(F).$$

But Vaisman proved [19] that if a compact LCK manifold of dimension at least 2 is of Kähler type, then the LCK metric is actually GCK. Hence, if F_0 has dimension at least 2, it follows that $i_0^*([\theta]) = 0$. From the exact sequence above, we see $[\theta]$ is a pullback, and hence, the above Lemma implies that g is GCK.

To prove 2), observe that if the genus of F is 0 then π^* is an isomorphism between $H^1(B)$ and $H^1(X)$, so the Lemma applies again.

If the genus is at least 2, we argue as follows. First, by Uniformization Theorem, after a conformal change of g, we may assume $g_{|F}$ has (negative) constant curvature. On the other hand, by [17], we get that $[\theta]_{|F}$ is the Poincaré dual of the character χ of $g_{|F}$ (see (1.1) for the definition of the character). But this character is trivial, since the Riemannian universal cover of $(F, g_{|F})$ is the Poincaré half plane with the metric of negative constant curvature with respect to which every homothety is an isometry. Hence, $[\theta]_{|F} = 0$, and so again $[\theta]$ is a pullback from B.

We now prove 3). As the $j-$invariant is a holomorphic map and B is compact, we see that all the fibers are isomorphic, and hence, by Fischer-Grauert theorem [7], the map π is a locally trivial fibration. Hence, X has a finite cover X' (which is still LCK, begin a cover of X) and is a principal elliptic bundle and the fiber F acts holomorphically on X'. Notice that the fiber F, which is a complex torus $T^1_{\mathbb{C}}$, acts holomorphically on X'. In particular, every $S^1 \subset F$ acts holomorphically on X'. At this point, we need the following result (which we reformulate in the present paper terms):

Theorem [13] *Let M be a compact complex manifold, equipped with a holomorphic S^1 -action and an LCK metric (not necessarily compatible). Suppose that this S^1-action lifts to the universal covering $\tilde{M}$ to a non-trivial action by homotheties. Then, M admits an LCK metric with an automorphic potential.*

We apply this result to $M = X'$. Hence, if X' is not GCK, it admits a Kähler covering with Kähler metric given by an automorphic global potential. On the other hand, a compact LCK manifold admitting a Kähler covering with automorphic global potential has a small deformation of its complex structure to one which has a compatible Vaisman metric (cf [14]). □

Remark Consider a Hopf surface X with fundamental group generated by $(z_1, z_2) \mapsto (\alpha z_1, \beta z_2)$, $0 < |\alpha| \leqslant |\beta| < 1$.

As shown in [5, Thm. 1], for any choice of α and β as above, X has a Vaisman metric, while, as shown in (e.g.,) [1, Prop. 18.2, p. 226], X has an elliptic fibration if and only if there are $m, n \in \mathbb{Z}$, $(m, n) \neq (0, 0)$ such that $\alpha^n = \beta^m$. Actually, more can be shown: if no (m, n) as above exists, then the Hopf surface has finitely many curves, two, more exactly!

Accordingly, for a "general choice" of (α, β), the resulting Hopf surface will be Vaisman but not elliptic.

Finally, let us give the

Proof of Corollary 1 Assume the product $X_1 \times X_2$ has a LCK structure. As X_1 and X_2 are regular compact Vaisman manifolds, they are total spaces of holomorphic submersions $\pi_i : X_1 \longrightarrow B_i, i = 1, 2$ onto (compact Hodge) manifolds B_1, B_2 with fibers elliptic curves F_1, F_2. But then

$$\pi : X_1 \times X_2 \longrightarrow B_1 \times B_2, \quad \pi(x_1, x_2) = (\pi_1(x_1), \pi_2(x_2))$$

is a holomorphic submersion with typical fiber $F_1 \times F_2$ which is a 2-dimensional torus and is of Kähler type. As the first Betti number of a compact Vaisman manifold is odd, $b_1(X_1 \times X_2)$ is even, and we see that $X_1 \times X_2$ is not biholomorphic to a Vaisman manifold. Then, from the above Theorem, it follows that $X_1 \times X_2$ is of Kähler type. But this forces X_1, X_2 to be of Kähler type as well, which is absurd. □

3 Appendix on elliptic bundles and elliptic curves

A. Let us first recall some facts about elliptic bundles. Fix a genus one curve E and let $E_0 = (E, O)$ be the elliptic curve obtained by fixing some arbitrary point $O \in E$. Fixing O allows us to give a group structure on E. The group of automorphisms $\mathrm{Aut}(E)$ is given by the extension

$$0 \longrightarrow \mathrm{Trans}(E) \longrightarrow \mathrm{Aut}(E) \longrightarrow \mathrm{Aut}(E_0) \longrightarrow 0$$

where $\mathrm{Trans}(E)$ is the subgroup of $\mathrm{Aut}(E)$ given by translations, and $\mathrm{Aut}(E_0)$ is the group of automorphisms of E fixing O. Now, the group $\mathrm{Aut}(E_0)$ is usually $\mathbb{Z}_2$ (and consists of the antipodal map $x \mapsto -x$) except for some cases when $\mathrm{Aut}(E_0)$ is finite of order 4 or 6 (these particular kind of elliptic curves are called "curves with complex multiplication"). See [1, p. 143].

For an arbitrary complex manifold M, let $\mathrm{PBun}_E(M)$, respectively, $\mathrm{Bun}_E(M)$ be the set of principal bundles, respectively, the set of elliptic bundles on M with fiber E. The above exact sequence implies

$$0 \longrightarrow \mathrm{PBun}_E(M) \longrightarrow \mathrm{Bun}_E(M) \longrightarrow H^1(M, \mathrm{Aut}(E_0))$$

See again [1, p. 143].

As $\mathrm{Aut}(E_0)$ is a finite group, we see that for any elliptic bundle $X \longrightarrow M$, there is a finite cover M' of M such that $X' = X \times_M M'$ is a principal bundle, in other words, any elliptic bundle has a finite cover which is an elliptic principal bundle (see [1, p. 147]).

B. We now recall some classical facts about the j-invariant. Let $\mathcal{E} = \mathbb{C}/\langle 1, \tau \rangle$, $\tau \in \mathbb{C}$, $\mathrm{Im}\tau > 0$ be a framed elliptic curve. Its j-invariant is the complex function

$$j(\mathcal{E}) = j(\tau) = 1728 \frac{g_2^3(\mathcal{E})}{\Delta(\mathcal{E})},$$

where

$$g_2(\mathcal{E}) = g_2(\tau) = 60 \sum_{(m,n)\in\mathbb{Z}\setminus\{0\}} (m+n\tau)^{-4},$$
$$g_3(\mathcal{E}) = g_3(\tau) = 160 \sum_{(m,n)\in\mathbb{Z}\setminus\{0\}} (m+n\tau)^{-6},$$

these two series being known to be absolutely convergent, and

$$\Delta(\mathcal{E}) = g_2^3(\mathcal{E}) - 27g_4^2(\mathcal{E}).$$

From the very definition, the j-invariant is an analytic function of τ.

If now $\pi : X \longrightarrow T$ is an analytic family of elliptic curves, one defines $J : T \longrightarrow \mathbb{C}$ by $J(t) = j(\mathcal{E}_t)$. As T is a manifold and hence locally simply connected, we may suppose the analytic family to be analytically framed. This implies that J is analytic, as a composition of the analytic maps $\tau \mapsto j(\tau)$ and the period map $t \mapsto \tau(t)$.

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References

1. Barth, W.P., Hulek, K., Peters, C.A.M., Van de Ven, A.: Compact complex surfaces. 2nd edn. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin (2004)
2. Belgun, F.A.: On the metric structure of non-Kähler complex surfaces. Math. Ann. **317**, 1–40 (2000)
3. Brunella, M.: Locally conformally Kähler metrics on Kato surfaces. Nagoya Math. J. **202**, 77–81 (2011)
4. Dragomir, S., Ornea, L.: Locally conformal Kähler geometry, Progress in Mathematics 155. Birkhäuser, Boston, Basel (1998)
5. Gauduchon, P., Ornea, L.: Locally conformally Kähler metrics on Hopf surfaces. Ann. Inst. Fourier **48**, 1107–1127 (1998)
6. Gini, R., Ornea, L., Parton, M., Piccinni, P.: Reduction of Vaisman structures in complex and quaternionic geometry. J. Geom. Phys. **56**(12), 2501–2522 (2006)
7. Fischer, W., Grauert, H.: Lokal-triviale familien kompakter komplexer mannigfaltigkeiten. Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. **II**, 89–94 (1965)
8. Ianuş, S., Ornea, L., Vuletescu, V.: Holomorphic and harmonic maps of locally conformal Kähler manifolds. Boll. Un. Mat. Ital. A (7) **9**(3), 569–579 (1995)
9. Marrero, J.C., Rocha, J.: Locally conformal Kähler submersions. Geom. Dedicata **52**(3), 271–289 (1994)
10. Oeljeklaus, K., Toma, M.: Non-Kähler compact complex manifolds associated to number fields. Ann. Inst. Fourier (Grenoble) **55**(1), 161–171 (2005)
11. Ornea, L., Verbitsky, M.: Structure theorem for compact Vaisman manifolds. Math. Res. Lett. **10**, 799–805 (2003)
12. Ornea, L., Verbitsky, M.: A report on locally conformally Kähler manifolds. Contemp. Math. **542**, 135–150 (2011)
13. Ornea, L., Verbitsky, M.: Automorphisms of locally conformally Kähler manifolds. Int. Math. Res. Not. **4**, 894–903 (2012)
14. Ornea, L., Verbitsky, M.: Topology of locally conformally Kähler manifolds with potential. Int. Math. Res. Not. **4**, 717–726 (2010)
15. Ornea, L., Verbitsky, M.: Locally conformal Kähler manifolds with potential. Math. Ann. **348**, 25–33 (2010)
16. Ornea, L., Verbitsky, M., Vuletescu, V.: Blow-ups of locally conformally Kähler manifolds. Int. Math. Res. Not. (to appear) arxiv:1108.4885. doi:10.1093/imrn/rns128
17. Parton, M., Vuletescu, V.: Examples of non-trivial rank in locally conformal Kähler geometry. Math. Z. **270**, 179–187 (2012)

18. Tsukada, K.: The canonical foliation of a compact generalized Hopf manifold. Differ. Geom. Appl. **11**(1), 13–28 (1999)
19. Vaisman, I.: On locally and globally conformal Kähler manifolds. Trans. Am. Math. Soc. **262**, 533–542 (1980)