Annals of Global Analysis and Geometry **16:** 383–398, 1998.

Compact Hyperhermitian–Weyl and Quaternion Hermitian–Weyl Manifolds

LIVIU ORNEA*
Faculty of Mathematics, University of Bucharest, 14, Academiei str, 70109 Bucharest, Romania (E-mail: lornea@geo.math.unibuc.ro)

PAOLO PICCINNI**
Dipartimento di Matematica, Università La Sapienza, Piazzale Aldo Moro 2, I-00185 Roma, Italy (E-mail: piccinni@mat.uniroma1.it)

Abstract. Let (M, g, H) be a quaternion Hermitian manifold. The additional datum of a torsion-free connection D preserving both the quaternionic structure H and the conformal class of g defines on M the structure of *quaternion Hermitian–Weyl manifold*. Under the compactness assumption of both M and the leaves of a canonical foliation, M is here shown to project on a *locally 3-Sasakian* orbifold P. Then M is proved to admit both a compatible global complex structure and a finite covering $\overline{M}$ carrying a *hyperhermitian–Weyl* structure. The uniqueness of the Weyl structure compatible with a given quaternion Hermitian metric and some restrictions on the Betti numbers are also obtained.

Mathematics Subject Classifications (1991): 53C15, 53C25, 53C55

Key words: conformally related metrics, hyperhermitian–Weyl manifold, locally 3-Sasakian manifold, quaternion Hermitian–Weyl manifold

1. Introduction and Statement of Results

In a previous paper [24] we studied the extension in quaternionic geometry of the locally conformal Kähler condition, already well understood on complex manifolds (cf., for example, the monograph [8]). The quaternionic situation presents of course the two possibilities of choosing whether the conformality is required with local hyperkähler or with local quaternion Kähler metrics. Manifolds M^{4n} carrying such types of metrics bear two remarkable canonical foliations $\mathcal{B}$, $\mathcal{D}$ – of dimensions one and four, respectively – and the dichotomy hyperkähler–quaternion Kähler is here reflected on different structures in the fibres of $\pi : M \to N = M/\mathcal{D}$, the projection to the leaf space.

The present work is devoted to the global geometry of compact locally conformal quaternion Kähler manifolds. The main properties we obtain are collected in Theorems A and B. They display a rather strong similarity between these manifolds

* Supported by M.U.R.S.T. and C.N.R.
** Supported by M.U.R.S.T. and I.N.d.A.M.

and the subclass of compact locally conformal hyperkähler manifolds. This is in sharp contrast with the quite different global geometries carried by compact hyperkähler and compact quaternion Kähler manifolds, the two non-locally conformal counterparts of the manifolds here studied and mentioned in the title.

We begin by recalling some basic definitions.

A *Weyl manifold* $(M, [g], D)$ consists of a conformal class $[g]$ of Riemannian metrics on the $\mathcal{C}^\infty$ manifold M and of a torsion-free connection D preserving $[g]$: then $Dg = \omega \otimes g$, $\omega =$ *associated 1-form* with respect to the representative g.

In quaternionic geometry one considers manifolds M^{4n} equipped with *hypercomplex structures* (I_1, I_2, I_3) or with *quaternionic structures* H. The former are by definition triples of global integrable complex structures I_α satisfying the quaternionic identities: $I_\alpha^2 = -\mathrm{id}$, $I_\alpha I_\beta = I_\gamma$, $(\alpha, \beta, \gamma) = (1, 2, 3)$ and cyclic permutations (we shall use the abbreviation "c.p." hereafter). The latter are 3-dimensional subbundles H of $\mathrm{End}(TM)$, locally spanned by (not necessarily integrable) almost complex structures I_1, I_2, I_3, again satisfying the quaternionic identities and related on the intersections of trivializing open sets by matrices of $SO(3)$. A Riemannian metric g on a hypercomplex manifold (M, I_1, I_2, I_3) is *hyperhermitian* if it is Hermitian, *hyperkähler* if it is Kähler, with respect to I_α, $\alpha = 1, 2, 3$. Similarly, on a quaternionic manifold (M, H) the metric g is *quaternion Hermitian* if it is Hermitian with respect to the local I_α, and *quaternion Kähler* if H is parallel with respect to the Levi–Civita connection ∇^g of g. We refer to [9] and to [3] for the basic theories of Weyl manifolds and of hypercomplex and quaternionic manifolds.

The requirement of compatibility between quaternionic and Weyl structures is expressed by the following definitions. A hyperhermitian manifold (M, g, I_1, I_2, I_3) is called *hyperhermitian–Weyl* if a torsion-free connection D is given on M satisfying $Dg = \omega \otimes g$, $DI_\alpha = 0$, for $\alpha = 1, 2, 3$. A quaternion Hermitian manifold (M, g, H) is said to be *quaternion Hermitian–Weyl* if a torsion-free connection D is given on M satisfying $Dg = \omega \otimes g$, $D_X H \subset H$ for any vector field X.

The associated 1-form ω of a hyperhermitian–Weyl M^{4n} (with the requirement M compact for $n = 1$) is necessarily closed [27]. Then M is hyperhermitian–Weyl if and only if g is *locally conformal hyperkähler*, i.e.,

$$g_{|U_i} = e^{f_i} g_i'$$

with g_i' hyperkähler metrics over open neighbourhoods $\{U_i\}$ covering M. The associated 1-form ω is then locally reconstructed as $\omega_{|U_i} = df_i$.

Similarly, a M^{4n}, $n \geq 2$, is quaternion Hermitian–Weyl if and only if it is *locally conformal quaternion Kähler*, i.e., $g_{|U_i} = e^{f_i} g_i'$, with g_i' quaternion Kähler. The differentials df_i again glue together to the associated 1-form ω.

We shall use both the "Hermitian–Weyl" and the "locally conformal" terminologies, keeping in mind that the former moves the accent from the metric properties to those of the canonically associated Weyl connection.

Due to the Einstein property of the local quaternion Kähler metrics, the quaternion Hermitian–Weyl manifolds are *Einstein–Weyl*, i.e., the Ricci tensor of D is a multiple of g. In particular, the hyperhermitian–Weyl manifolds are *Ricci-flat–Weyl*. We mention that on compact hyperhermitian–Weyl and compact quaternion Hermitian–Weyl manifolds a result of [13, p. 10], allows us to choose a representative g in the conformal class so that the associated 1-form ω, if not exact, is ∇^g-parallel. This choice enables us to recognize that compact hyperhermitian–Weyl and compact quaternion Hermitian–Weyl manifolds are endowed with the following canonical Riemannian foliations: a 1-dimensional foliation $\mathcal{B}$, generated by the *Lee vector field* $B = \omega^\sharp$; a 4-dimensional foliation $\mathcal{D}$, locally spanned by B, I_1B, I_2B, I_3B; a 2-dimensional foliation $\mathcal{V}$, spanned by B, JB in the hypercomplex case, or more generally when a global complex structure J exists compatible with the quaternionic one (cf. Theorem A, (ii)).

We can now state the theorems.

THEOREM A. *Let M^{4n} be a compact locally conformal quaternion Kähler manifold that is not quaternion Kähler and such that the foliations $\mathcal{B}$ and $\mathcal{D}$ have all closed leaves. Then:*

(i) *M admits a finite locally conformal hyperkähler covering $\overline{M}$ entering in the commutative diagram:*

$$\begin{array}{ccccc} \overline{M} & \xrightarrow{S^1} & \overline{P} & \xrightarrow{S^3/H} & \overline{N} \\ \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{S^1} & P & \xrightarrow{S^3/G} & N \end{array}$$

with finite coverings as vertical arrows and Riemannian submersions over orbifolds as horizontal arrows. The orbifolds $\overline{P}$, P carry, respectively, a globally and a locally 3-Sasakian structure (cf. Section 2), and project over quaternion Kähler orbifolds with positive scalar curvature $\overline{N}$, N. The fibres of these projections are 3-dimensional spherical space forms, respectively homogeneous S^3/H (for some finite subgroup $H \subset S^3$), and generally inhomogeneous S^3/G (here $G \subset SO(4)$), in the two cases.

Accordingly, the fibres of the composite horizontal maps are locally conformal hyperkähler Hopf surfaces $\mathbf{H} - 0/\Gamma$ for the map $\overline{M} \to \overline{N}$, and real Hopf 4-manifolds $\mathbf{R}^4 - 0/\Gamma$ for $M \to N$. The groups Γ, discrete and subgroups respectively of $GL(1, \mathbf{H})$ and $GL(1, \mathbf{H}) \cdot Sp(1)$, act without fixed points on the universal covering, preserving its hypercomplex or its quaternionic structure, in the two cases.

(ii) *M admits a global integrable compatible complex structure, it is locally conformal Kähler with respect to it, and projects in 1-dimensional complex tori over the twistor space Z of N.*

THEOREM B. *Let M^{4n} be a compact quaternionic manifold.*

(i) *For each quaternion Hermitian metric on M there is at most one compatible quaternion Hermitian–Weyl structure.*

(ii) *If a quaternion Hermitian–Weyl structure exists on M such that $\mathcal{B}$ and $\mathcal{D}$ are regular foliations, then the Betti numbers of M and $N = M/\mathcal{D}$ satisfy the relations:*

$$b_{2p}(M) = b_{2p+1}(M) = b_{2p}(N) - b_{2p-4}(N), \quad (0 \leq 2p \leq 2n-2),$$

$$b_{2n}(M) = 0,$$

$$\sum_{k=1}^{n-1} k(n-k+1)(n-2k+1)b_{2k}(M) = 0.$$

Let us mention a few points entering in the proofs.

A basic fact concerning compact quaternion Hermitian–Weyl and non-quaternion Kähler manifolds is the Ricci-flatness of the Weyl structure. This follows from a result of Gauduchon [13, p. 10], and it is proved also in [24]. Another ingredient is the notion of locally 3-Sasakian manifold, here introduced to describe the structure of the leaf space $M/\mathcal{B}$. The simplest examples of locally 3-Sasakian manifolds are quotients of spheres S^{4n+3}/G, where $G \subset SO(4)$, $G \not\subset S^3$ is a finite group acting freely and diagonally. The topological constrains in Theorem B extend results known in the hyperhermitian–Weyl case [11, 24]. Examples of compact quaternion Hermitian–Weyl manifolds are indicated in [23, 24].

It is also worth mentioning that the hypotheses of closed leaves and of regularity for foliations on compact manifolds assure to have leaf spaces that are respectively orbifolds or manifolds. The definition and basic properties of orbifolds, also in connection with foliations with all the leaves compact, can be found in [20, 34], or in the recent survey [6]. For convenience, we recall here that a k-dimensional foliation $\mathcal{F}$ on a $\mathcal{C}^\infty$ manifold M is said to be *regular* if each $p \in M$ has a neighbourhood U such that: (a) U is a *cubical neighbourhood centered at* p, i.e., the local coordinates $(x_1, \dots, x_n)$ of U satisfy $|x_i| \leq a$ and $m = (0, 0, \dots, 0)$; (b) U is *flat with respect to* $\mathcal{F}$, i.e., $\partial/\partial x_1, \dots, \partial/\partial x_k$ is a basis for the tangent spaces to $\mathcal{F}$ in U; and (c) each leaf of $\mathcal{F}$ intersects U in at most one *p-dimensional slice* $x_{k+1} = t_{k+1}, \dots, x_n = t_n$ ($t_{k+1}, \dots, t_n$ constants). This hypothesis, on compact manifolds M, allows us to recognize both the compactness of all the leaves and the property of $\mathcal{C}^\infty$ submersion of the projection $M \to M/\mathcal{F}$ to the leaf space, cf. [26].

2. Locally 3-Sasakian Manifolds

We introduce the following class of manifolds:

DEFINITION 2.1. *Let (P, g) be a Riemannian manifold with tangent bundle TP, and let $K \subset TP$ be a rank 3 vector subbundle. Then (P, g, K) is said to be a* locally 3-Sasakian manifold *if the following conditions are satisfied:*

(i) K is locally spanned by orthonormal Killing vector fields X_1, X_2, X_3, defined over open sets $U \subset P$ such that $[X_\alpha, X_\beta] = 2X_\gamma$ for $(\alpha, \beta, \gamma) = (1, 2, 3)$ and c.p. On the intersections $U \cap U'$:

$$X'_\lambda = \sum_\mu f_{\lambda\mu} X_\mu,$$

and $(f_{\lambda\mu}) : U \cap U' \to SO(3)$ are $\mathcal{C}^\infty$ functions.

(ii) The local tensor fields $F_\alpha = \nabla X_\alpha$, $\alpha = 1, 2, 3$ and ∇ = Levi–Civita connection of g, satisfy

$$(\nabla_Y F_\alpha)Z = \eta_\alpha(Z)Y - g(Y, Z)X_\alpha,$$

where $\eta_\alpha = X^\sharp_\alpha$.

When $K \to P$ is globally trivialized by such Killing vector fields X_1, X_2, X_3, then P is a *3-Sasakian manifold*. In [7] and the bibliography therein all the useful information is given concerning (globally) 3-Sasakian manifolds. In particular, examples of compact 3-Sasakian manifolds with a wide range of topologies have been constructed on structure $SO(3)$-bundles over orbifolds carrying a quaternion Kähler metric of positive scalar curvature. For each of these examples, a hyperhermitian–Weyl structure exists on any flat principal S^1-bundle over it. Recall also that 3-Sasakian manifolds are Einstein with positive scalar curvature [17]. Moreover, if the leaves of the foliation locally spanned by X_1, X_2, X_3 are compact, they project over positive quaternion Kähler orbifolds with fibres homogeneous 3-dimensional spherical space forms [7, 15].

The following properties, established in [7, 16] in the global case, hold good with respect to the local Killing vector fields X_1, X_2, X_3 on locally 3-Sasakian manifolds:

$$\nabla_{X_\alpha} X_\beta = X_\gamma, \tag{2.1a}$$

$$d\eta_\alpha(Y, Z) = 2g(F_\alpha Y, Z), \tag{2.1b}$$

$$g(F_\alpha Y, Z) + g(Y, F_\alpha Z) = 0, \tag{2.1c}$$

$$F_\alpha \circ F_\beta = X_\alpha \otimes \eta_\beta - F_\gamma, \tag{2.1d}$$

for any cyclic permutation of $(\alpha, \beta, \gamma) = (1, 2, 3)$ and any vector fields Y, Z.

LEMMA 2.2. *The transition functions of the vector bundle $K \to P$ over a locally 3-Sasakian manifold P can be chosen to be locally constant.*

Proof. Let (X_1, X_2, X_3), (X_1', X_2', X_3') be local orthonormal triples of Killing vector fields defining the local 3-Sasakian structure. Then $X_\lambda' = \sum_\rho f_{\lambda\rho} X_\rho$, $X_\mu' = \sum_\sigma f_{\mu\sigma} X_\sigma$ and hence:

$$[X_\lambda', X_\mu'] = \sum_{\rho,\sigma}[f_{\lambda\rho} X_\rho(f_{\mu\sigma}) - f_{\mu\rho} X_\rho(f_{\lambda\sigma})]X_\sigma + \sum_{\rho,\sigma} f_{\lambda\rho} f_{\mu\sigma}[X_\rho, X_\sigma].$$

Moreover, since $(f_{\lambda\mu}) \in SO(3)$ and for $(\lambda, \mu, \nu) = (1, 2, 3)$ and cyclic permutations:

$$\begin{aligned}\sum_{\rho,\sigma} f_{\lambda\rho} f_{\mu\sigma}[X_\rho, X_\sigma] &= \sum_{(\rho,\sigma,\tau)} (f_{\lambda\rho} f_{\mu\sigma} - f_{\lambda\sigma} f_{\mu\rho})[X_\rho, X_\sigma] \\ &= 2 \sum_{(\rho,\sigma,\tau)} (f_{\lambda\rho} f_{\mu\sigma} - f_{\lambda\sigma} f_{\mu\rho}) X_\tau = 2X_\nu'.\end{aligned}$$

It follows:

$$\sum_\rho [f_{\lambda\rho} X_\rho(f_{\mu\sigma}) - f_{\mu\rho} X_\rho(f_{\lambda\sigma})] = 0,$$

that is for any $\lambda, \mu, \sigma = 1, 2, 3$: $X_\lambda'(f_{\mu\sigma}) - X_\mu'(f_{\lambda\sigma}) = 0$, and then also: $X_\lambda(f_{\mu\sigma}') - X_\mu(f_{\lambda\sigma}') = 0$. It follows:

$$X_\lambda(f_{\sigma\mu}) - X_\mu(f_{\sigma\lambda}) = 0. \tag{2.2a}$$

On the other hand, the Killing condition

$$\sum_\mu X_\rho(f_{\lambda\mu}) g(X_\mu, X_\sigma) + \sum_\mu X_\sigma(f_{\lambda\mu}) g(X_\mu, X_\rho) = 0,$$

yields for all $\lambda, \rho, \sigma = 1, 2, 3$:

$$X_\rho(f_{\lambda\sigma}) + X_\sigma(f_{\lambda\rho}) = 0. \tag{2.2b}$$

Then from formulae (2.2a) and (2.2b) we have for all $\lambda, \rho, \sigma = 1, 2, 3$:

$$X_\sigma(f_{\lambda\rho}) = 0,$$

as to be proved. □

REMARK 2.3. The flatness of the vector bundle $K \to P$, given by Lemma 2.2, can be applied in particular to the case dim $P = 3$, so that the whole tangent bundle TP is flat. On the other hand, the 3-dimensional globally 3-Sasakian manifolds are known to reduce to homogeneous spherical space forms [32], so that it is natural to compare the flatness of TP with the fact that any 3-dimensional spherical space form is orientable [34, p. 452], and hence parallelizable by a classical theorem of Stiefel. In this respect, note that no parallelization of TP by a triple of orthonormal Killing vector fields giving a global 3-Sasakian structure is generally guaranteed.

This happens for the inhomogeneous 3-dimensional spherical space forms, that are parallelizable but locally and non-globally 3-Sasakian (cf. the discussion following Proposition 3.2).

The flatness of the vector bundle $K \to P$, given by Lemma 2.2, shows that the dichotomy hyperkähler–quaternion Kähler of quaternionic geometry finds in the above definitions of globally and locally 3-Sasakian manifolds only a partial odd-dimensional analogue. As will be better clarified by Lemma 4.1, locally 3-Sasakian manifolds parallel in some respects the role of *locally hyperkähler manifolds* in quaternionic geometry [3, 19].

By the mentioned curvature properties of globally 3-Sasakian manifolds, and by Myers' theorem, we have:

LEMMA 2.4. *Let (P, g, K) be a complete locally 3-Sasakian manifold. Then P is a compact Einstein manifold with positive scalar curvature.*

Any locally 3-Sasakian manifold bears a canonical 3-dimensional Riemannian foliation $\mathcal{K}$, locally spanned by the Killing vector fields X_1, X_2, X_3. If all the leaves of $\mathcal{K}$ are compact, then the leaf space $N = P/\mathcal{K}$ is an orbifold. For each section $\sigma : V \subset N \to P$ and triple X_1, X_2, X_3 of Killing vector fields defined on V the $(1,1)$-tensors $F_\alpha = \nabla X_\alpha$, $\alpha = 1, 2, 3$, define on V an almost hypercomplex structure J_1, J_2, J_3. This is given by the formula:

$$J_\alpha(Y_p) = d\pi\left(F_{\alpha|\sigma(p)}(\tilde{Y}_{\sigma(p)})\right),$$

where $\tilde{Y}_{\sigma(p)}$ is the unique horizontal lift of Y_p. Then local almost hypercomplex structures defined on N either by different sections $\sigma, \sigma' : V \cap V' \subset N \to P$ or by different local 3-Sasakian structures on $U \cap U' \subset P$ are related on $V \cap V'$ by matrices of $SO(3)$. Also, since $\mathcal{K}$ is spanned by local Killing vector fields, the metric g of P projects to a metric g_N that is quaternion Kähler in the quaternionic structure given by the local almost hypercomplex structures (J_1, J_2, J_3). Therefore, the fibrations studied in [7, 15] for global 3-Sasakian manifolds can be extended to our case as follows:

PROPOSITION 2.5. *Let (P, g, K) be a locally 3-Sasakian manifold such that all the leaves of $\mathcal{K}$ are compact. Then P projects over the quaternion Kähler orbifold $N = P/\mathcal{K}$ of positive scalar curvature, and the fibres are (generally inhomogeneous) 3-dimensional spherical space forms.*

3. Proof of Theorems A (ii) and B

Let now $\pi : M \to P$ be a flat principal S^1-bundle over a compact locally 3-Sasakian manifold P. If u is a closed 1-form on M defining the flat connection of the bundle, define the metric $g_M = \pi^* g_P + u \otimes u$. Then g_M is Hermitian with respect to the quaternionic structure H having the following compatible almost complex structures:

$$I_\alpha Y = -F_\alpha Y - \eta_\alpha(Y)B, \quad I_\alpha B = X_\alpha \tag{3.1a}$$

(Y horizontal vector field and $B = u^\sharp$). A quaternion Hermitian–Weyl structure is then defined on M by the torsion-free connection

$$D_X Y = \nabla^M_X Y - \frac{1}{2}\{u(X)Y + u(Y)X - g(X,Y)u^\sharp\}, \tag{3.1b}$$

where X, Y are any two vector fields on M. On the other hand, compact quaternion Hermitian–Weyl manifolds are known to be Ricci-flat–Weyl [13, 24], hence *locally conformal locally hyperkähler*. Note that this terminology does not require the existence of any global hypercomplex structure (that is assumed in the *locally conformal hyperkähler* case). The vector bundle $H \to M$ is thus flat with respect to the Levi–Civita connection ∇^M of g_M. Hence, similarly to the complex case [36], the following correspondence is deduced:

THEOREM 3.1. *The class of compact quaternion Hermitian–Weyl manifolds M, that are not quaternion Kähler and have the foliation $\mathcal{B}$ regular, coincides with that of flat principal S^1-bundles over compact locally 3-Sasakian manifolds $P = M/\mathcal{B}$.*

In particular, the compact hyperhermitian–Weyl manifolds M, not hyperkähler and having $\mathcal{B}$ regular, coincide with the flat principal S^1-bundles over compact globally 3-Sasakian manifolds $P = M/\mathcal{B}$.

Recall now that if the leaves of $\mathcal{D}$ are compact, there exists a fibration $M \to N$ over a quaternion Kähler orbifold of positive scalar curvature, with fibres real Hopf 4-manifolds, i.e., $(\mathbf{R}^4 - 0)/\Gamma$, Γ a discrete subgroup of $GL(1,\mathbf{H}) \cdot Sp(1) \cong CO^+(4)$ acting without fixed points (cf. [24] as well as the statement of Theorem A). This, together with the above statement and with Proposition 2.5, gives:

PROPOSITION 3.2. *Let M be a compact quaternion Hermitian–Weyl manifold having all the leaves of $\mathcal{B}$ and of $\mathcal{D}$ compact. Then the projection $M \to N = M/\mathcal{D}$ can be obtained by composition:*

$$M \xrightarrow{S^1} P \longrightarrow N$$

through the locally 3-Sasakian orbifold $P = M/\mathcal{B}$, that fibres over the quaternion Kähler basis N in generally inhomogeneous 3-dimensional spherical space forms.

If the foliations $\mathcal{B}$, $\mathcal{D}$ are assumed to be regular, then P, N are manifolds. However, even in the orbifold case, the quoted result from [24] assures that the leaves of $\mathcal{D}$ are Hopf real 4-manifolds. These are examples of *integrable* quaternionic manifolds, i.e., they admit a local coordinate system such that the Jacobian matrices of the coordinate transformations belong to the quaternionic group $GL(1, \mathbf{H}) \cdot Sp(1)$. Their universal covering $\mathbf{H} - 0$ is in fact, in accordance to a well known result, an open set of the quaternionic projective line $\mathbf{H}P^1$ (cf. [18] as well as [3, p. 411]). For a discussion of admissible groups acting on $\mathbf{H} - 0$ to obtain Hopf real 4-manifolds, see [24, 38].

Thus, in order to clarify the structure of compact quaternion Hermitian–Weyl manifolds, a more precise description of the projection $P \to N$ may be useful. An essential step is the study of the leaves of the foliation $\mathcal{K}$, that are 3-dimensional spherical space forms. Their classification, that goes back to the works of Seifert and Threlfall and of Hattori, is summarized in [39, pp. 226–227]. They also appear naturally in the context of 3-dimensional geometric structures according to Thurston (cf., for example, [34, pp. 449–457]).

We now recall some aspects of this classification in order to point out how the globally 3-Sasakian structure of S^3 induces similar structures on the space forms S^3/G with G finite subgroup of $SO(4)$. The *finite subgroups* $H \subset S^3$ (besides the identity, they are cyclic groups of any order, or binary dihedral, tetrahedral, octahedral and icosahedral groups) yield the *homogeneous* 3-dimensional spherical space forms S^3/H, all carrying a global 3-Sasakian structure (cf. [7, 32]).

The problem of the whole classification of 3-dimensional spherical space forms is in fact to classify all the finite subgroups $G \subset SO(4) \cong Sp(1) \cdot Sp(1)$ that act freely on S^3.

This can be done through the following result (cf. [34, thm. 4.10 and the subsequent classification]):

PROPOSITION 3.3. *Let G be a finite subgroup of $SO(4)$ acting freely on S^3. Then G is conjugate in $SO(4)$ to a subgroup of $\Gamma_1 = U(1) \cdot Sp(1)$ or of $\Gamma_2 = Sp(1) \cdot U(1)$.*

It is relevant to us that both Γ_1 and Γ_2 are isomorphic to $U(2)$ through the right and left isomorphisms $\mathbf{H} \cong \mathbf{C}^2$. It follows that any finite subgroup $\Gamma \subset \Gamma_1, \Gamma_2$ preserves two structures on $S^3 \subset \mathbf{C}^2$: the local 3-Sasakian structure induced by the hypercomplex structure of $\mathbf{C}^2$, and a global Sasakian structure induced by some complex structure of $\mathbf{C}^2$ belonging to the given hypercomplex structure. Now if the subgroup Γ is altered by conjugation in $SO(4)$ – and any finite group G acting freely on S^3 is thus obtained – the same mentioned structures on S^3 are preserved, *but the global Sasakian structure has to be looked at as induced by a conjugate complex structure on* $\mathbf{R}^4$.

Therefore:

PROPOSITION 3.4. *The compact leaves of $\mathcal{K}$ carry the structure of a locally 3-Sasakian 3-dimensional spherical space form with a global Sasakian structure.*

It follows that a locally 3-Sasakian manifold P having all the leaves of $\mathcal{K}$ compact admits a global unit vector field X that is Killing and Sasakian on the leaves. Since such a X belongs to the locally 3-Sasakian distribution $K \subset TP$, we say that the global Sasakian structure is *compatible* with the locally 3-Sasakian one. This gives:

COROLLARY 3.5. *Any locally 3-Sasakian manifold P having all the leaves of $\mathcal{K}$ compact admits a global compatible Sasakian structure.*

The above discussion enables us to complete the proof of statement (ii) in Theorem A. Let M be a compact quaternion Hermitian–Weyl and non-quaternion Kähler manifold such that $\mathcal{B}$ is a regular foliation and $\mathcal{K}$ has all the leaves compact on $P = M/\mathcal{B}$. Look at M as a flat S^1-bundle over P. Let X be the global compatible Sasakian structure on P given by Corollary 3.5. Formulae (3.1a) and (3.1b) allow us then to define a global compatible almost complex structure J on M, which is parallel with respect to the torsion-free connection D. It follows that J is integrable and compatible with both the quaternion Hermitian and the Weyl structures of M. The complex manifold (M, J) is therefore Hermitian–Weyl with respect to the conformal class $[g]$ and the connection D, and a generalized Hopf manifold with respect to any metric in $[g]$ making the 1-form ω parallel.

The twistor space Z of the quaternion Kähler base N, for such manifolds M, serves also as a Kähler–Einstein base of the complex tori fibration $\mathcal{V}$ induced on M by the compatible global complex structure J. Hence:

COROLLARY 3.6. *The structure of compact quaternion Hermitian–Weyl manifolds M satisfying the hypotheses of Proposition 3.2 is described by the diagram of sphere bundles:*

$$M \xrightarrow{S^1} P \xrightarrow{S^1} Z \xrightarrow{S^2} N,$$

where P, Z, N are orbifolds carrying respectively a locally 3-Sasakian, a complex Kähler, a quaternion Kähler structure, all Einstein with positive scalar curvature. The fibres of the composition $M \to Z$ are the tori $T^1_{\mathbf{C}}$ that are leaves of the foliation $\mathcal{V}$, those of $P \to N$ are generally inhomogeneous 3-dimensional spherical space forms S^3/G.

The proof of statement (ii) in Theorem B can now be carried out. The identities involving the single Betti numbers of M are obtained from the restrictions on Betti numbers of compact quaternion Kähler manifolds (see, for example, [3, pp. 417–419]) and the Gysin sequences of the fibrations. The point is the existence of a global compatible complex structure allowing to project M over the twistor space Z of the quaternion Kähler orbifold $N = M/\mathcal{D}$. In particular, $b_1(M) = 1$, an identity satisfied by any compact Ricci-flat–Weyl manifold [28, thm. 2.4]. Cf. also [24] for the corresponding identities in the hyperhermitian–Weyl case. The last identity is obtained by applying Salamon's constrains on compact positive quaternion Kähler manifolds to the same diagram described above (cf. [11] for the hyperhermitian–Weyl case).

As regards statement (i) in Theorem B, the uniqueness of the compatible Weyl structure, we first recall that on hypercomplex manifolds there is a unique torsion-free connection preserving the three complex structures I_α. This is the *Obata connection* $\overline{D}$ whose explicit expression is given, for example, in [1]. Thus, once a hyperhermitian g is chosen on a hypercomplex (M, I_1, I_2, I_3), there is at most one Weyl structure $([g], D)$ compatible with the I_α: $D = \overline{D}$ is the unique compatible Weyl structure if and only if $\overline{D}g = \omega \otimes g$.

On a quaternionic manifold (M, H) admitting a compatible torsion-free connection the space of such connections is an affine space modelled on the space of real 1-forms [1, p. 260]. However, once a quaternion Hermitian metric is fixed, again the uniqueness of a compatible Weyl structure holds. We thank S. Marchiafava for bringing this problem to our attention.

Clearly, the statement to be proved is equivalent to the uniqueness of a torsion-free connection D preserving both H and the conformal class $[g]$. Let D_1, D_2 be two such connections, so that $D_1 g = \omega_1 \otimes g$, $D_2 g = \omega_2 \otimes g$ and the Kähler 4-form Ω of (H, g) satisfies $d\Omega = \omega_1 \wedge \Omega = \omega_2 \wedge \Omega$ [27, p. 318]. Thus if $L : \Lambda^1 T^\star M \to \Lambda^5 T^\star M$ is the multiplication by Ω, it follows $L(\omega_1 - \omega_2) = 0$. The formal adjoint Λ of L satisfies $\Lambda L = (n-1)\mathrm{id}$ [4], so that L is injective. Thus $\omega_1 = \omega_2$ and then $D_1 = D_2$ by the formula:

$$D_X Y = \nabla_X Y - \frac{1}{2}\left(\omega(X)Y + \omega(Y)X - g(X, Y)\omega^\sharp\right)$$

which expresses D in terms of ω and of the Levi–Civita connection ∇ of g. The proof of Theorem B is now complete.

REMARK 3.7 Statement (i) in Theorem B can also be deduced from the following formula, relating two torsion-free connections preserving the quaternionic structure H on a manifold M with local compatible almost hypercomplex structures (I_1, I_2, I_3):

$$D_{2X}Y - D_{1X}Y = \xi(X)Y + \xi(Y)X - \sum_\alpha [\xi(I_\alpha X)I_\alpha Y + \xi(I_\alpha Y)I_\alpha X],$$

where ξ is any 1-form on M (cf. [1, prop. 5.1]). In fact, if D_1 and D_2 preserve also the conformal class $[g]$, one has:

$$\begin{aligned} D_{2X}Y - D_{1X}Y \;=\; & \frac{1}{2}\,[\omega_1(X) - \omega_2(X)]Y \\ & + \frac{1}{2}\,[\omega_1(Y) - \omega_2(Y)]X - g(X, Y)[\omega_1^\sharp - \omega_2^\sharp], \end{aligned}$$

and the two formulae together give $\xi = \omega_1 = \omega_2 = 0$ and $D_1 = D_2$.

4. Proof of Theorem A (i) and Some Diagrams

We prove first the following:

LEMMA 4.1. *Let $\overline{K} \to \overline{P}$ be the pullback of the locally 3-Sasakian vector bundle $K \to P$ to the universal covering $\overline{P}$ of the locally 3-Sasakian manifold P. Then $\overline{K}$ is globally trivialized by a global 3-Sasakian structure on $\overline{P}$.*

Proof. Since $K \to P$ is a flat vector bundle (Lemma 2.2), we know that its pullback to $\overline{K} \to \overline{P}$ is a trivial vector bundle. This fact is not sufficient to insure that a trivialization can be given by a global 3-Sasakian structure (cf. Remark 2.3). However, the following argument can be used. There is an induced locally 3-Sasakian structure on $\overline{P}$, whose Einstein metric insures the analiticity of the data. This allows us, according to a well known result by Nomizu [22], to extend any local Killing vector field uniquely to all of $\overline{P}$. This extension gives a global 3-Sasakian structure. In fact, one global Sasakian structure, compatible with the local 3-Sasakian one, is actually given by Corollary 3.5. If $\overline{X}_1$ is the corresponding global Killing vector field, the extension $\overline{X}_2$ of a second local Sasakian vector field turn out to remain in the vector bundle $\overline{K}$ everywhere, and to be normal to $\overline{X}_1$. Moreover, it follows easily from Lemma 2.2 that $\overline{X}_2$ generates a second global Sasakian structure. Thus, $\overline{X}_3 = [\overline{X}_1, \overline{X}_2]/2$ completes the global 3-Sasakian structure trivializing $\overline{K}$. □

The proof of statement (i) in Theorem A now goes as follows. Consider the projection $M \to P$ to the leaf space $P = M/\mathcal{B}$, assuming first $\mathcal{B}$ to be regular. P is then a compact Einstein manifold with positive Ricci curvature and Myers' theorem assures that the universal covering $\overline{P}$ is compact and the fundamental group of P is finite. It follows that the pullback $\overline{K} \to \overline{P}$ of the flat vector bundle $K \to P$ is trivial. Hence, by Lemma 4.1, $\overline{P}$ is globally 3-Sasakian. Look then at the pullback $\overline{M} \to \overline{P}$ of the S^1-bundle $M \to P$. Since this is a flat principal S^1-bundle over a globally 3-Sasakian manifold, the manifold $\overline{M}$ can be endowed with a structure of hyperhermitian–Weyl manifold (cf. the remark following Theorem 3.1). By construction such hyperhermitian–Weyl structure projects to the quaternion Hermitian–Weyl structure of M. Then, under the regularity assumption for the foliation $\mathcal{B}$, Proposition 2.5 and the discussion on 3-dimensional spherical space forms complete the proof. If $\mathcal{D}$ is also a regular foliation, the two bases $\overline{N}$ and N, both simply connected compact quaternion Kähler manifolds with positive scalar curvature, necessarily coincide.

Consider now the weaker assumption in the statement of Theorem A, namely that $\mathcal{B}$ has all the leaves compact. The proof can still proceed as indicated, but some attention has to be paid to the fact that P and N are now Riemannian orbifolds. (The reference [33] contains the extension of basic concepts of Riemannian geometry to orbifolds – originally called V-manifolds – while in [34] the theory of orbifold coverings and fundamental groups is developed.) Look first at the orbifold P, again compact with positive Ricci curvature. Its *universal orbifold covering* $\overline{P}^{\text{orb}}$ inherits a structure of complete Riemannian orbifold with positive Ricci curvature. Then a result of Borzellino [5, cor. 21] assures that the diameter of $\overline{P}^{\mathrm{orb}}$ is finite. Thus, $\overline{P}^{\mathrm{orb}}$ is compact (and P has a finite orbifold fundamental group). Look again at the flat vector bundle $K \to P$. Its pull-back $\overline{K} \to \overline{P}^{\mathrm{orb}}$ is a trivial vector bundle and the arguments used in Lemma 4.1 show that $\overline{P}^{\mathrm{orb}}$ is a globally 3-Sasakian orbifold. Then, as in the case of manifolds, a flat principal S^1-bundle $\overline{M} \to \overline{P}$ is obtained, and a hyperhermitian–Weyl structure is induced on $\overline{M}$. Note that $\overline{M}$, constructed as a hyperhermitian–Weyl orbifold, is actually a finite covering *manifold* of the original quaternion Hermitian–Weyl M. This completes the proof.

Thus, if also the leaves of $\mathcal{V}$ are all compact, the following diagram is deduced from Corollary 3.6:

$$\begin{array}{ccccccc} \overline{M} & \xrightarrow{S^1} & \overline{P} & \xrightarrow{S^1} & \overline{Z} & \xrightarrow{S^2} & \overline{N} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{S^1} & P & \xrightarrow{S^1} & Z & \xrightarrow{S^2} & N \end{array} \tag{4.1}$$

Note that, if Z,N are manifolds, they necessarily coincide with $\overline{Z}$, $\overline{N}$. The vertical arrows in the diagram are in any case finite coverings. The structures carried by the orbifolds in the diagram are as follows: $\overline{P}$, $\overline{Z}$, $\overline{N}$ are globally 3-Sasakian, Kähler-Einstein, quaternion Kähler structure, respectively, all with positive scalar curvature. P is locally 3-Sasakian. By composing the second and third horizontal arrows one obtains: $\overline{P} \to \overline{N}$ with fibres S^3/H, $P \to N$ with fibres S^3/G. The finite subgroups $H \subset S^3$, $G \subset SO(4)$ are as indicated in Section 3, and the fibres are then 3-dimensional spherical space forms, respectively homogeneous and generally inhomogeneous.

With the exceptions of $\overline{Z}$ and Z, the manifolds or orbifolds in the above diagram carry a structure described by a rank 3 real vector bundle: this is the quaternionic structure H for $\overline{M}$, M, $\overline{N}$, N and the globally or locally 3-Sasakian structure K of $\overline{P}$ and P. The associated S^2-bundles (a natural Euclidean metric is defined on both H and K) are the *twistor spaces*, combining in:

$$\begin{array}{ccccccc} Z_{\overline{M}} & \xrightarrow{S^1} & Z_{\overline{P}} & \xrightarrow{S^1} & Z_{\overline{Z}} & \xrightarrow{S^2} & Z_{\overline{N}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Z_M & \xrightarrow{S^1} & Z_P & \xrightarrow{S^1} & Z_Z & \xrightarrow{S^2} & Z_N \end{array} \tag{4.2}$$

that projects over diagram (4.1).

In particular, one has a diagram of sphere bundles:

$$\begin{array}{ccccccc} Z_M & \xrightarrow{S^1} & Z_P & \xrightarrow{S^1} & Z_Z & \xrightarrow{S^2} & Z_N \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{S^1} & P & \xrightarrow{S^1} & Z & \xrightarrow{S^2} & N \end{array}$$

that pointwise and up to finite coverings reduces to the lower part of the following one:

$$\begin{array}{ccccccc}
S^7 \times S^1 & \xrightarrow{S^1} & S^7 & \xrightarrow{S^1} & \mathbf{C}P^3 & \xrightarrow{S^2} & \mathbf{H}P^1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
S^3 \times S^2 \times S^1 & \xrightarrow{S^1} & S^3 \times S^2 & \xrightarrow{S^1} & S^2 \times S^2 & \xrightarrow{S^2} & S^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^3 \times S^1 & \xrightarrow{S^1} & S^3 & \xrightarrow{S^1} & S^2 & \xrightarrow{S^2} & pt
\end{array}$$

where the two compositions $S^3 \times S^2 \times S^1 \to S^2$ are both holomorphic maps with fibres complex Hopf surfaces [30]. We prove now the following:

THEOREM 4.2. *The product $S^3 \times S^2 \times S^1$ can be endowed with a structure of conformally Ricci-flat and non-conformally flat locally conformal Kähler manifold. This can be obtained as a product of S^1 with $S^3 \times S^2$, on which the Sasakian–Einstein metric $g = (2/3)g_0 - (2/9)\eta_0 \otimes \eta_0$ is defined through the standard metric g_0 induced by S^7 and the 1-form η_0 that is the dual with respect to g_0 of the Killing vector field generating the Hopf fibration $S^7 \to \mathbf{C}P^3$.*

Proof. Observe first that $S^3 \times S^2$ is diffeomorphic to the total space X^5 of the restriction of the Hopf fibration $S^7 \to \mathbf{C}P^3$ to the quadric surface $\mathbf{C}P^1 \times \mathbf{C}P^1$, imbedded in $\mathbf{C}P^3$ by the Segre map $\{[x_0, x_1], [y_0, y_1]\} \hookrightarrow [z_0 = x_0y_0, z_1 = x_1y_0, z_2 = x_0y_1, z_3 = x_1y_1]$. Look also at the fibration $\mathbf{C}P^3 \to \mathbf{H}P^1$ and fix $[y_0, y_1] = [1, 0]$ so that the line l=$[z_0, z_1, 0, 0]$ of $\mathbf{C}P^3$ projects to the point $[1, 0] \in \mathbf{H}P^1$. Its fibre S^3 in the Hopf fibration $S^7 \to \mathbf{H}P^1$ is the family of all the circles S^1 that are fibres over l of $S^7 \to \mathbf{C}P^3$. Letting $[y_0, y_1]$ vary in $\mathbf{C}P^1$, the stated equivalence is obtained.

Next, the metric g_0 induced by S^7 on X^5 turns out to be η-Einstein as a metric projecting with totally geodesic fibres to the Kähler–Einstein metric of the quadric $\mathbf{C}P^1 \times \mathbf{C}P^1$ [3, pp. 255–256]. This means that the Ricci tensor Ric_0 of g_0 satisfies $\mathrm{Ric}_0 = 2g_0 + 2\eta_0 \otimes \eta_0$, where η_0 is the dual 1-form of the Killing vector field ξ_0 projecting S^7 to $\mathbf{C}P^3$. This induced Hopf bundle inherits a Sasakian structure from S^7 [37, thm. 3.5 and the subsequent remark]. Then a computation shows that $g = (2/3)g_0 - (2/9)\eta_0 \otimes \eta_0$ is Sasakian and satisfies the Einstein condition $\mathrm{Ric} = 4g$.

Note also that the induced Hopf S^1-bundle $\beta : X^5 \to \mathbf{C}P^1 \times \mathbf{C}P^1$ has Chern class $c_1(\beta) = i^\star\alpha = a_1 + a_2$, where $i : \mathbf{C}P^1 \times \mathbf{C}P^1 \hookrightarrow \mathbf{C}P^3$ is the inclusion and α, a_1, a_2 are the canonical generators of the $H^2(\mathbf{C}P^3)$ and of the H^2 of the two factors $\mathbf{C}P^1$ in the quadric surface. Note furthermore that X^5 is diffeomorphic to the Stiefel manifold $V_2(\mathbf{R}^4)$ of the orthonormal 2-frames in $\mathbf{R}^4$, projecting in circles S^1 over $\mathbf{C}P^1 \times \mathbf{C}P^1$ (cf. [10, p. 277] or [2, pp. 95–96]). This is recognized by the Chern class $c_1(\gamma)$ of this latter circle bundle $\gamma : V_2(\mathbf{R}^4) \to \mathbf{C}P^1 \times \mathbf{C}P^1$, that is 1/2 of the first Chern class $c_1^\star$ of the Segre surface. Since $c_1^\star = 2(a_1 + a_2)$, it follows that the bundles β and γ are isomorphic. The structure of generalized Hopf manifold on $S^3 \times S^2 \times S^1$ is then deduced as in [37] by looking at it as a product of a circle with the Sasakian–Einstein manifold $S^3 \times S^2$. □

REMARK 4.3. The locally conformal Kähler structure just described on $S^3 \times S^2 \times S^1$ is not consistent with its twistor complex structure over the Hopf surface $S^3 \times S^1$. Indeed, the properties of Hermitian metrics on twistor spaces over oriented Riemannian 4-manifolds exclude the locally conformal Kähler possibility, at least by looking at metrics defined by means of the Levi–Civita connection [21]. On the other hand, by using the Weyl connection of the Hopf surface $S^3 \times S^1$, the lifted Hermitian metric on $S^3 \times S^2 \times S^1$ turns out to be standard and locally conformal semikähler, but not locally conformal Kähler. This is obtained from formulas in the appendix of [12], namely from its lemma 12 and corollary 2, pp. 618–619. We wish to thank Paul Gauduchon for a very helpful conversation about this point.

Acknowledgement

We wish to thank the referees for their useful observations, and in particular one of them for providing a simpler proof of Lemma 2.2, using only part of the locally 3-Sasakian structure.

References

1. Alekseevskij, D. V. and Marchiafava, S.: Quaternionic structures on a manifold and subordinated structures, *Ann. di Mat. Pura e Appl.* **171** (1996), 205–273.
2. Baum, H., Friedrich, T., Grunewald, R. and Kath, I.: *Twistors and Killing Spinors on Riemannian Manifolds*, Teubner Verlagsgesellschaft, Stuttgart, 1991.
3. Besse, A.: *Einstein Manifolds*, Springer-Verlag, Berlin, 1987.
4. Bonan, E.: Isomorphismes sur une variété presque hermitienne quaternionique, in Gentili, G., Marchiafava, S. and Pontecorvo, M. (eds.), *Proc. Meeting on Quaternionic Structures in Mathematics and Physics*, SISSA, September 1994, Electr. Library EMS, http://emis.csi.it:8888/proceedings/QSMP94/index.html.
5. Borzellino, J. E.: Orbifolds of maximal diameter, *Indiana Math. J.* **42** (1993), 37–53.
6. Boyer, Ch. P. and Galicki, K.: 3-Sasakian manifolds, Preprint, January 1998.
7. Boyer, Ch. P., Galicki, K. and Mann, B.: The geometry and topology of 3-Sasakian manifolds, *J. Reine Angew. Math.* **455** (1994), 183–220.
8. Dragomir, S. and Ornea, L., *Locally Conformal Kähler Geometry*, Progress in Mathematics, Vol. 155, Birkhäuser, Boston, 1998.
9. Folland, G. B.: Weyl manifolds, *J. Differential Geom.* **4**, (1970), 145–153.
10. Friedrich, Th. and Kath, I.: Einstein manifolds of dimension five with small first eigenvalue of the Dirac operator, *J. Differential Geom.* **29**, (1989), 263–279.
11. Galicki, K. and Salamon, S.: Betti numbers of 3-Sasakian manifolds, *Geom. Dedicata* **63**, (1996), 45–68.
12. Gauduchon, P.: Structures de Weyl et théorèmes d'annulation sur une variété conforme autoduale, *Ann. Scuola Norm. Sup. Pisa* **18**, (1991), 563–629.
13. Gauduchon, P.: Structures de Weyl–Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$, *J. Reine Angew. Math.* **469**, (1995), 1–50.
14. Hernandez, G.: On hyper f-structures, *Math. Ann.* **306** (1996), 205–230.

15. Ishihara, S.: Quaternion Kähler manifolds and fibred Riemannian spaces with Sasakian 3-structure, *Kodai Math. Sem. Rep.* **25**, (1973), 321–329.
16. Ishihara, S. and Konishi, M.: Fibred Riemannian space with triple of Killing vectors, *Kodai Math. Sem. Rep.* **25**, (1973), 175–189.
17. Kashiwada, T.: A note on a Riemannian space with Sasakian 3-structure, *Natur. Sci. Rep. Ochanomizu Univ.* **22**, (1971), 1–2.
18. Marchiafava, S.: Sulle varietà a struttura quaternionale generalizzata, *Rend. Mat. Appl.* **3**, (1970), 529–543.
19. McInnes, B.: Complex symplectic geometry and compact locally hyperkählerian manifolds, *J. Math. Phys.* **34** (1993), 4857–4871.
20. Molino, P.: *Riemannian Foliations*, Progress in Mathematics, Vol. 73, Birkhäuser, Boston, 1988.
21. Muskarov, O.: Almost Hermitian structures on twistor spaces and their types, *Atti Sem. Mat. Fis. Univ. Modena* **37** (1989), 285–297.
22. Nomizu, K.: On local and global existence of Killing vector fields, *Ann. of Math.* **72** (1960), 105–120.
23. Ornea, L. and Piccinni, P.: Weyl structures on quaternionic manifolds, in Gentili, G., Marchiafava, S. and Pontecorvo, M. (eds.), *Proc. Meeting on Quaternionic Structures in Mathematics and Physics*, SISSA, September 1994, Electr. Library EMS, http://emis.csi.it:8888/proceedings/QSMP94/index.html.
24. Ornea, L. and Piccinni, P.: Locally conformal Kähler structures in quaternionic geometry, *Trans. Amer. Math. Soc.* **349** (1997), 641–655.
25. Ornea, L. and Piccinni, P.: Induced Hopf bundles and Einstein metrics, in Szenthe, J. (ed.), *New Developments in Differential Geometry*, Budapest 1996, Kluwer Academic Publishers, Dordrecht, 1996, pp. 295-306.
26. Palais, R. S.: A global formulation of the Lie theory of transformation groups, *Mem. Amer. Math. Soc.* **22** (1957), 1–123.
27. Pedersen, H. and Poon, Y. S. and Swann, A.: The Einstein–Weyl equations in complex and quaternionic geometry, *Differential Geom. Appl.* **3** (1993), 309–321.
28. Pedersen, H. and Swann, A.: Einstein–Weyl geometry, Bach tensor and conformal scalar curvature, *J. Reine Angew. Math.* **441** (1993), 99–113.
29. Piccinni, P.: Manifolds with local quaternion Kähler structures, *Rend. Mat. Appl.* **17** (1997), to appear.
30. Pontecorvo, M.: Hermitian surfaces and a twistor space of algebraic dimension 2, *Proc. Amer. Math. Soc.* **113** (1991), 177–186.
31. Salamon, S.: Quaternionic Kähler manifolds, *Invent. Math.* **67** (1982), 143–171.
32. Sasaki, S.: Spherical space forms with normal contact metric 3-structure, *J. Differential Geom.* **6** (1972), 307–315.
33. Satake, I.: The Gauss–Bonnet theorem for V-manifolds, *J. Math. Soc. Japan* **9** (1957), 464–492.
34. Scott, P.: The geometry of 3-manifolds, *Bull. London Math. Soc.* **15**, (1983), 401–487.
35. Udrişte, C.: Structures presque coquaternioniennes, *Bull. Math. Soc. Sci. Math. R. S. Roumanie* **12** (1969), 487–507.
36. Vaisman, I.: Locally conformal Kähler manifolds with parallel Lee form, *Rend. Mat. Appl.* **12** (1979), 263–284.
37. Vaisman, I.: Generalized Hopf manifolds, *Geom. Dedicata* **13** (1982), 231–255.
38. Vaisman, I. and Reischer, C.: Local similarity manifolds, *Ann. Mat. Pura Appl.* **135** (1983), 279–291.
39. Wolf, J. A.: *Spaces of Constant Curvature*, McGraw-Hill, New York, 1967.

(Received September 9, 1997; revised version March 3, 1998)