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Non-zero contact and Sasakian reduction ☆

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Abstract

We complete the reduction of Sasakian manifolds with the non-zero case by showing that Willett's contact reduction is compatible with the Sasakian structure. We then prove the compatibility of the non-zero Sasakian (in particular, contact) reduction with the reduction of the Kähler (in particular, symplectic) cone. We provide examples obtained by toric actions on Sasakian spheres and make some comments concerning the curvature of the quotients.

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1. Introduction

1.1. Sasakian manifolds

We start by briefly recalling the notion of a Sasakian manifold, sending to [4,5] for more details and examples.

Definition 1.1. A *Sasakian* manifold is a $(2n+1)$-dimensional Riemannian manifold (M, g) endowed with a unitary Killing vector field ξ such that the curvature tensor of g satisfies the equation:

$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi \tag{1.1}$$

where η is the metric dual 1-form of ξ: $\eta(X) = g(\xi, X)$.

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It can be seen that η is a contact form (with Reeb field ξ). Using the Killing property of ξ and Eq. (1.1), one defines an almost complex structure on the contact distribution $\operatorname{Ker}\eta$, by (the restriction of) $\varphi = \nabla\xi$, where ∇ is the Levi-Civita connection of g.

The following formulae are then easily deduced:

$$\varphi\xi = 0, \qquad g(\varphi Y, \varphi Z) = g(Y, Z) - \eta(Y)\eta(Z). \tag{1.2}$$

The simplest compact example is the round sphere $S^{2n-1} \subset \mathbb{C}^n$, with the metric induced by the flat one of $\mathbb{C}^n$. The characteristic Killing vector field is $\xi_p = -i\vec{p}$, i being the imaginary unit. More general Sasakian structures on the sphere can be obtained by deforming this standard structure as follows. Let $\eta_A = \frac{1}{\sum a_j |z_j|^2}\eta_0$, for $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$. Its Reeb field is $R_A = \sum a_j(x_j\partial y_j - y_j\partial x_j)$. Clearly, η_0 and η_A underly the same contact structure. Define the metric g_A by the conditions:

- $g_A(X, Y) = \frac{1}{2}d\eta_A(IX, Y)$ on the contact distribution (here I is the standard complex structure of $\mathbb{C}^n$);
- R_A is normal to the contact distribution and has unit length.

It can be seen that $S_A^{2n-1} := (S^{2n-1}, g_A)$ is a Sasakian manifold (cf. [9]). It has recently been shown in [11] that each compact Sasakian manifold admits a CR-immersion in a S_A^{2N-1}.

Sasakian manifolds, especially the Sasakian–Einstein ones, seem to be more and more important in physical theories (connected with the Maldacena conjecture). Many new examples appeared lately, especially in the work of Ch.P. Boyer, K. Galicki and their collaborators.

This growing importance of Sasakian structures was the first motivation for extending in [8] the contact (zero) reduction to this metric setting, by showing that the contact reduction is compatible with the Sasakian data.

A good procedure for contact reduction away from zero was not available when the paper [8] was written. We here complete the missing picture by showing that Willett's recently defined non-zero reduction introduced in [13] is compatible with the Sasakian data.

1.2. Contact reduction

1.2.1. Contact reduction at 0 following [1,7]

Let (M^{2n-1}, η) be an exact contact manifold: this means that η is a contact form ($\eta \wedge (d\eta)^n \neq 0$), hence its kernel is a contact structure on M.

Let R be the Reeb vector field, characterized by the conditions $\eta(R) = 1$ and $d\eta(R, \cdot) = 0$. The flow of the (nowhere vanishing) Reeb vector field preserves the contact form η.

Let $\Phi : G \times M \to M$ be an action by strong contactomorphisms of a (finite dimensional) Lie group on M: for any $f \in G$, $f^*\eta = \eta$.[1] Such a G-action by strong contactomorphisms on (M, η) always admits an equivariant momentum map $J : M \to \mathfrak{g}^*$ given by evaluating the contact form on fundamental fields: $\langle J, \xi\rangle = \eta(\xi_M)$.[2] Note the main difference towards the symplectic case: an action by contactomorphisms is automatically Hamiltonian.

It can be seen that $0 \in \mathfrak{g}^*$ is a regular value for J if and only if the fundamental fields induced by the action do not vanish on the zero level set of J. In this case, the pull back of the contact form to $J^{-1}(0)$ is basic. Let $\pi_0 : J^{-1}(0) \to J^{-1}(0)/G$ and $\iota_0 : J^{-1}(0) \hookrightarrow M$ be the canonical projection (we shall always suppose that the considered actions are free and proper, although these hypothesis can be relaxed to deal with the category of orbifolds) and inclusion respectively. Albert's reduction theorem assures the existence of a unique contact form η_0 on $J^{-1}(0)/G$ such that $\pi_0^*\eta_0 = \iota_0^*\eta$. It can be seen that the contact structure of the quotient depends only on the contact structure on M.

The Sasakian version of this result states (cf. [8]) that if M is Sasakian and G acts by isometric strong contactomorphisms, then the metric also projects to the contact quotient and the whole structure is Sasakian.

[1] If the action is proper or G is compact, this is not more restrictive than asking G to preserve only the contact structure: in the first case, one uses a Palais type argument, in the second case an invariant contact form can be found by averaging.

[2] Here and in the sequel, for a $X \in \mathfrak{g}$, X_M denotes the fundamental field it induces on M.

1.2.2. Contact reduction away from zero following [13]

For $\mu \neq 0$, the restriction of the Reeb field is no longer basic on $J^{-1}(\mu)$ with respect to the action of G_μ, hence the above scheme does not apply. This situation was corrected by Albert, but in an unsatisfactory way, see [13] for examples. Willett's method, that we now describe, is more appropriate and was already used in [6] to extend the cotangent reduction theorems in the contact context. In the above setting, for a $\mu \in \mathfrak{g}^*$, Willett calls the *kernel group of* μ, the connected Lie subgroup K_μ of G_μ with Lie algebra $\mathfrak{k}_\mu = \ker(\mu|_{\mathfrak{g}_\mu})$. One can see that $\mathfrak{k}_\mu$ is an ideal in $\mathfrak{g}_\mu$, hence K_μ is a connected normal subgroup of G_μ. The contact quotient of M by G at μ is defined by Willett as

$$M_\mu := J^{-1}(\mathbb{R}_+\mu)/K_\mu.$$

If K_μ acts freely and properly on $J^{-1}(\mathbb{R}_+\mu)$, then J is transversal to $\mathbb{R}_+\mu$ and the pull back of η to $J^{-1}(\mathbb{R}_+\mu)$ is basic relative to the K_μ-action on $J^{-1}(\mathbb{R}_+\mu)$, thus inducing a 1-form η_μ on the quotient M_μ. If, in addition, $\ker\mu + \mathfrak{g}_\mu = \mathfrak{g}$ then the form η_μ is also a contact form. It is characterized, as usual, by the identity $\pi_\mu^*\eta_\mu = i_\mu^*\eta$, where $\pi_\mu : J^{-1}(\mathbb{R}_+\mu) \to M_\mu$ is the canonical projection and $i_\mu : J^{-1}(\mathbb{R}_+\mu) \hookrightarrow N$ is the canonical inclusion.

Remark 1.1. For $\mu = 0$, Albert's and Willett's quotients coincide.

In the next section we prove the compatibility of this procedure with the metric context.

2. Main results

2.1. The reduction theorem

Theorem 2.1. *Let (M, g, ξ, η) be a $(2n-1)$-dimensional Sasakian manifold, let G be a Lie group of dimension d acting on M by strong contactomorphisms. Let $J : M \to \mathfrak{g}^*$ be the momentum map associated to the action of G and let μ be an element of the dual $\mathfrak{g}^*$. We assume that*:

1. $\operatorname{Ker}\mu + \mathfrak{g}_\mu = \mathfrak{g}$.
2. *The action of K_μ on $J^{-1}(\mathbb{R}_+\mu)$ is proper and by isometries.*
3. *J is transverse to $\mathbb{R}_+\mu$.*

Then the contact quotient

$$M_\mu = J^{-1}(\mathbb{R}_+\mu)/K_\mu$$

is a Sasakian manifold with respect to the projected metric and Reeb field.

Proof. We already know that the reduced space M_μ is a contact manifold (see [13]). What is left to be proved is that the metric g and the Reeb field ξ project on M_μ, the latter onto a Killing field such that the curvature tensor of the projected metric satisfies formula (1.1).

From the transversality condition satisfied by the momentum map one knows that $J^{-1}(\mathbb{R}_+\mu)$ is an isometric Riemannian submanifold of M (which induced metric we also denote by g). As the flow of the Reeb field leaves invariant the level sets of the momentum J, one derives that the restriction of ξ is still a unit Killing field on $J^{-1}(\mathbb{R}_+\mu)$.

In order to establish the metric properties of the canonical projection $\pi_\mu : J^{-1}(\mathbb{R}_+\mu) \to M_\mu$, we have to understand the extrinsic geometry of the submanifold $J^{-1}(\mathbb{R}_+\mu) \subset M$. The first step is to find a basis in the normal bundle of $J^{-1}(\mathbb{R}_+\mu)$. To this end we look at the direct sum $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{m}$ where $\mu|_\mathfrak{m} = 0$ (such a decomposition exists, because $\operatorname{Ker}\mu + \mathfrak{g}_\mu = \mathfrak{g}$). Let $\mathfrak{m}_M = \{X_M \mid X \in \mathfrak{m}\}$ and recall that (see [13, Theorem 1]):

$$\big(T_x J^{-1}(\mathbb{R}_+\mu) \cap \operatorname{Ker}\eta_x\big) \oplus \mathbb{R}\xi_x \oplus \mathfrak{m}_M(x) = \big(T_x \Phi^{-1}(0) \cap \operatorname{Ker}\eta_x\big) \oplus \mathbb{R}\xi_x, \tag{2.1}$$

for any $x \in J^{-1}(\mathbb{R}_+\mu)$, where Φ is the momentum map associated to the action of K_μ on M.

Let now $\{X_1, \ldots, X_k\}$ and $\{Y_1, \ldots, Y_m\}$ be two bases in $\mathfrak{k}_\mu$ and, respectively, $\mathfrak{m}$. Without loss of generality, one may suppose that the fundamental fields $\{Y_{jM}\}_{j=1,m}$ form an orthogonal basis of $\mathfrak{m}_M$, g-orthogonal on $TJ^{-1}(\mathbb{R}_+\mu) \cap \operatorname{Ker}\eta$ and that $\{X_{iM}\}_{i=1,k}$ are mutually orthogonal.

With these hypotheses, one derives that $\{\varphi X_{iM}, \varphi Y_{jM}\}$ are linearly independent in each $x \in M$ and

$$g(\varphi Y_{jM}, W) = g(\varphi X_{iM}, W) = d\eta(W, X_{iM}) = -\langle dJ(W), X_i\rangle = \langle r\mu, X_i\rangle = 0$$

for any vector field W tangent to $J^{-1}(\mathbb{R}_+\mu)$. Therefore, for any i, j, the fields $\{\varphi X_{iM}, \varphi Y_{jM}\}$ belong to the normal bundle of $J^{-1}(\mathbb{R}_+\mu)$. A simple counting of the dimensions in the relation (2.1), together with the fact that $\{\varphi X_{iM}\}$ is a basis in the normal bundle of $T\Phi^{-1}(0)$ (see the proof of [8, Theorem 3.1]), imply that $\{\varphi X_{iM}, \varphi Y_{jM}\}$ is indeed a basis of the normal bundle of $J^{-1}(\mathbb{R}_+\mu)$.[3]

Let ∇, ∇^M be the Levi-Civita covariant derivatives of $J^{-1}(\mathbb{R}_+\mu)$ and M respectively and let A_i, A_j be the Weingarten operators associated to the unitary normal sections $\varphi X_{iM}/\|X_{iM}\|$, $1 \leqslant i \leqslant k$, $\varphi Y_{jM}/\|Y_{jM}\|$, $1 \leqslant j \leqslant m$. By applying the Weingarten formula and the relation (1.1), one obtains, for any X, Y, Z tangent to $J^{-1}(\mathbb{R}_+\mu)$:

$$g(A_iY, Z) = \|X_{iM}\|^{-1}\big\{g(X_{iM}, Y)\eta(Z) - g(\varphi\nabla^M_Y X_{iM}, Z)\big\},$$
$$g(A_jY, Z) = \|Y_{jM}\|^{-1}\big\{g(Y_{jM}, Y)\eta(Z) - g(\varphi\nabla^M_Y Y_{jM}, Z)\big\}.$$

As K_μ acts by strong contact isometries, the metric g projects on a metric g^{M_μ} on M_μ with respect to which the canonical projection π_μ becomes a Riemannian submersion. We now show that the vertical distribution $\mathcal{V}$ is locally generated by the vector fields $\{X_{iM}\}$. We have indeed:

$$T_x\pi_\mu\big(X_{iM}(x)\big) = T_x\pi_\mu\big(\dot{c}(0)\big) = (\pi_\mu \dot{\circ} c)(0)$$

where $c(t) = \Phi(\exp tX_{iM}, x)$.

But $(\pi_\mu \circ c)(t) = \pi_\mu(x)$ for any t and then

$$T_x\pi_\mu\big(X_{iM}(x)\big) = 0 \quad \text{for any } x \in J^{-1}(\mathbb{R}_+\mu).$$

This proves that $\{X_{iM}\}_{1\leqslant i\leqslant k} \subset \mathcal{V}_x$ and, as $\dim \mathcal{V}_x = k$, it implies that $\{X_{iM}\}$ generate $\mathcal{V}$.

The formulae $\mathcal{L}_{X_{iM}}\xi = 0$ for $i = 1, \ldots, k$ prove that ξ is a projectable vector field and its projection ζ is a unit Killing field on the reduced space M_μ.

Let X, Y, Z be vector fields orthogonal to ζ. Using O'Neill's formulae (see [3, (9.28f)]) we derive:

$$g^{M_\mu}\big(R^{M_\mu}(X, \zeta)Y, Z\big) = g\big(R(X^h, \xi)Y^h, Z^h\big) + 2g\big(A(X^h, \xi), A(Y^h, Z^h)\big)$$
$$- g\big(A(\xi, Y^h), A(X^h, Z^h)\big) + g\big(A(X^h, Y^h), A(\xi, Z^h)\big),$$

where X^h denotes the horizontal lift of the vector field X, A is O'Neill's $(1, 2)$ tensor field given by the relation: $A(Z^h, X^h) = \textit{vertical part of}\, \nabla^M_{Z^h} X^h$ and R the curvature tensor of the connection ∇ on $J^{-1}(\mathbb{R}_+\mu)$. On the other hand:

$$g(\nabla_{Z^h}\xi, X_{iM}) = g(\varphi Z^h, X_{iM}) = d\eta(X_{iM}, Z^h) = \langle dJ(Z^h), X_{iM}\rangle = r\langle\mu, X_{iM}\rangle = 0,$$

and hence:

$$R^{M_\mu}(X, \zeta)Y = R(X^h, \xi)Y^h.$$

This completes the proof. □

Remark 2.1. Under the hypothesis of the above theorem, the dimension of the reduced space is $2n - d - m - k + 1$.

2.2. Compatibility with the Kähler reduction

We now analyze the compatibility of the non-zero Sasakian reduction with Kähler reduction using the cone construction. In particular, we obtain a relation between non-zero contact reduction and symplectic reduction.

Let $\mathcal{C}(M) = M \times \mathbb{R}_+$ be the cone over M endowed with the Kähler metric $r^2g + dr^2$. The action of G on M lifts to an action on $\mathcal{C}(M)$ by holomorphic isometries which commute with the translations along the generators (see [8] e.g.). Similarly, the action of K_μ lifts to the cone, the lifted action being the restriction of the above.

[3] $\{\varphi X_{iM}, Y_{jM}\}$ is also a basis for $T^\perp J^{-1}(\mathbb{R}_+\mu)$. Our choice is only technically motivated.

Let $J_s : \mathcal{C}(M) \to \mathfrak{g}^*$, resp. $\Phi_s : \mathcal{C}(M) \to \mathfrak{k}_\mu{}^*$ be the symplectic momentum map associated to the G-action, resp. K_μ-action on the cone. The differentials of these two momentum maps are related by the transpose ι^t of the natural inclusion $\iota : \mathfrak{k}_\mu \hookrightarrow \mathfrak{g}$, namely $T\Phi_s = \iota^t \circ TJ_s$.

We now embed M in the cone as $M \times \{1\}$ and observe that the contact momentum maps are the restrictions of the symplectic ones: $J = J_s|_{M\times\{1\}}$, resp. $\Phi = \Phi_s|_{M\times\{1\}}$. Clearly J and Φ are the contact momentum maps associated to the G, resp. K_μ-action on $M \times \{1\}$. Moreover, we have

$$\Phi = \iota^t \circ J. \tag{2.2}$$

On the other hand, we recall (see [8]) that the reduced space at 0 of the Kähler cone is the Kähler cone of the Sasakian reduced space at 0:

$$\Phi_s^{-1}(0)/K_\mu = \mathcal{C}\big(\Phi^{-1}(0)/K_\mu\big).$$

We are now prepared to prove:

Theorem 2.2. *Let (M, g, ξ, η) be a Sasakian manifold, let G be a Lie group acting on M by strong contactomorphisms and μ an element of $\mathfrak{g}^*$. Suppose that:*

- 0 *is a regular value for J_s.*
- $\operatorname{Ker}\mu + \mathfrak{g}_\mu = \mathfrak{g}$.
- *K_μ acts properly and by isometries on $\Phi^{-1}(0)$.*
- *J is transverse to $\mathbb{R}_+\mu$ and to $\mathbb{R}_-\mu$.*

Then the cone over the Sasakian quotient of M at 0 *with respect to the K_μ action is the disjoint union of the Kähler cones over the Sasakian quotients M_μ and $M_-\mu$ and the cone over the co-isotropic submanifold $J^{-1}(0)/K_\mu$:*

$$\big(\mathcal{C}(M)\big)_0 = \mathcal{C}\big(\Phi^{-1}(0)/K_\mu\big) = \mathcal{C}(M_\mu) \cup \mathcal{C}\big(J^{-1}(0)/K_\mu\big) \cup \mathcal{C}(M_-\mu).$$

Proof. Since ι^t is surjective, from (2.2) and from 0 being a regular value for J_s (and hence also for J) it follows that 0 is a regular value for Φ_s and hence for Φ.

As $\Phi^{-1}(0) = J^{-1}(\mathbb{R}\mu)$ (cf. [13, proof of Theorem 2]) and K_μ acts on $J^{-1}(0)$ and on $J^{-1}(\mathbb{R}_+\mu)$, we have the partition

$$\Phi^{-1}(0)/K_\mu = J^{-1}(\mathbb{R}_+\mu)/K_\mu \cup J^{-1}(0)/K_\mu \cup J^{-1}(\mathbb{R}_-\mu)/K_\mu.$$

We note that $(M, g, -\xi)$ is also a Sasakian manifold on which G acts by Sasakian automorphisms and the associated momentum map is $-J$. Then, if the quotient $M_{-\mu} := J^{-1}(\mathbb{R}_-\mu)/K_\mu$ exists (or, equivalently, the quotient of (M, g, ξ) at $-\mu$), it will be a Sasakian manifold according to our previous theorem. But note that two of the hypothesis of the theorem are not automatically satisfied in both cases: if K_μ acts properly on $J^{-1}(\mathbb{R}_+\mu)$ it does not necessarily act properly on $J^{-1}(\mathbb{R}_-\mu)$ and similarly for the transversality condition.

Now $J^{-1}(0)/K_\mu$ is a manifold on which the one-form η is projected. However, it is no longer a contact form and its differential will not be, in general, an exact symplectic one. Indeed, using Albert [1, Propositions 1,2], $J^{-1}(0)/K_\mu$ is contact or symplectic if and only if

$$T_x(K_\mu \cdot x) = \operatorname{Ker}(T\eta_x|_{T_xJ^{-1}(0)\cap \operatorname{Ker}\eta_x}).$$

But, in general, one has $\operatorname{Ker}(T\eta_x|_{T_xJ^{-1}(0)\cap \operatorname{Ker}\eta_x}) = T_x(G\cdot x)$. However, this implies that $J^{-1}(0)/K_\mu$ is a co-isotropic submanifold with respect to the contact form of $\Phi^{-1}(0)/K_\mu$. □

Remark 2.2. Forgetting the metric and *mutatis mutandis*, the result of Theorem 2.2 remains valid for contact manifolds.

3. Examples: actions of tori on spheres

In Willett's reduction scheme, the smallest dimension of G which produces non-trivial examples is 2. We here present some complete computations for various actions of $G = T^2$ on $M = S^7$ with the standard Sasakian structure given by the contact form $\eta = \sum (x_j\,dy_j - y_j\,dx_j)$. When possible, we briefly discuss also the reduction at zero with the same group and the cone construction (the notations for the momentum maps will be the ones used in the previous section). Generalizations to S^{2n-1} are also indicated.

Note that our examples show the dependence of the dimension of the quotient on the choice of μ.

Example 3.1. Let first T^2 act on S^7 by

$$\big((e^{it_0}, e^{it_1}), (z_0, \dots, z_3)\big) \mapsto (e^{it_0} z_0, e^{it_0} z_1, e^{it_1} z_2, e^{it_1} z_3).$$

Since G is commutative, $\mathfrak{g}_\mu = \mathfrak{g} = \mathbb{R}^2$.

For any $(r_1, r_2) \in \mathfrak{g}$ the associated infinitesimal generator is given by

$$(r_1, r_2)_{S^7}(z) = r_1(-y_0\partial_{x_0} + x_0\partial_{y_0}) + r_1(-y_1\partial_{x_1} + x_1\partial_{y_1}) + r_2(-y_2\partial_{x_2} + x_2\partial_{y_2}) + r_2(-y_3\partial_{x_3} + x_3\partial_{y_3})$$

and the momentum map $J : S^7 \to (\mathbb{R}^2)^*$ reads $J(z) = \langle (|z_0|^2 + |z_1|^2, |z_2|^2 + |z_3|^2), \cdot\rangle$.

Let $\mu : \mathbb{R}^2 \to \mathbb{R}$, $\mu = \langle v, \cdot\rangle$, $v \in \mathbb{R}^2 \setminus \{0\}$ fixed. Then:

$$J^{-1}(\mathbb{R}_+\mu) = \begin{cases} S^3\big(\sqrt{\frac{v_1}{v_1+v_2}}\big) \times S^3\big(\sqrt{\frac{v_2}{v_1+v_2}}\big), & \text{if } v_1, v_2 > 0,\\ S^3\big(\sqrt{\frac{v_1}{v_1+v_2}}\big), & \text{if } v_1 > 0, v_2 = 0,\\ S^3\big(\sqrt{\frac{v_2}{v_1+v_2}}\big), & \text{if } v_1 = 0, v_2 > 0. \end{cases}$$

For $v = (1, 0)$ $J^{-1}(\mathbb{R}_+\mu) = S^3$, $\operatorname{Ker}\mu = \mathfrak{k}_\mu = \{0\} \times \mathbb{R}$, $K_\mu = \{e\} \times S^1$. The action of K_μ on $J^{-1}(\mathbb{R}_+\mu)$ is trivial and hence $M_\mu = S^3$. In this case 0 is not a regular value of Φ—the momentum map associated to the K_μ action but, nevertheless, $\Phi^{-1}(0)$ is a submanifold of S^7 and hence the reduced space at zero, $\Phi^{-1}(0)/K_\mu$ is a Sasaki manifold. As $\Phi^{-1}(0) = S^3$ and $\mathcal{C}(S^n) = \mathbb{R}^{n+1} \setminus \{0\}$, we obtain that $(\mathcal{C}(S^7))_0 = \mathbb{R}^4 \setminus \{0\}$. Note that for this choice of μ reducing and taking the cone are commuting operations exactly as in the zero case.

For $v = (1, 1)$ we obtain: $J^{-1}(\mathbb{R}_+\mu) = S^3(1/\sqrt{2}) \times S^3(1/\sqrt{2})$, $\mathfrak{k}_\mu = \{(-x, x) \mid x \in \mathbb{R}\}$, $K_\mu = \{(e^{-it}, e^{it}) \mid e^{it} \in S^1\}$. The action of K_μ on $J^{-1}(\mathbb{R}_+\mu)$ is given by

$$\big((e^{-it}, e^{it}), z\big) \mapsto (e^{-it} z_0, e^{-it} z_1, e^{it} z_2, e^{it} z_3),$$

thus $M_\mu = S^2 \times S^3$.

We can generalize this example for $M = S^{2n+1}$ by considering the action

$$\big((e^{it_0}, e^{it_1}), z\big) = (e^{it_0} z_0, e^{it_0} z_1, e^{it_1} z_2, \dots, e^{it_1} z_n).$$

Now the momentum map is $J(z) = \langle (|z_0|^2 + |z_1|^2, \sum |z_k|^2), \cdot\rangle$. For μ as above, we have:

$$J^{-1}(\mathbb{R}_+\mu) = \begin{cases} S^3\big(\sqrt{\frac{v_1}{v_1+v_2}}\big) \times S^{2n-3}\big(\sqrt{\frac{v_2}{v_1+v_2}}\big), & \text{if } v_1, v_2 > 0,\\ S^3\big(\sqrt{\frac{v_1}{v_1+v_2}}\big), & \text{if } v_1 > 0, v_2 = 0,\\ S^{2n-3}\big(\sqrt{\frac{v_2}{v_1+v_2}}\big), & \text{if } v_1 = 0, v_2 > 0. \end{cases}$$

For the same particular choices of μ as above, we obtain as reduced spaces respectively S^3, S^{2n-3} or $S^3 \times \mathbb{C}P^{n-2}$.

Example 3.2. Let now the action be given by

$$\big((e^{it_0}, e^{it_1}), z\big) \mapsto (e^{-it_0} z_0, e^{it_0} z_1, e^{it_1} z_2, e^{it_1} z_3).$$

The infinitesimal generator of the action will be

$$(r_1, r_2)_{S^7}(z) = r_1(y_0\partial_{x_0} - x_0\partial_{y_0}) + r_1(-y_1\partial_{x_1} + x_1\partial_{y_1}) + r_2(-y_2\partial_{x_2} + x_2\partial_{y_2}) + r_2(-y_3\partial_{x_3} + x_3\partial_{y_3}).$$

The momentum map is $J(z) = \langle(|z_1|^2 - |z_0|^2, |z_2|^2 + |z_3|^2), \cdot\rangle$ and

$$J^{-1}(\mathbb{R}_+\mu) = \left\{z \in S^7 \mid \exists s > 0 \text{ such that } \begin{cases} |z_1|^2 - |z_0|^2 - sv_1 = 0, \\ |z_2|^2 + |z_3|^2 - sv_2 = 0. \end{cases}\right\} \tag{3.1}$$

For $v = (1, 0)$ we obtain

$$J^{-1}(\mathbb{R}_+\mu) = \{z \in S^7 \mid z_2 = z_3 = 0, |z_1| > |z_0|\} = S^3 \setminus \{|z_1| \leqslant |z_0|\}.$$

The action of $K_\mu = \{e\} \times S^1$ on $J^{-1}(\mathbb{R}_+\mu)$ is trivial, thus $M_\mu = S^3 \setminus \{|z_1| \leqslant |z_0|\}$, an open submanifold of S^3. For $v = (1, 1)$, solving for s the equations in (3.1) gives $s \in (0, 1/2]$. Hence:

$$\begin{aligned} J^{-1}(\mathbb{R}_+\mu) &\simeq \left(S^1\left(\frac{1}{\sqrt{2}}\right) \times S^5\left(\frac{1}{\sqrt{2}}\right)\right) \Big\backslash \left\{z \in S^7 \mid |z_0|^2 = \frac{1}{2}\right\} \\ &\simeq S^1\left(\frac{1}{\sqrt{2}}\right) \times \left(S^5\left(\frac{1}{\sqrt{2}}\right) \Big\backslash S^1\left(\frac{1}{\sqrt{2}}\right)\right) \end{aligned}$$

an open submanifold of the product of spheres.

The action of K_μ on $J^{-1}(\mathbb{R}_+\mu)$ is given by

$$\big((e^{-it}, e^{it}), z\big) \mapsto (e^{it}z_0, e^{-it}z_1, e^{it}z_2, e^{it}z_3).$$

Let A denote the set $\{z \in S^7(\sqrt{2}) \mid 0 < |z_2|^2 + |z_3|^2 \leqslant 1\}$. Obviously, the above action of K_μ can be understood on the whole $\mathbb{C}^4$ and, as such, restricts to an action on A. Then M_μ is diffeomorphic with $(S^1 \times S^5) \cap A/K_\mu$. To identify the quotient, let $g : (S^1 \times S^5) \cap A \to (S^1 \times S^5) \cap A$ be given by

$$(z_0, z_1, z_2, z_3) \mapsto (z_0, z_1^{-1}, z_2, z_3).$$

g induces a map from $((S^1 \times S^5) \cap A)/S^1$ (with respect to the diagonal action of S^1) to $((S^1 \times S^5) \cap A)/K_\mu$. The map

$$(z_0, \ldots, z_3) \mapsto (\bar{z}_1 z_0, z_1, \bar{z}_1 z_2, \bar{z}_1 z_3)$$

is a diffeomorphism of $(S^1 \times S^5) \cap A$ equivariant with respect to the diagonal action of S^1 and the action of S^1 on the first factor. Hence M_μ is diffeomorphic to $S^5(1/\sqrt{2}) \setminus \mathrm{pr}\{z \in S^7 \mid |z_0|^2 = 1/2\} \simeq S^5(1/\sqrt{2}) \setminus S^1(1/\sqrt{2})$, where $\mathrm{pr} : \mathbb{C}^4 \to \mathbb{C}^3$, $\mathrm{pr}(z_0, \ldots, z_3) = (z_0, z_2, z_3)$.

If we change the action on z_0 with $e^{-ikt}z_0$, the reduced space will be the above one quotiented by $\mathbb{Z}^k$ (see also [8, Example 4.2]).

Example 3.3. Let us take this time:

$$\big((e^{it_0}, e^{it_1}), z\big) \mapsto (e^{it_0}z_0, e^{it_1}z_1, e^{it_1}z_2, e^{it_1}z_3),$$

whose infinitesimal generator is

$$(r_1, r_2)_{S^7}(z) = r_1(-y_0\partial x_0 + x_0\partial y_0) + r_2\sum_{j=1}^{3}(-y_j\partial x_j + x_j\partial y_j).$$

The momentum map is:

$$J(z) = \big\langle\big(|z_0|^2, |z_1|^2 + |z_2|^2 + |z_3|^2\big), \cdot\big\rangle.$$

For $J^{-1}(\mathbb{R}_+\mu)$ we obtain the following possibilities:

$$J^{-1}(\mathbb{R}_+\mu) = \begin{cases} S^1\big(\sqrt{\frac{v_1}{v_1+v_2}}\big) \times S^5\big(\sqrt{\frac{v_2}{v_1+v_2}}\big), & \text{if } v_1, v_2 > 0, \\ S^5\big(\sqrt{\frac{v_2}{v_1+v_2}}\big), & \text{if } v_1 = 0, v_2 > 0, \\ S^1\big(\sqrt{\frac{v_1}{v_1+v_2}}\big), & \text{if } v_2 = 0, v_1 > 0. \end{cases} \tag{3.2}$$

In particular, for $v=(1,0)$, $M_\mu = S^1$, for $v=(0,1)$, $M_\mu = S^5$ and for $v=(1,1)$ one obtains the same quotient as in the preceding example.

Example 3.4. Considering the weighted action of T^2 on S^7 given this time by

$$\big((e^{it_0}, e^{it_1}), z\big) \mapsto (e^{it_0\lambda_0}z_0, e^{it_1\lambda_1}z_1, z_2, z_3),$$

one obtains the momentum map

$$J(z) = \big\langle \big(\lambda_0|z_0|^2, \lambda_1|z_1|^2\big), \cdot \big\rangle.$$

For $v=(0,1)$ and λ_1 strictly positive, the reduced space is $S^5 \setminus S^3$ if $\lambda_0 \neq 0$ and $S^7 \setminus S^5$ if $\lambda_0 = 0$.

The cone construction is verified in this case. Indeed, $J^{-1}(0) = S^3$ and

$$\big(\mathcal{C}(S^7)\big)_0 \simeq \mathcal{C}(S^5) = \mathcal{C}(S^3) \cup \mathcal{C}(S^5 \setminus S^3).$$

If $v=(1,1)$ and $\lambda_0, \lambda_1 > 0$,

$$J^{-1}(\mathbb{R}_+\mu) = \left\{ z \in S^7 \;\middle|\; |z_1| = \sqrt{\frac{\lambda_0}{\lambda_1}}|z_0|,\ z_0 \neq 0 \right\} = S^7 \cap (\mathbb{C}^* \times A) \tag{3.3}$$

where A is the ellipsoid of equation

$$|z_1|^2\left(1 + \frac{\lambda_1}{\lambda_0}\right) + |z_2|^2 + |z_3|^2 = 1.$$

The action of K_μ on $J^{-1}(\mathbb{R}_+\mu)$ is given by

$$\big((e^{-it}, e^{it}), z\big) \mapsto (e^{-it\lambda_0}z_0, e^{it\lambda_1}z_1, z_2, z_3)$$

and the reduced space

$$M_\mu = \bigcup_{(z_2,z_3)\in \mathrm{pr}(J^{-1}(\mathbb{R}_+\mu))} S^1(\beta^{-\lambda_0}\alpha^{\lambda_1}) \times \big\{(z_2, z_3)\big\}$$

where $\mathrm{pr}: \mathbb{C}^4 \to \mathbb{C}^2$, $\mathrm{pr}(z_0, \ldots, z_3) = (z_2, z_3)$, $\beta = \sqrt{\frac{\lambda_0(1-|z_2|^2-|z_3|^2)}{\lambda_0+\lambda_1}}$ and $\alpha = \sqrt{\frac{\lambda_1(1-|z_2|^2-|z_3|^2)}{\lambda_0+\lambda_1}}$.

If $[z] = [z']$ in the reduced space then $z_2 = z_2'$ and $z_3 = z_3'$. So let (z_2, z_3) be fixed in $\mathrm{pr}(J^{-1}(\mathbb{R}_+\mu))$. $z \in J^{-1}(\mathbb{R}_+\mu)$ and $\mathrm{pr}(z) = (z_2, z_3)$ imply $|z_0| = \alpha$ and $|z_1| = \beta$. The action of K_μ on $J^{-1}(\mathbb{R}_+\mu)$ is in fact the diagonal action of S^1 on the first two coordinates. Let $f: (S^1(\alpha) \times S^1(\beta) \times \{(z_2, z_3)\})/S^1 \to S^1(\alpha^{\lambda_1}\beta^{-\lambda_0})$ be the map given by

$$[z] \mapsto z_0^{\lambda_1} z_1^{\lambda_0}.$$

One can easily check that f is a diffeomorphism.

In the previous examples, the Reeb flow on the reduced space is the restriction of the canonical one of the standard sphere. In this latter case, we obtain a non-standard Reeb flow.

We now write the flow of the Reeb field of the reduced contact form on M_μ (for $v=(1,1)$). Let $r(t) = \left(\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\right)$, $Z = (z_0^2, z_0^3)^t$. Then the flow is written as

$$\varphi_t = \big(Ae^{i(a+bt)}, R(t)Z\big),$$

where

$$\begin{aligned} &A = \|z_0^0\|^{\lambda_1}\|z_0^1\|^{\lambda_0}, \\ &a = \lambda_1 v_0 + \lambda_0 v_1, \quad \text{with } v_0 = \arg(z_0^0),\ v_1 = \arg(z_0^1), \\ &b = \lambda_1 + \lambda_0, \\ &R(t) = \mathrm{diag}\big(r(t), r(t)\big). \end{aligned}$$

4. The sectional curvature of the quotient

4.1. Contact CR submanifolds

In order to evaluate the sectional curvature of the Sasakian reduced space, both at 0 and away from 0, it will be convenient to place ourselves in a slightly more general situation. We first recall the following definition (see e.g. [2]):

Definition 4.1. Let (M, g_M, ξ) be a Sasakian manifold. An isometric submanifold N is called *contact CR* or *semi-invariant* if it admits two mutually orthogonal distributions D and $D^\perp$, such that:

(1) TN decomposes orthogonally as: $TN = D \oplus D^\perp \oplus \langle \xi \rangle$ and
(2) $\varphi D = D, \quad \varphi D^\perp \subseteq T^\perp N$.

We see that, in general, the normal bundle of the submanifold also splits into two orthogonal distributions: $\varphi D^\perp$ and its orthogonal complement that we denote by ν and which is invariant at the action of φ. We then have:

$$TM_{|N} = D \oplus D^\perp \oplus \langle \xi \rangle \oplus \varphi D^\perp \oplus \nu.$$

For a vector field V normal to N we shall denote $\bar{V}$, respectively $\tilde{V}$ its component in $\varphi D^\perp$, respectively in ν.

Such submanifolds have been extensively studied in the last thirty years.

Obviously, very natural examples are the level sets of Sasakian momentum maps. To better mimic our situation, we moreover make the following:

Assumption. There exists a Riemannian submersion $\pi : N \to P$ over a Sasakian manifold (P, g_P, ζ) such that:

(1) $D \oplus \langle \xi \rangle$ represents the horizontal distribution of the submersion; (and hence $D^\perp$ represents the vertical distribution of the submersion);
(1) The two Reeb fields are π-related: ξ is basic and projects over ζ.

This situation was already considered by Papaghiuc in [12], on the model of Kobayashi's paper [10] where the similar setting was discussed in Kählerian context.

Let $\phi := \nabla^P \zeta$ and observe that in our assumption we have $(\phi X)^h = \varphi X^h$.

We want to relate the sectional curvature of planes generated by orthonormal pairs $\{X, \phi X\}$, respectively $\{X^h, \varphi X^h\}$. This is usually known as φ-sectional curvature, the analogue in Sasakian geometry of holomorphic sectional curvature; it completely determines the curvature tensor, cf. [4], so it is worth having information about it.

We first apply (as in the proof of Theorem 2.1) O'Neill's formula to relate the curvatures of N and P. For X tangent to P and orthogonal to ζ, (this is not restrictive, as the planes passing through the Reeb field have sectional curvature 1 on a Sasakian manifold), using the anti-symmetry of the tensor A, we obtain:

$$R^N(X^h, \varphi X^h, X^h, \varphi X^h) - R^P(X, \phi X, X, \phi X) = -3\|A(X^h, \varphi X^h)\|_N^2, \tag{4.1}$$

where the sub-index refers to the norm with respect to g^N.

The next step is to apply the Gauss equation to the Riemannian submanifold N of M:

$$\begin{aligned} &R^M(X^h, \varphi X^h, X^h, \varphi X^h) - R^N(X^h, \varphi X^h, X^h, \varphi X^h) \\ &\quad = \|h(X^h, \varphi X^h)\|_M^2 - g_M\big(h(X^h, X^h), h(\varphi X^h, \varphi X^h)\big). \end{aligned} \tag{4.2}$$

We now need to relate the tensors A and h. To this end, we write hE, respectively vE for the horizontal, respectively vertical part of a tangent (to N) vector field E and we first decompose

$$\nabla^M_{X^h}(\varphi Y^h) = h\nabla^M_{X^h}(\varphi Y^h) + A(X^h, \varphi Y^h) + h(X^h, \varphi Y^h).$$

Then we use the formulae $(\nabla^M_E \varphi)F = \eta(F)E - g_M(E, F)\xi$ (see [4]) and $(\nabla^M_E \varphi)F = \nabla^M_E(\varphi F) - \varphi\nabla^M_E F$ to express $\nabla^M_{X^h}(\varphi Y^h)$. Finally, equaling the tangent and normal parts in the equation we obtain this way, we arrive at the following relations:

$$A(X^h, \varphi Y^h) = \nu\varphi h(X^h, Y^h),$$
$$h(X^h, \varphi Y^h) = \varphi A(X^h, Y^h) + \varphi h\widetilde{(X^h, Y^h)}. \tag{4.3}$$

Note that if $\varphi D^\perp = T^\perp N$ (i.e., $\nu = \{0\}$), and this is the case when N is the *zero* level set of a Sasakian momentum map, the above relations simplify to:

$$A(X^h, \varphi Y^h) = \varphi h(X^h, Y^h),$$
$$h(X^h, \varphi Y^h) = \varphi A(X^h, Y^h). \tag{4.4}$$

In the general case, from (4.3) we easily derive:

$$h(\varphi X^h, \varphi Y^h) = \overline{h(X^h, Y^h)} - h\widetilde{(X^h, Y^h)},$$

and hence

$$g_M\big(h(\varphi X^h, \varphi Y^h), h(X^h, Y^h)\big) = \big\|\overline{h(X^h, Y^h)}\big\|_M^2 - \big\|h\widetilde{(X^h, Y^h)}\big\|_M^2. \tag{4.5}$$

From Eq. (1.2) it follows that on the orthogonal complement of ξ, the tensor φ acts like an isometry. Therefore, using again (4.3), we derive:

$$\big\|h(X^h, \varphi Y^h)\big\|_M^2 = \big\|A(X^h, Y^h)\big\|_M^2 + \big\|h\widetilde{(X^h, Y^h)}\big\|_M^2,$$
$$\big\|A(X^h, \varphi Y^h)\big\|_M^2 = \big\|\overline{h(X^h, Y^h)}\big\|_M^2. \tag{4.6}$$

Let us denote $K_\phi^P(X)$, respectively $K_\varphi^M(X^h)$ the sectional curvature of the plane $\{X, \phi X\}$, respectively $\{X^h, \varphi X^h\}$. Adding Eqs. (4.1), (4.2) and using (4.5), (4.6), we finally obtain (taking again into account the anti-symmetry of A):

$$K_\phi^P(X) = K_\varphi^M(X^h) + 4\big\|\overline{h(X^h, X^h)}\big\|_M^2 - 2\big\|h\widetilde{(X^h, X^h)}\big\|_M^2. \tag{4.7}$$

4.2. The curvature of the quotient

In general, from Eq. (4.7) one hopes to deduce the positivity of the φ-sectional curvature of the quotient. This depends on the extrinsic geometry of the level set, which is a data additional to the reduction scheme: the second fundamental form of the level set cannot be entirely expressed in terms of the action. But in some particular cases, one is able to derive a conclusion.

Obviously the simplest situation occurs when $J^{-1}(\mathbb{R}_+\mu)$ is totally geodesic in M: then the φ-sectional curvatures of M and M_μ are equal. In fact, one is only interested in the vanishing of $h(X^h, Y^h)$, which, by the first equation in (4.3), is implied by the vanishing of O'Neill's integrability tensor A. This is a rather strong condition, implying that $J^{-1}(\mathbb{R}_+\mu)$ is a locally a (not necessarily Riemannian) product and cannot be predicted by the action. Other conditions on the second fundamental form which are common in Riemannian and Cauchy–Riemann submanifold theory, see e.g. [2], (mixed totally geodesic, (contact)-totally umbilical, extrinsic sphere etc.) and permit some speculations in (4.7) or even the computation of the Ricci curvature of the quotient, seem to be artificial in this context, as not directly expressible in terms of the action.

We apply the above computation for N being $J^{-1}(0)$ and for P being the respective reduced space. Then Eq. (4.7) implies:

Proposition 4.1. *The reduced space at* 0 *of a Sasakian manifold with positive φ-sectional curvature* (*in particular of an odd sphere with the standard Sasakian structure*) *has strictly positive φ-sectional curvature.*

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