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Locally conformal Kähler manifolds with potential

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Abstract A locally conformally Kähler (LCK) manifold M is one which is covered by a Kähler manifold $\widetilde{M}$ with the deck transformation group acting conformally on $\widetilde{M}$. If M admits a holomorphic flow, acting on $\widetilde{M}$ conformally, it is called a Vaisman manifold. Neither the class of LCK manifolds nor that of Vaisman manifolds is stable under small deformations. We define a new class of LCK-manifolds, called LCK manifolds with potential, which is closed under small deformations. All Vaisman manifolds are LCK with potential. We show that an LCK-manifold with potential admits a covering which can be compactified to a Stein variety by adding one point. This is used to show that any LCK manifold M with potential, $\dim M \geq 3$, can be embedded into a Hopf manifold, thus improving similar results for Vaisman manifolds Ornea and Verbitsky (Math Ann 332:121–143, 2005).

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1 Introduction

1.1 Motivation

The study of algebraic geometry of compact non-Kähler manifolds is stymied by the lack of metric structures and unability to apply most differential geometric and analytic arguments. However, in many examples of non-Kähler manifold, the metric structures arise naturally. The most common example of these is given by the locally conformally Kähler (LCK) geometry. An LCK manifold is one admitting a Kähler covering $\widetilde{M}$ with the deck transformation group acting on $\widetilde{M}$ by conformal transforms. The first examples of non-Kähler manifolds were Hopf manifolds, that is, quotients of $\mathbb{C}^n \backslash 0$ by a cyclic group $\langle A \rangle$ generated by an invertible linear operator $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, with all eigenvalues < 1. Such a quotient is obviously locally conformally Kähler (see e.g. [8]). For a long time it was conjectured that all non-Kähler complex surfaces are LCK (this conjecture was disproven in [3]). Still, the ubiquity of LCK manifolds is hard to overestimate.

LCK manifolds have been widely studied in the last 30 years. They share some properties with Kähler manifolds (they are stable under blow-up, satisfy an embedding theorem of Kodaira type, cf. [15] and the last section here), but this geometry also contrasts Kähler geometry. A striking feature is that a deformation of an LCK manifold is not LCK. Even the smaller (and better understood) subclass of Vaisman manifolds is not stable under deformations (both results were proved in [3] by finding appropriate counterexamples). On the other hand, it was proven in [21] that the Kähler metric of the universal covering of a Vaisman manifold admits a Kähler potential. This property is stable under small deformations and hence a Vaisman manifold deforms to a LCK one (but not necessarily a Vaisman one).

This suggests the need for a new subclass of LCK manifolds, called LCK manifolds with potential, stable under deformations and containing Vaisman manifolds. It is the aim of this paper to study the LCK manifolds with potential. We obtain a Kodaira-type embedding theorem for LCK-manifolds with potential (Theorem 3.4), embedding each into an appropriate Hopf manifold, not necessarily of Vaisman type.

1.2 Basic definitions

Let (M, I, g) be a connected complex Hermitian manifold of complex dimension at least 2. Denote by ω its fundamental Hermitian two-form, with the convention $\omega(X, Y) = g(X, IY)$.

Definition 1.1 (M, I, g) is called **locally conformally Kähler (LCK)** if there exists a *closed* one-form θ (called the Lee form) such that

$$d\omega = \theta \wedge \omega.$$

Equivalently, any cover $\widetilde{M}$ of M on which the pull-back $\widetilde{\theta}$ of θ is exact (in particular, the universal cover) carries a Kähler form $\Omega = e^{-f}\widetilde{\omega}$, where $\widetilde{\theta} = df$, and such that $\pi_1(M)$ acts on $\widetilde{M}$ by holomorphic homotheties. Conversely, a manifold admitting such a Kähler covering is necessarily locally conformally Kähler.

For many equivalent definitions and examples, the reader is referred to [7,15].

The most important subclass of LCK manifolds is defined by the parallelism of the Lee form with respect to the Levi-Civita connection of g [7,19].

Definition 1.2 An LCK manifold (M, I, g) is called **a Vaisman manifold** if $\nabla\theta = 0$, where ∇ is the Levi-Civita connection of g.

According to [9], the compact Vaisman manifolds can be defined as LCK manifolds admitting a conformal holomorphic flow which does not induce an isometry on $\widetilde{M}$.

Vaisman geometry is intimately related to Sasakian geometry (see [14]). On the other hand, the Kähler form Ω on the universal covering of a Vaisman manifold admits a Kähler potential (cf. [21] and the next section).

2 LCK manifolds with potential

2.1 LCK manifolds and monodromy

The main object of this paper is described in the following

Definition 2.1 An **LCK manifold with potential** is a manifold which admits a Kähler covering $(\widetilde{M}, \Omega)$ and a smooth function $\varphi : \widetilde{M} \to \mathbb{R}^{>0}$ (the **LCK potential**) satisfying the following conditions:

1. φ is proper, i.e. its level sets are compact;
2. The monodromy map τ acts on φ by multiplication with a constant $\tau(\varphi) = const \cdot \varphi$;
3. φ is a Kähler potential, i.e. $-\sqrt{-1}\,\partial\overline{\partial}\varphi = \Omega$.

Remark 2.2 Let Γ be the monodromy (deck transformation) group of $\widetilde{M} \to M$, and $\chi : \Gamma \longrightarrow \mathbb{R}^{>0}$ be a homomorphism mapping $\gamma \in \Gamma$ to the number $\frac{\gamma(\varphi)}{\varphi}$. Consider the covering $\widetilde{M}' := \widetilde{M}/\ker\chi$. Clearly, φ defines a Kähler potential φ' on $\widetilde{M}'$, and φ' satisfies assumptions (1)–(3) of Definition 2.1. Therefore, we may always choose the covering $\widetilde{M}$ in such a way that $\chi : \Gamma \longrightarrow \mathbb{R}^{>0}$ is injective. Further on, we shall always assume that χ is injective whenever an LCK manifold with potential is considered.

Remark 2.3 Any closed complex submanifold of an LCK-manifold with potential is also an LCK-manifold with potential.

Note that, by [21, Proposition 4.4], all compact Vaisman manifolds do have a potential. Indeed, on $\widetilde{M}$, we have $\theta = d\varphi$, where $\varphi = \Omega(\theta, I(\theta))$. Using the parallelism, one shows that $\Omega = e^{-\varphi}\widetilde{\omega}$ is a Kähler form with potential φ.

The two non-compact, non-Kähler, LCK manifolds constructed recently by J. Renaud in [16] do also have a potential.

In both cases the monodromy is isomorphic to $\mathbb{Z}$, hence, by Proposition 2.5 below, condition (1) in the definition is fulfilled.

Claim 2.4 Let M be a compact LCK-manifold with potential, $\widetilde{M}$ the corresponding cover with monodromy Γ, and $\chi : \Gamma \longrightarrow \mathbb{R}^{>0}$ the injective character constructed above. Then the image of χ is discrete in $\mathbb{R}^{>0}$, in other words, Γ is isomorphic to $\mathbb{Z}$.

Proof Let γ be a non-trivial element. Then φ defines a continuous and proper map

$$\widetilde{M}/\langle\gamma\rangle \xrightarrow{\varphi_1} \mathbb{R}^{>0}/\langle\varphi(\gamma)\rangle \cong S^1.$$

Since $\mathbb{R}^{>0}/\langle\varphi(\gamma)\rangle$ is compact, and φ_1 is proper, $\widetilde{M}/\langle\gamma\rangle$ is also compact. On the other hand, $\Gamma/\langle\gamma\rangle$ acts freely on $\widetilde{M}/\langle\gamma\rangle$. Therefore, $\Gamma/\langle\gamma\rangle$ is finite. □

The converse is also true, hence:

Proposition 2.5 *Let M be a compact LCK-manifold, $\widetilde{M} \longrightarrow M$ its Kähler covering, $\chi : \Gamma \longrightarrow \mathbb{R}^{>0}$ the character defined above, and φ a Kähler potential on $\widetilde{M}$ which satisfies $\gamma(\varphi) = \chi(\gamma)\varphi$. Assume that χ is injective. Then φ is proper if and only if the image of χ is discrete in $\mathbb{R}^{>0}$.*

Proof The "only if" part is proven in Claim 2.4. Assume, conversely, that the image of χ is discrete. Then Γ is isomorphic to $\mathbb{Z}$, and φ defines a map

$$\widetilde{M}/\langle\gamma\rangle \xrightarrow{\varphi_1} \mathbb{R}^{>0}/\langle\varphi(\gamma)\rangle,$$

where γ is the generator of Γ. Since $\widetilde{M}/\langle\gamma\rangle = M$, and M is compact, this map is proper. Its fibers are naturally identified with the level sets of φ, and therefore they are compact, and φ is proper. □

2.2 Deformations of LCK manifolds with potential

With an argument similar to the one in [15, Theorem 4.5], noting that small deformations of positive potentials remain positive, we derive

Theorem 2.6 *The class of compact LCK manifolds with potential is stable under small deformations.*

Proof Let (M, I') be a small deformation of (M, I). Then φ is a proper function on $(\widetilde{M}, I')$ satisfying $\gamma(\varphi) = \chi(\gamma)\varphi$. It is strictly plurisubharmonic because a small deformation of a strictly plurisubharmonic function is again strictly plurisubharmonic (note that the fundamental domain of the monodromy action is compact). Therefore, $(\widetilde{M}, I')$ is Kähler, and φ is an LCK-potential on $(\widetilde{M}, I')$. □

Hence, in particular, any compact Vaisman manifold deforms to a LCK one. This explains a posteriori why the construction in [8] worked for deforming the Vaisman structure of a Hopf surface of Kähler rank 1 to a non-Vaisman LCK structure on a Hopf surface of Kähler rank 0 which, moreover, by [3], does not admit any Vaisman metric.

On the other hand, not all LCK manifolds admit an LCK potential. For example, one may consider the blow up of a point on a compact Vaisman manifold. Moreover, the LCK structure of the Inoue surfaces do not admit potential, since they can be deformed to the non-LCK type Inoue surface $S^+_{n;p,q,r,u}$ with $u \in \mathbb{C} \setminus \mathbb{R}$ (cf. [3]).

3 Holomorphic contractions on Stein varieties

3.1 Filling a Kähler covering of an LCK-manifold with potential

Theorem 3.1 *Let M be a compact LCK manifold with a potential,* $\dim M \geq 3$*, and $\widetilde{M}$ the corresponding covering. Then $\widetilde{M}$ is an open subset of a Stein variety $\widetilde{M}_c$, with at most one singular point. Moreover, $\widetilde{M}_c \backslash \widetilde{M}$ is just one point.*

Proof Consider the subset

$$\widetilde{M}(a) = \{x \in \widetilde{M} \mid \varphi(x) > a\}.$$

By the properties of the potential function, this set is holomorphically concave. By Rossi-Andreotti-Siu theorem (cf. [17, Theorem 3, p. 245] and [1, Proposition 3.2]) $\widetilde{M}(a)$ can be compactified: it is an open subset of a Stein variety $\widetilde{M}_c$, with (at most) isolated singularities.[1] The ring of holomorphic functions on the "filled in" (see also [12]) part of $\widetilde{M}(a)$ is identified with the ring of CR-functions on the level set $\varphi^{-1}(a)$. Therefore, one could extend the embedding $\widetilde{M}(a) \to \widetilde{M}_c$ to $\widetilde{M} \to \widetilde{M}_c$, filling all $\widetilde{M}$ to a Stein variety with at most isolated singularities.

We show that $\widetilde{M}_c$ is obtained from $\widetilde{M}$ by adding only one point.

The deck transformation (monodromy) group $\Gamma \cong \mathbb{Z}$ acts on $\widetilde{M}_c$ by non-trivial conformal automorphisms. Let γ be its generator, which is a contraction[2] (that is, γ maps Ω to $\lambda \cdot \Omega$ with $\lambda < 1$). Consider a holomorphic function f on $\widetilde{M}_c$. The variety $\widetilde{M}_c$ is covered by a sequence of compact subsets $B_r = \widetilde{M}_c \backslash \widetilde{M}(r)$. Since γ maps B_r to $B_{\lambda r}$, the sequence $\gamma^n f$ is uniformly bounded on compact sets. Therefore, it has a subsequence converging to a holomorphic function, in the topology of uniform convergence on compact sets. It is easy to see that the limit reaches its maximum on any

[1] It is here that the restriction on the dimension is essential.

[2] It has to be a contraction or an expansion. We make the first choice.

B_r, hence by the maximum principle the limit is constant. On the other hand, $\sup \gamma^n f$ stays the same, hence the limit is the same for all subsequences of $\gamma^n f$. We obtain that $\gamma^n f$ converges to a constant.

The set $Z := \widetilde{M}_c \backslash \widetilde{M}$ is by construction compact and fixed by Γ. Therefore, $\sup_Z |\gamma^n f| = \sup_Z |f|$ and $\inf_Z |\gamma^n f| = \inf_Z |f|$ for any holomorphic function f. Since $\gamma^n f$ converges to a constant, we obtain that $\inf_Z |f| = \sup_Z |f|$. This implies that all globally defined holomorphic functions take the same value in all points of Z. The variety $\widetilde{M}_c$ is Stein, and therefore for any pair $x, y \in \widetilde{M}_c$, $x \neq y$, there exists a global holomorphic function taking distinct values at x, y. Therefore, Z is just one point.

We obtained that $\widetilde{M}_c$ is a one-point compactification of $\widetilde{M}$. This proves Theorem 3.1. □

Proposition 3.2 *In the assumptions of Theorem 3.1, denote the point $\widetilde{M}_c \backslash \widetilde{M}$ by z. Let $\mathfrak{m}$ be the ideal of z, and γ the monodromy generator, which acts on $\widetilde{M}_c$. Then γ acts with eigenvalues strictly smaller than 1 on the cotangent space $\mathfrak{m}/\mathfrak{m}^2$.*

Proof It will be enough to carry on the argument on the tangent space. In the smooth situation, if γ fixes a point x and maps an open neighborhood U of x into a compact neighborhood of x contained in U, then γ will act on $T_x U$ as desired, as the Schwarz Lemma implies. In general, we already know that $\lim \gamma^n f = const$ for any holomorphic function f. Now let v be an eigenvector of $d_x\gamma \in \operatorname{End} T_x U$ with eigenvalue λ, such that $d_x f(v) \neq 0$. Then $d_x(\gamma^n f)(v) = \lambda^n d_x f(v)$. As $\gamma^n f$ converges to a constant, $\lambda^n d_x f(v)$ converges to 0, and hence $|\lambda| < 1$. □

3.2 Holomorphic flow and $\mathbb{Z}$-equivariant maps

By Proposition 3.2, the formal logarithm of γ converges, and γ is obtained by integrating a holomorphic flow.

Using an argument similar to [20] we show that γ acts with finite Jordan blocks on the formal completion $\hat{\mathcal{O}}_z$ of the local ring $\mathcal{O}_z$ of germs of analytic functions in $z \in \widetilde{M}_c$. But more is true: the formal solution of the ODE is in fact analytic, and hence γ acts with finite Jordan blocks on $\hat{\mathcal{O}}_z$.

Theorem 3.3 *Let ν be a holomorphic flow on a Stein variety S, acting with eigenvalues strictly smaller than 1 on the tangent space $T_s S$ for some $s \in S$. Then there exists a sequence $V_n \subset \mathcal{O}_s S$ of finite-dimensional subspaces in the ring of germs of analytic functions on S such that the s-adic completion of $\oplus V_n$ equals the completion $\hat{\mathcal{O}}_s S$ of $\mathcal{O}_s S$ and all of the V_n are preserved by ν (which acts by linear transformations of V_n).*

Proof Taking an appropriate embedding of S, we may assume that S is an analytic subvariety of an open ball B in $\mathbb{C}^n$. Since S is Stein, the natural restriction map $T_s B \to T_s B|_S$ is surjective. Therefore, ν can be extended to a holomorphic flow on B. As a consequence, it is enough to prove the theorem assuming S smooth.

We take S to be an open ball in $\mathbb{C}^n$ and $s = 0$. By the Poincaré-Dulac theorem (cf. [2, Sect. 23, p. 181]), after a biholomorphic change of coordinates, ν can be given the following normal form:

$$\nu(x_i) = \lambda_i x_i + P(x_{i+1}, \dots, x_n)$$

where $P(x_{i+1}, \dots, x_n)$ are resonant polynomials[3] corresponding to the eigenvalues λ_i.

Let V_{λ_i} be the spaces generated by the coordinate functions x_i corresponding to the eigenvalue λ_i and resonant monomials. By construction, ν preserves each V_{λ_i}. Taking a set $V_{\lambda_{i_1}}, \dots, V_{\lambda_{i_k}}$, we see that ν also preserves the space $V_{\lambda_{i_1},\dots,\lambda_{i_k}}$ generated by $q_1, \dots, q_k$, with $q_j \in V_j$. But a completion of all these $V_{\lambda_{i_1},\dots,\lambda_{i_k}}$ is, by construction, equal with $\hat{\mathcal{O}}_s S$. □

Picking enough holomorphic functions in these finite-dimensional eigenspaces, we find an embedding $\widetilde{M}_c \hookrightarrow \mathbb{C}^n$ such that the monodromy Γ acts on $\mathbb{C}^n$ equivariantly. Then M is embedded to $(\mathbb{C}^n \backslash 0)/\Gamma$, where Γ acts linearly with eigenvalues strictly smaller than 1.

We shall call such a quotient a *Hopf manifold*; to avoid confusion with class 0 Hopf surfaces of Kodaira - "linear Hopf manifold". It can be viewed as a generalization in any dimension of the class 0 Hopf surface. Clearly, such a Hopf manifold is LCK ([8]) but, in general, not Vaisman.

Using Sternberg's normal form of holomorphic contractions (cf. [18]), Belgun also showed that Hopf manifolds are LCK and classified the Vaisman ones (they are precisely those for which the normal form of the contraction is resonance free), relating them to Sasakian geometry (see [4]).

As a result, we obtain the following:

Theorem 3.4 *Any compact LCK manifold M with potential,* $\dim M \geq 3$*, can be embedded into a linear Hopf manifold.*

Remark 3.5 The converse is also true, of course.

This gives a new version of the Kodaira-type theorem on immersions of Vaisman manifolds to Vaisman-type Hopf manifolds (cf. [15]). In fact, a much stronger result is obtained.

Theorem 3.6 *Let M be a compact Vaisman manifold. Then M can be embedded into a Vaisman-type Hopf manifold $H = (\mathbb{C}^n \backslash 0)/\langle A \rangle$, where A is a diagonal linear operator on $\mathbb{C}^n$ with all eigenvalues < 1.*

Proof Let $S_a := \varphi^{-1}(a)$ be the level set of φ, and V the space of CR-holomorphic functions on S. Using the solution of the $\overline{\partial}$-Neumann problem [12], we identify the L^2-completion of V with the space of L^2-integrable holomorphic functions on the pseudoconvex domain bounded by S_a.

[3] We recall, *loc. cit.*, that an n-tuple $\lambda = (\lambda_1, \dots, \lambda_n)$ is called resonant if there exists an integral relation $\lambda_s = (\lambda, m) := \sum m_i \lambda_i$, with $m = (m_1, \dots, m_n)$, all $m_i \geq 0$ and $|\, m \,| := \sum m_i \geq 2$. Correspondingly, if the eigenvalue λ of an operator A is resonant and $x^m = x_1^{m_1} \cdots x_n^{m_n}$ is a monomial, x_i being the coordinates of a vector x in a fixed eigenbasis $\{e_i\}$, we say that the vector-valued monomial $x^m e_s$ is resonant if $\lambda_s = (m, \lambda)$, $|\, m \,| \geq 2$.

Let $X := \theta^\sharp$ be the holomorphic vector field dual to the Lee form. According to the general results from Vaisman geometry ([7, p. 37]), X is holomorphic. Therefore, X acts on V in a natural way. Denote by g the natural metric on S_a. Since X acts on $\widetilde{M}$ by homotheties ([7, p. 37]), we have $X(g) = cg$. Therefore, X acts on V as a self-dual operator. Then, X is diagonal on any finite-dimensional subspace in V preserved by X. This is true for the spaces $V_{\lambda_{i_1},...,\lambda_{i_k}}$ used to construct the embedding $\widetilde{M}_c \longrightarrow \mathbb{C}^n$. □

After this work was completed, Professor Hajime Tsuji communicated to us the reference to [11]. In this work, Masahide Kato obtains a sufficient condition for a compact complex manifold Z of dimension ≥ 4 to dominate bimeromorphically a subvariety of a Hopf manifold, in terms of a certain effective divisor on D and a flat line bundle on Z.

Also, in [10], the following theorem was obtained: "Let X be a complex space with a fixed point z, equipped with a holomorphic contraction, that is, an invertible morphism $\psi : X \longrightarrow X$ such that, for any sufficient small neighbourhood U of z, $\psi^k(U)$ lies inside U for k sufficiently large, and, for all $x \in X$,

$$\lim_{k \longrightarrow \infty} \psi^k(x) = z.$$

Then $(X\backslash z)/\langle\psi\rangle$ can be embedded into a Hopf manifold".

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