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Blow-ups of Locally Conformally Kähler Manifolds

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A locally conformally Khler (LCK) manifold is a complex manifold which admits a covering endowed with a Kähler metric with respect to which the covering group acts through homotheties. We show that the blow-up of a compact LCK manifold along a complex submanifold admits an LCK structure if and only if this submanifold is globally conformally Kähler. We also prove that a twistor space (of a compact four-manifold, a quaternion-Kähler manifold, or a Riemannian manifold) cannot admit an LCK metric, unless it is Kähler.

1 Introduction

1.1 Bimeromorphic maps and locally conformally Kähler structures

A locally conformally Kähler (LCK) manifold is a complex manifold M, $\dim_{\mathbb{C}} M > 1$, admitting a Kähler covering $(\tilde{M}, \tilde{\omega})$, with the deck transform group acting on $(\tilde{M}, \tilde{\omega})$ by holomorphic homotheties. Unless otherwise stated, we shall consider only compact LCK manifolds.

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In this paper, we are interested in the birational (or, more precisely, bimeromorphic) geometry of LCK manifolds.

An obvious question arises immediately.

Question 1.1. Let $X \subset M$ be a complex subvariety of an LCK manifold, and $M_1 \longrightarrow M$ a blow-up of M in X. Would M_1 also admit an LCK structure? □

When X is a point, the question is answered in affirmative by Tricerri [13] and Vuletescu [16]. When $\dim X > 0$, the answer is not immediate. To state it properly, we recall the notion of a *weight bundle* of an LCK manifold. Let $(\tilde{M}, \tilde{\omega})$ be the Kähler covering of an LCK manifold M, and $\pi_1(M) \longrightarrow \operatorname{Map}(\tilde{M}, \tilde{M})$ the deck transform map. Since $\rho^*(\gamma)\tilde{\omega} = \mathsf{const} \cdot \tilde{\omega}$, this constant defines a character $\pi_1(M) \xrightarrow{\chi} \mathbb{R}^{>0}$, with $\chi(\gamma) := \frac{\rho^*(\gamma)\tilde{\omega}}{\tilde{\omega}}$.

Definition 1.2. Let L be the one-dimensional local system on M with monodromy defined by the character χ. We think of L as of a real bundle with a flat connection. This bundle is called *the weight bundle* of M. □

One may think of the Kähler form $\tilde{\omega}$ as of an L-valued differential form on M. This form is closed, positive, and of type (1,1). Therefore, for any smooth complex subvariety $Z \subset M$ such that $L|_Z$ is a trivial local system, Z is Kähler.

The following two theorems describe how the LCK property behaves under blow-ups.

Theorem 1.3. Let $Z \subset M$ be a compact complex submanifold of an LCK manifold, and M_1 the blow-up of M with center in Z. If the restriction $L|_Z$ of the weight bundle is trivial as a local system, then M_1 admits an LCK metric. □

Proof. See Corollary 2.11. ■

A similar question about blow-downs is also answered.

Theorem 1.4. Let $D \subset M_1$ be an exceptional divisor on an LCK manifold, and let M be the complex variety obtained as a contraction of D. Then the restriction $L|_D$ of the weight bundle to D is trivial. □

Proof. Theorem 2.9. ■

This result is quite unexpected, and leads to the following theorem about a special class of LCK manifolds called *Vaisman manifolds* (Section 2).

Claim 1.5. Let M be a Vaisman manifold. Then any bimeromorphic contraction $M \longrightarrow M'$ is trivial. Moreover, for any positive-dimensional submanifold $Z \subset M$, its blow-up M_1 does not admit an LCK structure. □

Proof. Corollary 2.13 ■

1.2 Positive currents on LCK manifolds

The proofs of Theorems 1.4 and 1.3 are purely topological. However, they were originally obtained using a less elementary argument involving positive currents.

We state this argument here, omitting minor details of the proof, because we think that this line of thought could be fruitful in other contexts too; for more information and missing details, the reader is referred to [2–4].

A current is a form taking values in distributions. The space of (p, q)-currents on M is denoted by $D^{p,q}(M)$. A *strongly positive current* is a linear combination

$$\sum_I \alpha_I (z \wedge \bar{z})_I,$$

where α_I are positive, measurable functions, and the sum is taken over all multi-indices I. An integration current of a closed complex subvariety is a strongly positive current. In this paper, we shall often omit "strongly", because we are only interested in strong positivity.

It is easy to define the de Rham differential on currents, and check that its cohomology coincides with the de Rham cohomology of the manifold.

Currents are naturally dual to differential forms with compact support. This allows one to define an integration (pushforward) map of currents, dual to the pullback of differential forms. This map is denoted by π_*, where $\pi : M \longrightarrow N$ is a proper morphism of smooth manifolds.

Now, let $\pi : M \longrightarrow N$ be a blow-up of a subvariety $Z \subset N$ of codimension k, and ω a Kähler form on M. Then $(\pi_*\omega)^k$ has a singular part which is proportional to the integration current of Z.

This follows from the Siu's decomposition of positive currents [3]. Demailly's results on intersection theory of positive currents [2] are used to multiply the currents, and the rest follows because the Lelong numbers of $\pi_*\omega$ along Z are nonzero.

Applying this argument to a birational contraction $M \xrightarrow{\phi} M'$ of an LCK manifold M, and denoting by $\tilde{M} \xrightarrow{\tilde{\phi}} \tilde{M}'$ the corresponding map of coverings, we obtain a closed, positive current $\xi := \tilde{\phi}_* \tilde{\omega}$ on $\tilde{M}'$, with the deck transform map ρ acting on ξ by homotheties. Then ρ would also act by homotheties on the current ξ^k, $k = \dim Z$, where Z is the exceptional set of $\tilde{\phi}$.

Applying the above result to decompose ξ^k onto its absolutely continuous and singular part, we obtain that the current of integration $[Z]$ of Z is mapped to $\mathsf{const}[Z]$ by the deck transform action. Since the current of integration of Z is mapped by the deck transform to the current of integration of $\tilde{\phi}(Z) = Z$, the constant const is trivial; this implies that $\pi(Z) \subset M'$ is Kähler, with the Kähler metric obtained in the usual way from $\tilde{\omega}$.

1.3 Fujiki class C and LCK geometry

A compact complex variety X is said to belong to *Fujiki class C* if X is bimeromorphic to a Kähler manifold. The Fujiki class C manifolds are closed under many natural operations, such as taking a subvariety, or the moduli of subvarieties, and play an important role in Kähler geometry.

This notion has a straightforward LCK analog.

Definition 1.6. Let M be a compact complex variety. It is called *a locally conformally class C variety* if it is bimeromorphic to an LCK manifold. □

The importance of the Fujiki class C notion was emphasized by a more recent work of Demailly and Păun [4], who characterized class C manifolds in terms of positive currents. Recall that a *Kähler current* is a positive (A (1, 1) current T is *positive* if $T(\alpha) \geq 0$ for any $(n-1, n-1)$ form, then $n = \dim_{\mathbb{C}} M$. We write $T - T' \geq 0$ when $T - T'$ is a positive current.), closed (1,1)-current ϕ on a complex manifold M such that $\phi \geq \omega$ for some Hermitian form ω on M.

Demailly and Păun have proved that a compact complex manifold M belongs to class C if and only if it admits a positive Kähler current.

For an LCK manifold, an analog of a Kähler current is provided by the following notion (motivated by Definition 2.2).

Definition 1.7. Let M be a compact complex manifold, θ be a closed real 1-form on M, Ξ be a positive, real (1,1)-current satisfying $d\Xi = \theta \wedge \Xi$ and $\Xi \geq \omega$ for some Hermitian form ω on M. Then Ξ is called *an LCK current*. □

It would be interesting to know if an LCK-analog of the Demailly–Păun theorem is true.

Question 1.8. Let M be a complex compact manifold. Determine whether the following conditions are equivalent.

(i) M belongs to locally conformally class C.

(ii) M admits an LCK current. □

2 Blow-ups and Blow-downs of LCK Manifolds

We start by repeating (in a more technical fashion) the definition of an LCK manifold given in Section 1. See [5] for more details and several other versions of the same definition, all of them equivalent.

Definition 2.1. A LCK manifold is a complex manifold X covered by a system of open subsets U_α endowed with *local* Kähler metrics g_α, conformal on overlaps $U_\alpha \cap U_\beta$: $g_\alpha = c_{\alpha\beta} g_\beta$. □

Note that, in complex dimension at least 2, as we always assume, $c_{\alpha\beta}$ are positive constants. Moreover, they obviously satisfy the cocycle condition. Interpreted in cohomology, the cocycle $\{c_{\alpha\beta}\}$ determines a closed one-form θ, called *the Lee form*. Hence, locally $\theta = df_\alpha$. It is easily seen that $e^{-f_\alpha} g_\alpha = e^{-f_\beta} g_\beta$ on $U_\alpha \cap U_\beta$, and thus determine a *global* metric g which is conformal on each U_α with a Kähler metric. One obtains the following equivalent:

Definition 2.2. A Hermitian manifold M is LCK if its fundamental two-form ω satisfies:

$$d\omega = \theta \wedge \omega, \quad d\theta = 0 \tag{2.1}$$

for a *closed* one-form θ. □

If θ is exact, then M is called *globally conformally Kähler* (GCK).

As we work with compact manifolds and, in general, the topology of compact Kähler manifolds is very different from the one of compact LCK manifolds, we always assume $\theta \neq 0$ on M.

Let $\Gamma \longrightarrow \tilde{M} \stackrel{\pi}{\longrightarrow} M$ be the universal cover of M with deck group Γ. As $\pi^*\theta$ is exact on $\tilde{M}$, $\pi^*\omega$ is globally conformal with a Kähler metric $\tilde{\omega}$. Moreover, Γ acts by holomorphic homotheties with respect to $\tilde{\omega}$. This defines a character

$$\chi : \Gamma \longrightarrow \mathbb{R}^{>0}, \quad \gamma^*\tilde{\omega} = \chi(\gamma)\tilde{\omega}. \tag{2.2}$$

It can be shown that this property is indeed an equivalent definition of LCK manifolds, see [11].

Clearly, an LCK manifold M is GCK if and only if Γ acts trivially on $\tilde{\omega}$ (i.e., $\operatorname{im}\chi = \{1\}$).

A particular class of LCK manifolds are the *Vaisman manifolds*. They are LCK manifolds with the Lee form parallel with respect to the Levi-Civita connection of the LCK metric. The compact ones are mapping tori over the circle with Sasakian fibre, see [10]. The typical example is the Hopf manifold, diffeomorphic to $S^1 \times S^{2n-1}$.

On a Vaisman manifold, the vector field $\theta^\sharp - \sqrt{-1}J\theta^\sharp$ generates a one-dimensional holomorphic, Riemannian, totally geodesic foliation. If this is regular and if M is compact, then the leaf space B is a Kähler manifold.

Example 2.3. On a Hopf manifold $\mathbb{C}^n \setminus \{0\}/\langle z_i \mapsto 2z_i\rangle$, the LCK metric $\frac{\sum dz_i \otimes d\bar{z}_i}{|\sum z_i\bar{z}_i|^2}$ is Vaisman and regular; the leaf space is $\mathbb{C}P^{n-1}$.

We refer to [5] or to the more recent [11] for more details about LCK geometry. □

It is known, [13, 16], that the blow-up *at points* preserve the LCK class. The present paper is devoted to the blow-up of LCK manifolds along subvarieties. In this case, the situation is a bit more complicated and a discussion should be made according to the dimension of the submanifold.

Definition 2.4. Let $Y \stackrel{j}{\hookrightarrow} M$ be a complex subvariety. We say that Y is *of induced globally conformally Kähler type* (IGCK) if the cohomology class $j^*[\theta]$ vanishes, where $[\theta]$ denotes the cohomology class of the Lee form on M. □

Remark 2.5. Note that a IGCK-submanifold of an LCK manifold is always Kähler. □

Remark 2.6. By a theorem of Vaisman [14], any LCK metric on a compact complex manifold Y of Kähler type is GCK if $\dim_{\mathbb{C}} Y > 1$. Therefore, the IGCK condition above for smooth Y with $\dim_{\mathbb{C}} Y > 1$ is equivalent to Y being Kähler. □

Remark 2.7. Note that there may exist curves on LCK manifolds which are not IGCK, despite being obviously of Kähler type. For instance, if M is a regular Vaisman manifold, and if Y is a fiber of its elliptic fibration, then Y is not IGCK, as any compact complex subvariety of a compact Vaisman manifold has an induced Vaisman structure (see, e.g., [15, Proposition 6.5]). □

The main goal of this paper is to prove the following two theorems.

Theorem 2.8. Let M be an LCK manifold, $Y \subset M$ be a smooth complex IGCK subvariety, and let $\tilde{M}$ be the blow-up of M centered in Y. Then $\tilde{M}$ is LCK. □

Proof. See the argument after Lemma 3.4. ■

Theorem 2.9. Let M be a complex variety, and $\tilde{M} \longrightarrow M$ the blow-up of a compact subvariety $Y \subset M$. Assume that $\tilde{M}$ is smooth and admits an LCK metric. Then the blow-up divisor $\tilde{Y} \subset \tilde{M}$ is a IGCK subvariety. □

Proof. See Remark 3.3. ■

Remark 2.10. In the situation described in Theorem 2.9, the variety $\tilde{Y}$ is of Kähler type, because it is IGCK. When Y is smooth, Y is Kähler, as shown by Blanchard [1, Théorème II.6]. Together with Remark 2.6, this implies the following corollary. □

Corollary 2.11. Let M be an LCK manifold, and $Y \subset M$ a smooth compact subvariety, such that the blow-up of M in Y admits an LCK metric. If $\dim_{\mathbb{C}}(Y) > 1$, then Y is a IGCK subvariety. □

Remark 2.12. Note that, from [15, Proposition 6.5], a compact complex submanifold Y of a compact Vaisman manifold is itself Vaisman, and θ represents a nontrivial class in the cohomology of Y, so there are no IGCK submanifolds of proper dimension $\dim_{\mathbb{C}}(Y) > 0$. This implies the following corollary. □

Corollary 2.13. The blow-up of a compact Vaisman manifold along a compact complex submanifold Y of dimension at least 1 cannot have an LCK metric. □

The proofs of these two theorems and of the corollary will be given in Section 3. As a by-product of our proof, we obtain the following:

Corollary 2.14. Let M be a twistor space of any of the following types: of a half-conformally flat four-dimensional Riemannian manifold, of a quaternionic-Kähler manifold, of a conformally flat manifold. If M admits a LCK metric, then this metric is actually GCK. □

Proof. See Corollary 3.2. ■

Remark 2.15. (i) A similar, weaker result is proved in [8]. Namely, the twistor space of half-conformally flat four-dimensional Riemannian manifolds with large fundamental group cannot admit LCK metrics with automorphic potential on the covering. The proof uses different techniques from ours, and which cannot be generalized neither to higher dimensions nor to quaternionic Kähler manifolds.

(ii) It was known from [6, 9] that the natural metrics (with respect to the twistor submersion) cannot be LCK. Our result refers to any metric on the twistor space, not necessarily related to the twistor submersion. On the other hand, as shown by Hitchin, the twistor space of a compact four-dimensional manifold is not of Kähler type, unless it is biholomorphic to $\mathbb{C}P^3$ or to the flag variety F_2 [7]. □

Remark 2.16. So far, we were unable to deal with the reverse statement of Theorem 2.8, namely, to determine whether a smooth bimeromorphic contraction of an LCK manifold is always LCK. In the particular case when an exceptional divisor is contracted to a point, this has been proved to be true by Tricerri [13]; we conjecture that in the general case this is false, but we are not able to find any example.

For GCK (i.e., Kähler) manifolds, the answer is well known: blow-downs of Kähler manifolds can be non-Kähler, as one can see from any example of a Moishezon manifold. □

Remark 2.17. We summarize the case of blow-up of curves on LCK manifolds. Since rational curves are simply connected, they are IGCK submanifolds, so blowing-up a rational curve on a LCK manifold always yields a manifold of LCK type. The case of the elliptic curves was partially tackled in Corollary 2.13. If Y is a curve of arbitrary genus contained in an exceptional divisor of a blow-up, then it is also automatically a IGCK subvariety since the exceptional divisor is so; hence again, blowing it up yields a manifold of LCK type.

To our present knowledge, the only examples of curves Y on LCK manifolds M with genus $g(Y) \geq 2$ are curves belonging to some exceptional divisors. It would be interesting to prove that this is the case in general, or to build out a counter-example. □

3 The Proofs

Lemma 3.1. Let M be an LCK manifold, B be a path connected topological space and let $\pi : M \longrightarrow B$ be a continuous map. Assume that either

(i) B is an irreducible complex variety, and π is proper and holomorphic.

(ii) π is a locally trivial fibration with fibers which are complex subvarieties of M.

Suppose also that the map

$$\pi^* : H^1(B) \longrightarrow H^1(M)$$

is an isomorphism, and the fibers of π are positive-dimensional. Then the LCK structure on M is actually GCK. □

Proof. Denote by θ the Lee form of M, and let $\tilde{M}$ be the minimal GCK covering of X, that is, the minimal covering $\tilde{M} \longrightarrow M$ such that the pullback of θ is exact. Since $H^1(B) \cong H^1(M)$, there exists a covering $\tilde{B} \longrightarrow B$ such that the following diagram is commutative, and the fibers of $\tilde{\pi}$ are compact:

$$\begin{array}{ccc} \tilde{M} & \longrightarrow & M \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \tilde{B} & \longrightarrow & B \end{array}$$

Let $\tilde{B}_0 \subset \tilde{B}$ be the set of regular values of $\tilde{\pi}$, and let $F_b := \tilde{\pi}^{-1}(b)$ be the regular fibers of $\tilde{\pi}$, $\dim_{\mathbb{C}} F_b = k$. Since $\tilde{B}_0$ is connected, all F_b represent the same homology class in $H_{2k}(\tilde{M})$.

Denote the Kähler form of $\tilde{M}$ by $\tilde{\omega}$, conformally equivalent to the pullback of the Hermitian form on X.

Since all F_b represent the same homology class, the Riemannian volume

$$\mathrm{Vol}_{\tilde{\omega}}(F_b) := \int_{F_b} \tilde{\omega}^k$$

is independent from $b \in \tilde{B}_0$. This gives (recall the definition of the character χ in (2.2))

$$\mathrm{Vol}_{\tilde{\omega}}(F_b) = \int_{F_b} \tilde{\omega}^k = \int_{F_{\gamma^{-1}(b)}} \gamma^* \tilde{\omega}^k = \int_{F_{\gamma^{-1}(b)}} \chi(\gamma)^k \tilde{\omega}^k = \chi(\gamma)^k \mathrm{Vol}_{\tilde{\omega}}(F_b),$$

hence the constant χ_γ is equal to 1 for all $\gamma \in \Gamma$. Therefore, $\tilde{\omega}$ is Γ-invariant, and M is GCK. ■

The above lemma immediately implies Corollary 2.14.

Corollary 3.2. Let Z be the twistor space of M, understood in the sense of Corollary 2.14. Assume that Z admits an LCK metric. Then this metric is GCK. □

Proof. There is a locally trivial fibration $Z \longrightarrow M$, with complex analytic fibers which are compact symmetric Kähler spaces, hence Lemma 3.1 can be applied. ■

Remark 3.3. In the same way one deals with the blow-ups: the fibers over an exceptional set of a blow-up map are positive-dimensional. Therefore, Lemma 3.1 implies Theorem 2.9. □

We can now give the following:

Proof of Corollary 2.13. If $\dim_{\mathbb{C}}(Y) > 1$, then the result follows from Corollary 2.11 and Remark 2.12. In the case $\dim_{\mathbb{C}}(Y) = 1$ we cannot use this argument directly—see Remark 2.7—so in this case we argue as follows.

Assume $\tilde{M}$ has an LCK metric $\tilde{\omega}$ with Lee form $\tilde{\eta}$. By Theorem 2.9, the restriction $\tilde{\eta}_{|Z}$ to the exceptional divisor Z is exact. Hence, after possibly making a conformal change of the LCK metric, we can assume $\tilde{\eta}_{|V} = 0$ where V is a neighborhood of Z. In particular, $\tilde{\eta}$ will be the pull-back of a one-form η on M. On the other hand, $\tilde{\omega}$ gives rise to a current on $\tilde{M}$ (see also Section 1.2) and its pushforward defines an LCK positive $(1, 1)$ current Ξ on M with associate Lee form η. Clearly, $\eta_{|Y} = 0$.

Possibly conformally changing now Ξ, we can assume that η is the unique harmonic form (with respect to the Vaisman metric of M) in its cohomology class. Possibly $\eta_{|Y}$ is no longer zero, but remains *exact*.

We now show that η is basic with respect to the canonical foliation $\mathcal{F}$ generated on M by $\theta^\sharp - \sqrt{-1}J\theta^\sharp$. Indeed, from [14], we know that any harmonic form on a compact

Vaisman manifold decomposes as a sum $\alpha + \theta \wedge \beta$ where α and β are basic and transversally (with respect to $\mathcal{F}$) harmonic forms. In particular, as a transversally harmonic function is constant, we have

$$\eta = \alpha + c \cdot \theta, \tag{3.1}$$

where $c \in \mathbb{R}$ and α is basic, transversally harmonic (see [12] for the theory of basic Laplacian and basic cohomology, etc.).

Let now S^1 denote the unique homology class in $H_1(M)$ (call it *the fundamental circle of* θ) such that $\int_{S^1} \theta = 1$ and $\int_{S^1} \alpha = 0$ for every basic cohomology class α.

As any complex submanifold of a compact Vaisman manifold is tangent to the Lee field and hence Vaisman itself, Y is Vaisman with Lee form $\theta_{|Y}$. Hence we deduce that the fundamental circle of θ is the image of the fundamental circle of $\theta_{|Y}$ under the natural map $H_1(Y) \to H_1(M)$.

We now integrate (3.1) on any $\gamma \in H_1(Y)$ and take into account that $\eta_{|Y}$ is exact to get $c = 0$. Hence, η basic. It can then be treated as a harmonic one-form on a Kähler manifold (or use the existence of a transversal dd^c-lemma). This implies $d^c\eta = 0$.

But then one obtains a contradiction, as follows. Letting J to be the almost complex structure of M, we see, on the one hand, we have

$$\int_M \mathrm{d}(\Xi^{n-1}) \wedge J(\theta) = \int_M (n-1)\Xi^{n-1} \wedge \theta \wedge J(\theta) > 0,$$

since Ξ is positive. On the other hand, since $\mathrm{d}(J(\theta)) = 0$, it follows that $\mathrm{d}(\Xi^{n-1}) \wedge J(\theta)$ is exact so $\int_M \mathrm{d}(\Xi^{n-1}) \wedge J(\theta) = 0$, a contradiction. ∎

The following result is certainly well known, but since we were not able to find out an exact reference we include a proof here.

Lemma 3.4. Assume (U, g) is a Kähler complex manifold, $Y \subset U$ a compact submanifold and let $c : \tilde{U} \longrightarrow U$ be the blow-up of U along Y. Then, for any open neighborhood $V \supset Y$, there is a Kähler metric $\tilde{g}$ on $\tilde{U}$ such that

$$\tilde{g}_{|\tilde{U} \setminus c^{-1}(V)} = c^*(g_{|U \setminus V}). \qquad \square$$

Proof. (due to Păun; see also [16]).

1. There is a (non-singular) metric on $\mathcal{O}_{\tilde{U}}(-D)$ (where D is the exceptional divisor of the blow-up) such that:

(a) its curvature is zero outside $c^{-1}(V)$, and

(b) its curvature is strictly positive at every point of D and in any direction tangent to D.

Indeed, if such a metric is found, everything follows, as the curvature of this metric plus a sufficiently large multiple of $c^*(g)$ will be positive definite on $\tilde{U}$.

2. To finish the proof, we note that the existence of a metric h with property 1(b) is clear, due to the restriction of $\mathcal{O}_{\tilde{U}}(-D)$ to D.

Now let α be its curvature; then $\alpha - i\partial\bar{\partial}\tau = -[D]$ for some function τ, with at most logarithmic poles along D, bounded from above, and nonsingular on $\tilde{U} \setminus D$. Consider the function $\tau_0 := \max(\tau, -C)$, where C is some positive constant, big enough such that on $\tilde{U} \setminus c^{-1}(V)$ we have $\tau > -C$. Clearly, on a (possibly smaller) neighborhood of D we will have $\tau_0 = -C$, such that the new metric $\mathrm{e}^{-\tau_0}h$ on $\mathcal{O}_{\tilde{U}}(-D)$ also satisfies 1(a). ■

Now we can give the following:

Proof of Theorem 2.8. Let $c: \tilde{M} \longrightarrow M$ be the blow up of M along the submanifold Y. Let g be a LCK metric on M and let θ be its Lee form. Since Y is IGCK we see $\theta_{|Y}$ is exact. Let U be a neighborhood of Y such that the inclusion $Y \hookrightarrow U$ induces an isomorphism of the first cohomology. Then $\theta_{|U}$ is also exact, so, after possibly conformally rescaling g, we may assume $\theta_{|U} = 0$ and hence $g_{|U}$ is Kähler. In particular, $\mathrm{supp}(\theta) \cap U = \emptyset$. Now choose a smaller neighborhood V of Y and apply Lemma 3.4. We get a Kähler metric $\tilde{g}$ on $\tilde{U}$ which equals $c^*(g)$ outside $c^{-1}(V)$, so it glues to $c^*(g)$ giving a LCK metric on $\tilde{M}$. ■

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