Reduction of Sasakian manifolds

Gueo Grantcharov[a)]
Department of Mathematics, The University of Connecticut, 196 Auditorium Road, Stores, Connecticut 06269-3009

Liviu Ornea[b)]
Faculty of Mathematics, University of Bucharest, 14 Academiei str., 70109 Bucharest, Romania

(Received 7 February 2001; accepted for publication 9 May 2001)

We show that the contact reduction can be specialized to Sasakian manifolds. We prove that the Sasakian reduction is compatible with the Kähler reduction both in the cone construction and in the Boothby–Wang fibration. In particular, applying Futaki's results, we obtain a sufficient condition for the reduced space of a regular Sasakian–Einstein manifold to be Sasakian–Einstein. We present examples of Sasakian–Einstein manifolds obtained by S^1 reduction of standard Sasakian spheres. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1386636]

I. INTRODUCTION

The reduction technique was naturally extended from symplectic to contact structures by Geiges in Ref. 1 and Albert in Ref. 2. On the other hand, Boyer, Galicki and B. Mann defined in Ref. 3 a moment map for 3-Sasakian manifolds, thus extending the reduction procedure for nested metric contact structures. Quite surprisingly, a reduction scheme for Sasakian manifolds (contact manifolds endowed with a compatible Riemannian metric satisfying a curvature condition), was still missing.

In this note, based on the preprint,[4] we fill the gap by defining a Sasakian moment map and constructing the associated reduced space (compare with Ref. 5; here we focus on the Riemannian aspects). We then relate Sasakian reduction to Kähler reduction *via* the Kähler cone over a Sasakian manifold and *via* the Boothby–Wang fibration. Further, we derive a condition, similar to Futaki's, for the reduced manifold of a regular Sasakian–Einstein manifold to be Sasakian–Einstein. We end this paper with some completely worked examples of the S^1-reduction of standard Sasakian spheres. Most of the reduced Sasakian structures that we obtain are Einstein. Some of them are among the examples considered in Ref. 6, however our methods allow a much simpler check of the Einstein condition.

Sasakian manifolds seem to be more and more important in superconformal field theories, being connected with the Maldacena conjecture. One of our examples, a S^1 bundle over $S^2 \times S^2$, already appeared in Ref. 7, where the Kähler reduction of the cone over a Sasakian manifold is implicitly used.

II. SASAKIAN MANIFOLDS

In this section we briefly recall the notion of a Sasakian manifold. The definition we give is not the standard one, but is suited for our purpose. For more details, we refer to Refs. 8 and 9.

Definition 2.1: A *Sasakian* manifold is a $(2n+1)$-dimensional Riemannian manifold (N,g) endowed with a unitary Killing vector field ξ such that the curvature tensor of g satisfies the equation

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.1}$$

where η is the metric dual 1-form of ξ: $\eta(X) = g(\xi,X)$.

[a)]Electronic mail: geogran@math.uconn.edu
[b)]Electronic mail: lornea@imar.ro

Let $\varphi=\nabla\xi$, where ∇ is the Levi-Civita connection of g. The following formulas are then easily deduced:

$$\varphi\xi=0, \quad g(\varphi Y,\varphi Z)=g(Y,Z)-\eta(Y)\eta(Z), \quad d\eta(X,Y)=g(\phi X,Y). \tag{2.2}$$

It can be seen that η is a contact form on N, whose Reeb field is ξ (it is also called the characteristic vector field). Moreover, the restriction of φ to the contact distribution $\eta=0$ is an almost complex structure satisfying the "integrability" condition (called normality) $[\varphi,\varphi]+2\,d\eta\otimes\xi=0$.

The simplest example is the standard sphere $S^{2n+1}\subset\mathbb{C}^{n+1}$, with the metric induced by the flat one of $\mathbb{C}^{n+1}$. The characteristic Killing vector field is $\xi_p=-i\vec{p}$, i being the imaginary unit; this is the standard Sasakian structure of the odd sphere. Other Sasakian structures on the sphere can be obtained by D-homothetic transformations (cf. Ref. 10). Also, the unit sphere bundle of any real space form is Sasakian.

More generally, the quantization bundle of a compact Kähler manifold naturally carries a Sasakian structure. The converse construction, possible when the characteristic field is regular, is known as the Boothby–Wang fibration. Precisely, the following result (the metric part is due to Morimoto and Hatakeyama) is available (cf. Refs. 11 or 9).

Theorem 2.1: Let (P,h) be a Hodge manifold. There exists a principal circle bundle $\pi:N\to P$ and a connection form η in it, with curvature from the pull-back of the Kähler form of P, which is a contact form on S. Let ξ be the vector field dual to η with respect to the metric $g=\pi^*h+\eta\otimes\eta$. Then (N,g,ξ) is Sasakian.

The following equivalent definition puts Sasakian geometry into the framework of holonomy groups. Let $C(N)=N\times\mathbb{R}_+$ be the cone over (N,g). Endow it with the warped-product cone metric $C(g)=r^2g+dr^2$. Let $R_0=r\,\partial r$ and define on $C(N)$ the complex structure J acting like this (with obvious identifications): $JY=\varphi Y-\eta(Y)R_0$, $JR_0=\xi$. We have the following.

Theorem 2.2 Ref. 9: (N,g,ξ) is Sasakian if and only if the cone over N $(C(N),C(g),J)$ is Kählerian.

III. MAIN RESULTS

A. Direct construction

In this section we show that the contact map defined in Ref. 1 is compatible with the Sasakian metric and the contact reduced space is indeed Sasakian.

Theorem 3.1: Let (N,g,ξ) be a compact $2n+1$-dimensional Sasakian manifold and G a compact d-dimensional Lie group acting on N by contact isometries. Suppose $0\in\mathfrak{g}^*$ is a regular value of the associated moment map μ. Then the reduced space $M=N//G:=\mu^{-1}(0)/G$ is a Sasakian manifold of dimension $2(n-d)+1$.

Proof: By Ref. 1, the contact moment map $\mu:N\to\mathfrak{g}^*$ is defined by

$$<\mu(x),\underline{X}>=\eta(X),$$

for any $\underline{X}\in\mathfrak{g}$ and X the corresponding field on N. We know that the reduced space is a contact manifold, *loc. cit*. Hence we only need to check that (1) the Riemannian metric is projected on M and (2) the field ξ projects to a unitary Killing field on M such that the curvature tensor of the projected metric satisfies formula (2.1).

To this end, we first describe the metric geometry of the Riemannian submanifold $\mu^{-1}(0)$.

Let $\{\underline{X}_1,\ldots,\underline{X}_d\}$ be a basis of $\mathfrak{g}$ and let $\{X_1,\ldots,X_d\}$ be the corresponding vector fields on N. Since 0 is a regular value of μ, $\{X_{ix}\}$ is a linearly independent system in each $T_x\mu^{-1}(0)$. From the very definition of the moment map we have $\eta_p(X_i)=\mu(p)(X_i)=0$, hence $X_i\perp\xi$. As G acts by contact isometries, we have

$$\mathcal{L}_{X_i}g=0, \quad \mathcal{L}_{X_i}\eta=0 \quad i=1,\ldots,d. \tag{3.1}$$

Note that these also imply $[X_i,\xi]=\mathcal{L}_{X_i}\xi=0$.

We observe that $\mu^{-1}(0)$ is an isometrically immersed submanifold of N (we denote the induced metric also with g) whose tangent space in each point is described by $Y\in T_x\mu^{-1}(0)$ if and only if $d\mu_x(Y)=0$. Hence, by the definition of the moment map, the vector fields ξ and X_i are tangent to $\mu^{-1}(0)$. Moreover, for any Y tangent to $\mu^{-1}(0)$, one has $g(\varphi X_i,Y)=d\eta(Y,X_i)=d\mu(Y)=0$; hence the vector fields $\{X_i\}$ produce a local basis (not necessarily orthogonal) of the normal bundle of $\mu^{-1}(0)$. The shape operators $A_i:=A_{\varphi X_i}$ of this submanifold in N are computed as follows [we let ∇, ∇^N be the Levi Civita covariant derivatives of $\mu^{-1}(0)$, resp. N]:

$$\begin{aligned}g(A_iY,Z)&=-g(\nabla^N_Y(\|X_i\|^{-1}\varphi X_i),Z)=-g(Y(\|X_i\|^{-1})\varphi X_i,Z)-g(\|X_i\|^{-1}\nabla^N_Y(\varphi X_i),Z)\\&=-\|X_i\|^{-1}g(\nabla^N_Y(\varphi X_i),Z)=-\|X_i\|^{-1}g(\nabla^N_Y(\varphi)X_i+\varphi\nabla^N_YX_i,Z)\\&=-\|X_i\|^{-1}g(\eta(X_i)Y-g(X_i,Y)\xi+\varphi\,\nabla^N_YX_i,Z)\\&=\|X_i\|^{-1}\{g(X_i,Y)\,\eta(Z)-g(\varphi\,\nabla^N_YX_i,Z\}.\end{aligned}\tag{3.2}$$

In particular, for the corresponding quadratic second fundamental forms we get

$$h_i(Y,\xi)=\|X_i\|^{-1}g(X_i,Y),\qquad h_i(\xi,\xi)=0.\tag{3.3}$$

Consequently, one easily obtains the following: *the restriction of the vector field ξ is Killing on $\mu^{-1}(0)$ too*.

Using the Gauss equation of a submanifold,

$$R^N(X,Y,Z,W)=R^{\mu^{-1}(0)}(X,Y,Z,W)+g(h(X,Z),h(Y,W))-g(h(X,W),h(Y,Z)),$$

and the formula (3.2) we now compute the needed part of the curvature tensor of $\mu^{-1}(0)$ at a fixed point $p\in\mu^{-1}(0)$. We take X,Y,Z orthogonal to ξ_p and obtain

$$\begin{aligned}g(R^{\mu^{-1}(0)}(X,\xi)Y,Z)-g(R^N(X,\xi)Y,Z)&=-\sum_{i=1}^{d}\|X_i\|^{-2}\{h_i(X,Y)h_i(\xi,Z)-h_i(X,Z)h_i(\xi,Y)\}\\&=-\sum_{i=1}^{d}\|X_i\|^{-2}\{g(X_i,Z)g(\nabla^N_XX_i,\varphi Y)\\&\quad-g(X_i,Y)g(\nabla^N_XX_i,\varphi Z)\}\end{aligned}\tag{3.4}$$

(note that $\nu_i=\|X_i\|^{-1}\varphi X_{ip}$ are chosen to be orthonormal in p; this is always possible pointwise by an appropriate choice of the initial $\underline{X}_i$).

Now let $\pi:\mu^{-1}(0)\to M$ and endow M with the projection g^M of the metric g such that π becomes a Riemannian submersion. This is possible because G acts by isometries. In this setting, the vector fields X_i span the vertical distribution of the submersion, while ξ is horizontal and projectable (because $\mathcal{L}_{X_i}\xi=0$). Denote with ζ its projection on M. ζ is obviously unitary. To prove that ζ is Killing on M, we just observe that $\mathcal{L}_\zeta g(Y,Z)=\mathcal{L}_\xi g(Y^h,Z^h)$, where Y^h denotes the horizontal lift of Y. Finally, to compute the values $R^M(X,\zeta)Y$ of the curvature tensor of g^M, we use the O'Neill formula [cf. Ref. 12 Eq. (9.28f)]:

$$\begin{aligned}g^M(R^M(X,\zeta)Y,Z)=&\,g(R^{\mu^{-1}(0)}(X^h,\xi)Y^h,Z^h)+g(A(X^h,\xi),A(Y^h,Z^h))-g(A(\xi,Y^h),A(X^h,Z^h))\\&+g(A(X^h,Z^h),A(\xi,Z^h)),\end{aligned}$$

where X,Y,Z are unitary, normal to ζ and the O'Neill (1,2) tensor A is defined as: $A(Z^h,X^h)=$vertical part of $\nabla_{Z^h}X^h$. Using the Gauss formula and (3.3), we obtain

$$g(\nabla_{Z^h}\xi, X_i) = g(\varphi Z^h, X_i) = -g(Z^h, \varphi X_i) = 0;$$

hence $\nabla_{Z^h}\xi$ has no vertical part and $A(Z^h,\xi)=0$. Thus

$$R^M(X,\zeta)Y = R^{\mu^{-1}(0)}(X^h,\xi)Y^h = R^N(X^h,\xi)Y^h$$

because of (3.4) and the fact that X^h, Y^h are normal to all X_i. Hence

$$R^M(X,\zeta)Y = g(\xi,Y^h)X^h - g(X^h,y^h)\xi = g^M(\zeta,Y)X - g^M(X,Y)\zeta,$$

which proves that (M,g^M,ζ) is a Sasakian manifold. □

B. Compatibility with the cone construction

In the following we relate Sasakian reduction to Kähler reduction by using the cone construction. Roughly speaking, we prove that reduction and taking the cone are commuting operations.

Let $\omega = dr\wedge\eta + r^2\, d\eta$ be the Kähler form of the cone $C(N)$ over a Sasakian manifold (N,g,ξ). If ρ_t are the translations acting on $C(N)$ by $(x,r)\mapsto(x,tr)$, then the vector field $R_0 = r\,\partial r$ is the one generated by $\{\rho_t\}$. Moreover, the following two relations are useful:

$$\mathcal{L}_{R_0}\omega = \omega, \quad \rho_t^*\omega = t\omega. \tag{3.5}$$

Suppose that a compact Lie group G acts on $C(N)$ by holomorphic isometries, commuting with ρ_t. This ensures a corresponding action of G on N. In fact, we can consider $G \cong G\times\{Id\}$ acting as $(g,(x,r))\times(gx,r)$.

Suppose that a moment map $\Phi: C(N)\to\mathfrak{g}$ exists.

As above, let $\{\underline{X}_1,\ldots,\underline{X}_d\}$ be a basis of $\mathfrak{g}$ and let $\{X_1,\ldots,X_d\}$ be the corresponding vector fields on $C(N)$. We see that X_i are independent on r, hence can be considered as vector fields on N. Furthermore, the commutation of G with ρ_t implies

$$\Phi(\rho_t(p)) = t\Phi(p). \tag{3.6}$$

Now we imbed N in the cone as $N\times\{1\}$ and let $\mu := \Phi|_{N\times\{1\}}$. This is the moment map of the action of G on N. To see this, recall the definition of the symplectic moment map $\Phi = (\Phi_1,\ldots,\Phi_d)$: Φ_i is given up to constant by $d\Phi_i(Y) = \omega(X_i, Y)$. Here we uniquely determine Φ_i by imposing the condition $\eta(X_i) = \Phi|_{N\times\{1\}}$. This immediately implies that the Reeb field of N is orthogonal to the vector fields X_i since $g(\xi,X_i) = \eta(X_i) = 0$. As G acts by isometries on $C(N)$, we may project the cone metric to a metric on $N//G\times\mathbb{R}_+$ which we denote by g_0. Then $g_0(Y,Z) = C(g)(Y^h,Z^h)$, where Y^h, Z^h are the unique vector fields on $\Phi^{-1}(0)$ orthogonal to all of X_i which project on Y, Z (we call them horizontal).

Let $P = \Phi^{-1}(0)/G$ be the reduced Kähler manifold. The key remark is that because of (3.6), $\Phi^{-1}(0)$ is the cone $N'\times\mathbb{R}_+$ over $N' = \{x\in N; (x,1)\in\Phi^{-1}(0)\}$. Moreover, since the actions of G and ρ_t commute, one has an induced action of G on N'. Then

$$\Phi^{-1}(0)/G \cong (N'\times\mathbb{R}_+)/G \cong N'/G\times\mathbb{R}_+.$$

The manifold $N'//G\times\mathbb{R}_+$ is Kähler, as the reduction of a Kähler manifold, but we still have to check that this Kähler structure is a cone one. For the more general, symplectic case, this was done in Ref. 13. Let g_0 be the reduced Kähler metric and g' be the Sasakian reduced metric on $N'//G$. It is easily seen that the lift of g_0 to $\Phi^{-1}(0)$ coincides with the lift of the cone metric $r^2 g' + dr^2$ on horizontal fields. This implies that the cone metric coincides with g_0.

Summing up we have proved the following.

Theorem 3.2: Let (N,g,ξ) be a Sasakian manifold and let $(C(N), C(g), J)$ be the Kähler cone over it. Let a compact Lie group G act by holomorphic isometries on $C(N)$ and commuting with the action of the 1-parameter group generated by the field R_0. If a moment map with regular value 0 exists for this action, then a moment map with regular value 0 exists also for the induced action of G on N. Moreover, the reduced space $C(N)//G$ is the Kähler cone over the reduced Sasakian manifold $N//G$.

The advantage of defining the Sasakian reduction *via* Kähler reduction, as done in Ref. 3 for 3-Sasakian manifolds, is the avoiding of curvature computations. However, as we shall see, the direct construction is easily applicable.

C. Compatibility with the Boothby–Wang fibration and obtaining Sasakian–Einstein spaces by reduction

Let (N,g,ξ) be a compact Sasakian manifold with a regular characteristic vector field and let $\pi:N\rightarrow P$ be the corresponding Boothby–Wang fibration over the Hodge base P. Let G be a compact Lie group acting on N by Sasakian transformations. By (3.1), it preserves the fibers of π; hence it induces an action by Kähler transformations on P. If we denote with ${}^N\mu$, ${}^P\mu$ the corresponding moment maps, using the relation (cf. Theorem 2.1) $g(X,\varphi Y)=\pi^*\omega(X,Y)$ and the definitions of the respective moment maps, it is easy to see that we have an S^1 subbundle ${}^N\mu^{-1}(0)\rightarrow {}^P\mu^{-1}(0)$ and an S^1 bundle $N//G\rightarrow P//G$. Finally one can check that 0 is a regular value for an induced moment map on the base using the relation

$$d\,{}^N\mu(X)=d\,\eta(X,\cdot)=\pi^*\omega(X,\cdot)=\pi^*d\,{}^P\mu$$

The details being easily settled, we can state the following.

Proposition 3.1: Let G be a compact Lie group acting by Sasakian transformations on the total space of a Boothby–Wang fibration $\pi:N\rightarrow P$. Then there exists a Boothby–Wang fibration of the reduced spaces $N//G\rightarrow P//G$.

On the other hand, if the Hodge base of a Boothby–Wang fibration is Kähler–Einstein, then the total space is Sasakian–Einstein, as proved in Ref. 9. According to Futaki (cf. Ref. 14, Corollary 7.3.4), if one reduces a Kähler–Einstein manifold, the reduced space is still Einstein if and only if the length of the multivector $X_1\wedge\cdots\wedge X_d$ is constant on the level set of the moment map. Hence, one way to obtain Sasakian–Einstein metrics *via* reduction is to start with a regular Sasakian–Einstein manifold and with a Sasakian action inducing on the Hodge base a Kähler action of Futaki's type. The precise result is the following.

Theorem 3.3: Let G be a compact Lie group acting by Sasakian transformations on the regular Sasakian–Einstein manifold N having 0 as a regular value of the corresponding moment map μ. If the length of the multivector $X_1\wedge\cdots\wedge X_d$ is constant on $\mu^{-1}(0)$ then the reduced space is Sasakian–Einstein.

Proof: First observe that the Boothby–Wang fibration $\pi:N\rightarrow P$ has a Kähler–Einstein base, with positive Ricci curvature, according to Ref. 9, Theorem 2.5(iv). Now, from the equations (3.1) we see that the S^1 action on $\mu^{-1}(0)$ commutes with the G-action. In particular the multivector $X_1\wedge\cdots\wedge X_d$ is projectable to $\mu^{-1}(0)/S^1$ for the projection $\pi:\mu^{-1}(0)\rightarrow\mu^{-1}(0)/S^1$. In view of the preceding section, this is a restriction of the Boothby–Wang fibration to the corresponding zero-sets of the moment maps on N and P, respectively. Since it is a Riemannian submersion and X_i are orthogonal to ξ, the lengths of the projected vectors are preserved, as well as the length of the projected multivector. Now we can apply Futaki's result (Ref. 14, Corollary 7.3.4) in order to conclude that the base for the reduced Boothby–Wang fibration $N//G\rightarrow P//G$ is Kähler–Einstein. Then, again according to Ref. 9, Theorem 2.5(iv), the reduced Sasakian manifold is Sasakian–Einstein. □

We apply this result in the examples of the last section.

In the same spirit, combining Ref. 9, Theorem 2.5(iii) (stating that the base of the Boothby–Wang fibration is Fano if and only if the Ricci curvature of the total space is >-2) and Ref. 14, Corollary 7.3.3 (asserting that the reduced space of a Fano manifold is again Fano), we obtain the following.

Proposition 3.2: Let G be a compact Lie group acting by Sasakian transformations on the regular Sasakian–Einstein manifold N having 0 as a regular value of the corresponding moment map μ. Then N has Ricci curvature $\mathrm{Ric}>-2$ if and only if $N//G$ has Ricci curvature $\mathrm{Ric}>-2$.

IV. EXAMPLES: S^1 ACTIONS ON SASAKIAN SPHERES

Example 4.1: Let us start with $S^7\subset\mathbb{C}^4$ with its standard Sasakian structure. Let the complex coordinates of $\mathbb{C}^4$ be $(z_0,\ldots,z_3)$, with $z_j=x_j+iy_j$. The contact form on S^7 can then be written as

$$\eta=\sum_{j=0}^{3}(x_j\,dy_j-y_j\,dx_j),$$

and its Reeb field is

$$\xi=\sum_{j=0}^{3}(x_j\,\partial y_j-y_j\,\partial x_j).$$

Let S^1 act on S^7 by $e^{it}\mapsto(e^{-it}z_0,e^{-it}z_1,e^{it}z_2,e^{it}z_3)$. The associated field of this action is (in real coordinates)

$$X_0=-(x_0\,\partial y_0-y_0\,\partial x_0)-(x_1\,\partial y_1-y_1\,\partial x_1)+(x_2\,\partial y_2-y_2\,\partial x_2)+(x_3\,\partial y_3-y_3\,\partial x_3).$$

The moment map $\mu:S^7\rightarrow\mathbb{R}$ reads as

$$\mu(z)=\eta_z(X_0)=-|z_0|^2-|z_1|^2+|z_2|^2+|z_3|^2,$$

with zero level set

$$\{z\in S^7;\quad |z_0|^2+|z_1|^2=|z_2|^2+|z_3|^2\}=S^3\left(\frac{1}{\sqrt{2}}\right)\times S^3\left(\frac{1}{\sqrt{2}}\right).$$

Clearly μ is nondegenerate on $\mu^{-1}(0)$.

The reduced space can be identified with $S^3\times S^3/S^1$ which, by Ref. 6, is diffeomorphic with $S^2\times S^3$. [In this case, one can also avoid the topological arguments in Ref. 6 and identify the reduced space by observing that the following diffeomorphism of $S^3\times S^3$: $(z_0,z_1,z_2,z_3)\mapsto(z_0z_3+\overline{z_1z_2},z_0z_2-\overline{z_1z_3},z_2,z_3)$ is equivariant with respect to the previous S^1 action which, restricted to the second factor of the product, is the usual action inducing the Hopf fibration; *mille grazie* to Rosa Gini and Maurizio Parton for letting us know it (Ref. 15).]

The reduced Sasakian structure obtained in this way on $S^2\times S^3$ is directly checked to be Einstein and to project on the Kähler Einstein metric of $\mathbb{C}P^1\times\mathbb{C}P^1$ making the fiber map be a Riemannian submersion. As by Ref. 6 such an Einstein metric is unique, our reduced Sasakian structure coincides with the Sasakian structure found in Ref. 16 viewing $S^2\times S^3$ as a minimal submanifold of S^7, the total space of the pull-back over $\mathbb{C}P^1\times\mathbb{C}P^1$ of the Hopf bundle $S^7\rightarrow\mathbb{C}P^3$. The same Einstein–Sasakian metric on $S^2\times S^3$ also appears in Ref. 10, constructed by a different approach. In Example 4.3 we will generalize this structure by making use of Theorem 3.2.

Example 4.2: Consider again S^7 as the starting Sasakian manifold, but let S^1 act by $e^{it}\mapsto(e^{-kit}z_0,e^{it}z_1,e^{it}z_2,e^{it}z_3)$, $k\in\mathbb{Z}_+$. Now $\mu^{-1}(0)\cong S^1(\sqrt{k/k+1})\times S^5(\sqrt{1/k+1})$. In order to identify the reduced space, we regard the $k:1$ mapping,

$$S^1\times S^5\ni(z_0,z_1,z_2,z_3)\mapsto((z_0)^{-k},z_1,z_2,z_3)\in S^1\times S^5.$$

It induces a k:1 map from $M=S^1\times S^5/S^1$, where S^1 acts diagonally, to the reduced space $\mu^{-1}(0)/S^1$ with the action given above. As in Ref. 15, the map

$$(z_0,\ldots,z_3)\mapsto(z_0,\overline{z_0}z_1,\overline{z_0}z_2,\overline{z_0}z_3)$$

is an equivariant diffeomorphism of $S^1\times S^5$, equivariant with respect to the diagonal action of S^1 and the action of S^1 on the first factor. Hence M is diffeomorphic to S^5 and the reduced Sasakian space is $S^5/\mathbb{Z}_k$. Again, we shall see below that the metric is actually Sasakian–Einstein and it is the same as that found in Ref. 6.

Example 4.3: In general, consider the weighted action of S^1 on $S^{2n-1}\subset\mathbb{C}^n$ by

$$(e^{it},(z_0,\ldots,z_{n-1}))\mapsto(e^{\lambda_0 it}z_0,\ldots,e^{\lambda_n it}z_{n-1}),$$

where $(\lambda_0,\ldots,\lambda_{n-1})\in\mathbb{Z}^n$. The associated moment map,

$$\mu(z)=\lambda_0|z_0|^2+\cdots+\lambda_n|z_{n-1}|^2,$$

is regular on $\mu^{-1}(0)$ for any $(\lambda_0,\ldots,\lambda_{n-1})$ such that $\lambda_0\cdots\lambda_{n-1}\neq 0$, $(\lambda_0,\ldots,\lambda_{n-1})=1$ and at least two λ's have different signs (compare with the 3-Sasakian case where the weights obey to more restrictions; cf. Ref. 3).

Now take $\lambda_0=\cdots=\lambda_k=a$ and $\lambda_{k+1}=\cdots=\lambda_{n-1}=-b$, $a,b\in\mathbb{Z}_+$ relatively prime. Then $\mu^{-1}(0)\cong S^{2k+1}(\sqrt{a/a+b})\times S^{2(n-k)-1}(\sqrt{b/a+b})$. Moreover the length of the induced vector field X_0 on μ^{-1} is easily calculated to be $(a^2+b^2)/2$. Now we can apply Theorem 3.2 to deduce that the reduced metric is Sasakian–Einstein. Note that the induced metric on $\mu^{-1}(0)$ coincides with the product metric of the standard metrics of the two factors. Moreover we see that the reduced space is diffeomorphic with an S^1 factor of the above product of spheres given by the following action:

$$(e^{it},(x,y))\mapsto(e^{iat}x,e^{-ibt}y).$$

One can now adapt the arguments of Ref. 6, Corollary 2.2 and prove that the reduced spaces are S^1 bundles over $\mathbb{C}P^k\times\mathbb{C}P^{n-k-1}$ and, for $1\leqslant k$, $4<n$, they are not homeomorphic to each other in general. These are the examples considered in Ref. 6. However, for $k=1$, $n=2$, the reduced space is always diffeomorphic with $S^2\times S^3$. Hence, one obtains an infinite family of Sasakian structures on $S^2\times S^3$.

ACKNOWLEDGMENTS

Some of the results in this article were communicated at the 2nd Meeting on Quaternionic Structures in Mathematics and Physics, Rome 1999. This research was initiated during the authors visit at the *Abdus Salam International Center for Theoretical Physics*, Trieste, in summer 1999. The authors thank the Institute for support and excellent environment. The second author also acknowledges financial and technical support from the *Erwin Schrödinger Institute*, Vienna, in September 1999, during the ''Special holonomies'' semester. Both authors are grateful to Kris Galicki and Henrik Pedersen for many illuminating conversations on Sasakian geometry and related themes. Both authors are EDGE members, partially supported by the European Contract ''Human Potential Programme Research Training Network'' HPRN-CT-2000-00101.

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