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Oeljeklaus-Toma manifolds and locally conformally Kähler metrics. A state of the art

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To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. We review several properties about Oeljeklaus-Toma manifolds, especially the locally conformally Kähler ones, with focus on the non-existence of certain complex submanifolds.

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1. Introduction

The idea of associating compact complex manifolds to number fields is present since the very beginnings of complex geometry. If one was to write a history of this ideas, he would probably start from elliptic curves, which subtle links to number theory were felt by L. Kronecker and K. Weierstrass, would then include H. Weyl, whose research on complex tori have roots in the study of number fields units, and would then arrive to A. Weil who extended this line of research to Kähler manifolds.

The goal of the present paper is to give an account on the recent progress in a highly interesting class of compact complex manifolds associated to certain number fields introduced by K. Oeljeklaus & M. Toma in 2005. Despite being a relatively new topic, this kind of manifolds already provided a number of surprising results in the non-Kähler geometry, as we shall see below.

2. Basic facts from algebraic number theory

We recall (cf. e.g. [7]) that an (abstract) number field is a finite extension K of $\mathbb{Q}$; it follows that K is isomorphic (as $\mathbb{Q}-$algebras) to $\mathbb{Q}[X]/(f)$ where $f \in \mathbb{Z}[X]$ is a (monic) irreducibile polynomial. An abstract number field K can be embedded into $\mathbb{C}$

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by mapping $X(\mathrm{mod} f) \in K$ to α, where α is a root of f. It follows that K has exactly n embeddings into $\mathbb{C}$, where $n = \deg(f)$. Usually, one divides the roots of f into two subsets: the real ones, and call the corresponding embeddings *real embeddings* of K, and the complex, non-real ones, that come in pairs of conjugate numbers (and call the resulting embeddings accordingly, *complex embeddings*). We shall denote by s the number of real embeddings and by $2t$ the number of complex ones; hence $n = s + 2t$.

An *algebraic integer* of K is an element $a \in K$ satisfying a monic equation with integer coefficients. The set of all algebraic integers of K forms a ring, usually denoted by $\mathcal{O}_K$. For instance, if $p > 2$ is some prime number and $K = \mathbb{Q}(\zeta_p)$ (where ζ_p is a primitive root of unity of order p) then $\mathcal{O}_K = \mathbb{Z}[\zeta_p]$. But in the general case, such nice descriptions of the ring of integers are no longer available. Eventually, let us recall that seen as $\mathbb{Z}-$module, $\mathcal{O}_K$ is free of rank n.

The invertible elements of $\mathcal{O}_K$ are called *units*, and the (multiplicative) group of units is denoted $\mathcal{O}_K^*$. By the celebrated Dirichlet's units theorem, $\mathcal{O}_K^*$ is a group of rank $s + t - 1$. For instance, if $K = \mathbb{Q}(\sqrt{3})$ then any solution $(a, b) \in \mathbb{Z}^2$ of the Pell equation

$$x^2 - 3y^2 = 1$$

will define a unit $a + b\sqrt{3} \in \mathcal{O}_K^*$. By contrast, in $K = \mathbb{Q}(i\sqrt{3})$ the only units are ± 1 and $\pm\epsilon$, where ϵ is a non-real root of unity of order 3. Again, in the general case, there are no immediate descriptions of the group of units.

3. Oeljeklaus-Toma manifolds

3.1. The construction

The following construction was done in [8].

Fix a number field K with s real embeddings and $2t > 0$ complex embeddings. Suppose the embeddings $\sigma_1, \dots, \sigma_n$ of K are labelled in such a way that the first s ones are real, and $\sigma_{s+k} = \overline{\sigma}_{s+t+k}$ for all $k,\ 1 \le k \le t$.

We say that a unit $u \in \mathcal{O}_K^*$ is *totally positive* if $\sigma_i(u) > 0$ for all real embeddings $\sigma_i, 1 \le i \le s$. The set $\mathcal{O}_K^{*,+}$, of totally positive units form a subgroup of $\mathcal{O}_K^*$, obviously of finite index - since for any unit u, its square u^2 is totally positive.

Let $\mathbb{H} = \{z \in \mathbb{C}\,;\ \mathrm{Im}\, z > 0\}$ be the upper half-plane. For any $a \in \mathcal{O}_K$ denote by T_a the automorphism of $\mathbb{H}^s \times \mathbb{C}^t$ given by

$$T_a(z_1, \dots, z_{t+s}) = (z_1 + \sigma_1(a), \dots, z_{s+t} + \sigma_{s+t}(a))\,.$$

Similary, for any totally positive unit u, let R_u be the automorphism of $\mathbb{H}^s \times \mathbb{C}^t$ defined by

$$R_u(z_1, \dots, z_{t+s}) = (\sigma_1(u)z_1, \dots, \sigma_{s+t}(u)z_{t+s})\,.$$

Note that the totally positivity of u is needed for R_u to act on $\mathbb{H}^s \times \mathbb{C}^t$.

The above maps define for any subgroup $U \subset \mathcal{O}_K^{*,+}$ a fixed-point-free action of the semidirect product $U \ltimes \mathcal{O}_K$ on $\mathbb{H}^s \times \mathbb{C}^t$. The main point is that one can always find subgroups U such that the above action is also discrete and cocompact; such subgroups are called *admissible subgroups*. Note that if U is an admissible subgroup then necessarily one has $\mathrm{rank}_{\mathbb{Z}}(U) + \mathrm{rank}_{\mathbb{Z}}(\mathcal{O}_K) = 2(s+t)$, hence $\mathrm{rank}_{\mathbb{Z}}(U) = s$. This explains why the condition $t > 0$ is needed: otherwise we would have that the rank of $\mathcal{O}_K^*$ is $s-1$, strictly less than s, and hence admissible subgroups could not exist.

By definition, if U is an admissible subgroup, the compact quotient

$$\mathbb{H}^s \times \mathbb{C}^t / U \ltimes \mathcal{O}_K$$

is called *an Oeljeklaus-Toma manifold* and is usually denoted by $X(K,U)$.

Remark 3.1. For $s = t = 1$, one recovers the familiar Inoue surface S_M, [5]. This is known to be (real) homogeneous, indeed a solvmanifold. Accordingly, H. Kasuya proved the following:

Proposition 3.2. [6, §6] *Oeljeklaus-Toma manifolds are solvmanifolds.*

Indeed, Kasuya proved that

$$X(K,U) = G/U \ltimes \mathcal{O}_K, \quad \text{with} \quad G = \mathbb{R}^s \ltimes_\phi (\mathbb{R}^s \times \mathbb{C}^t).$$

Here ϕ acts as follows:

$$\phi(t_1, \ldots, t_s) = \operatorname{diag}\left(e^{t_1}, \ldots, e^{t_s}, e^{\psi_1 + \sqrt{-1}\phi_1}, \ldots, e^{\psi_t + \sqrt{-1}\phi_t}\right),$$

where $\psi_k = \frac{1}{2}\sum_1^s b_{ik} t_i$, $\varphi_k = \sum_1^s c_{ik} t_i$, with the coefficients b_{ik}, c_{ik} given by expressing $|\sigma_{s+k}(a)| = e^{\frac{1}{2}\sum_1^s b_{ik} t_i}$, and hence $\sigma_{s+k}(a) = e^{\frac{1}{2}\sum_1^s b_{ik} t_i + \sum_1^s c_{ik} t_i}$.

The natural complex structure on $\mathbb{R}^s \ltimes_\phi (\mathbb{R}^s \times \mathbb{C}^t)$ is seen to descend to the quotient and to be integrable, but the induced complex structure is G left-invariant and *not* G right-invariant, and hence $X(K,U)$ is not a complex Lie group. This is in accordance with the result proven in [8] (that we also recall below, see 3.5) that the biholomorphism group of $X(K,U)$ is discrete.

3.2. Basic invariants

We next investigate the basic invariants of Oeljeklaus-Toma manifolds. We start by looking at their Betti numbers.

Theorem 3.3. ([8]) *If K is a number field with s real embeddings and t complex embeddings, and if U is an admissible subgroup of $\mathcal{O}_K^*$, then:*

a) $b_1(X(K,U)) = s$;

b) if, in addition, there is no proper subfield $L \subset K$ such that $U \subset \mathcal{O}_L^*$, then

$$b_2(X(K,U)) = \binom{s}{2}.$$

Sketch of proof. The basic idea to compute $H^i(X(K,U),\mathbb{Q})$ is as follows. Since the universal cover of $X(K,U)$ is contractible, one is reduced to compute the group cohomology $H^i(U \ltimes \mathcal{O}_K, \mathbb{Q})$. Next, as $U \ltimes \mathcal{O}_K$ is a semidirect product of two abelian groups, and since the cohomology of abelian groups is well-known, one can simply use the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(U, H^q(\mathcal{O}_K, \mathbb{Q})) \Rightarrow H^{p+q}(U \ltimes \mathcal{O}_K, \mathbb{Q}).$$

Now the claims of a) and b) follow by a careful inspection of the differentials in the above spectral sequence.

Alternatively, one can prove a) directly, along the following lines. An immediate computation shows that

$$T_a R_u T_a^{-1} R_u^{-1} = T_{(1-u)a}$$

for any $u \in U$ and any $a \in \mathcal{O}_K$. Now one can check that the subgroup of $\mathcal{O}_K$ generated by elements of the form $(1-u)a$ is of finite index, so U is a quotient of $H_1(X(K,U),\mathbb{Z})$ by a finite subgroup; consequently, the first Betti number will equal the rank of U.

Remark 3.4. In fact, one can explicitely exhibit s linearly independent closed 1-forms on $X(K,U)$. Indeed, if we let $z_k = x_k + iy_k$ for all k, then the differential forms

$$\omega_k = \frac{1}{y_k} dy_k, k = 1, \ldots, s \tag{3.1}$$

defined on $\mathbb{H}^s \times \mathbb{C}^t$ are $U \ltimes \mathcal{O}_K-$invariant, hence descending to forms on $X(K,U)$.

Next, we look at some analytical invariants.

Theorem 3.5. ([8]) *On any Oeljelkaus-Toma manifold $X = X(K,U)$ the holomorphic vector bundles: Ω^1_X, the holomorphic tangent bundle $\mathcal{T}_X$ and any positive power $\mathcal{K}_X^n$ of the canonical bundle have no global holomorphic sections. Consequently, $H^{1,0}(X) = H^0(X, \Omega^1_X) = 0$, X has finitely many automorphisms and its Kodaira dimension is $-\infty$.*

By contrast, $h^{0,1}(X) = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X) \geq s$. In particular, since $h^{0,1} \neq h^{1,0}$, it follows that for $s > 0$ the manifold X cannot carry Kähler metrics.

Sketch of proof. The assertions on the absence of global sections of all the vector bundles in the statement follow in a rather direct way: one shows that the corresponding bundles on the universal cover have no non-zero global sections which are invariant under $U \ltimes \mathcal{O}_K$. The key ingredient is the following fact: if one factors $\mathbb{H}^s \times \mathbb{C}^t$ by $\mathcal{O}_K$ only (hence getting a non-compact manifold, which covers X), the quotient has no global non-constant holomorphic function (exactly as in the compact case).

4. Oeljeklaus-Toma manifolds and locally conformally Kähler geometry

4.1. LCK geometry

At this point, we recall the notion of *locally conformally Kähler manifold, LCK* for short, see [3]. By definition, a hermitian metric g on a complex manifold X is LCK if X can be covered by open subsets

$$X = \bigcup_{\alpha \in A} U_\alpha$$

with the property that on each U_α there exists a Kähler metric g_α wich is conformal to the restriction of g to U_α,

$$g_\alpha = e^{-f_\alpha} g_{|U_\alpha}$$

for some smooth function f_α defined on U_α. If one of the U_α equals the whole X, we say that g is *globally conformally Kähler*, GCK for short.

There are at least two different, equivalent ways, of saying that a hermitian metric g is LCK. One of them is as follows. Let g be a hermitian metric on the complex manifold X and let ω be its associated Kähler form,

$$\omega(X, Y) = g(XJY)$$

where J is the almost-complex structure of X. Then g is LCK if and only if there exists a closed 1-form θ (called *the Lee form* of g) such that

$$d\omega = \theta \wedge \omega.$$

Notice that g is GCK if and only if θ is exact.

An equivalent definition is as follows. Let $\widetilde{X}$ be the universal cover of X. Then X has an LCK metric iff $\widetilde{X}$ has a Kähler metric Ω upon which the fundamental group of X (seen as the group of deck transforms of $\widetilde{X}$) acts by homotheties,

$$\gamma^*(\Omega) = \chi(\gamma)\Omega, \forall \gamma \in \pi_1(X) \tag{4.1}$$

for some $\chi(\gamma) \in \mathbb{R}_{>0}$. Notice that in order to obtain non-GCK metrics on X, at least one $\chi(\gamma)$ above should be different from 1.

This last way of characterizing LCK manifolds is particularly useful in exhibiting examples. For instance, we can see that the so-called *diagonal Hopf manifolds* are LCK. Recall that such a manifold is by definition the quotient of $\mathbb{C}^n \setminus \{0\}$ under the action of $\mathbb{Z}$ generated by the map

$$(z_1, \ldots, z_n) \mapsto (\alpha z_1, \ldots, \alpha z_n)$$

where $\alpha \in \mathbb{C}, |\alpha| \neq 1$. Clearly, in this way, the action of $\mathbb{Z}$ is by homotheties with respect to the standard flat metric on $\mathbb{C}^n \setminus \{0\}$,

$$\omega_{flat} = dz_1 \wedge d\overline{z}_1 + \cdots + dz_n \wedge d\overline{z}_n.$$

In fact, all Hopf manifolds $\mathbb{C}^n \setminus \{0\}/\langle A \rangle$, with A being a linear operator with eigenvalues of strictly smaller than 1 absolute values are LCK, see [11].

Locally conformally Kähler metrics were introduced for the first time by I. Vaisman in the mid 80's. Since then, by the effort of many people, it was shown that almost all non-Kähler compact complex surfaces have LCK metrics, see [1], [2]. Still, in higher dimensions, until the paper of Oeljeklaus-Toma appeared, the only known LCK structures known were basically Hopf manifolds (and their complex submanifolds).

Theorem 4.1. ([8]) *Let K be a number field with $t = 1$ complex embeddings. Then, for any admissible group of totally positive units U, the manifold $X(K, U)$ has an LCK metric.*

Proof. Let $H : \mathbb{H}^s \times \mathbb{C} \to \mathbb{C}$ be the map

$$H(z_1, \ldots, z_s, z_{s+1}) = \prod_{i=1}^{s} \frac{1}{Im(z_i)} + |z_{s+1}|^2.$$

By direct computation, one checks that H is a *Kähler potential,* that is, its associated $(1, 1)-$ form

$$\omega = \sqrt{-1}\partial\overline{\partial}H$$

is a Kähler metric. Clearly, any translation T_a ($a \in \mathcal{O}_K$) leaves ω invariant, while for any $u \in U$ we have

$$R_u^*(\omega) = |\sigma(u)|^2\omega$$

where σ is the only (up to complex conjugation) complex embedding of K.

We see that $U \ltimes \mathcal{O}_K$ acts by homotheties upon ω, hence $X(K,U)$ has a LCK metric. On the other hand, this metric will not be GCK, as $X(K,U)$ cannot carry Kähler metrics.

Remark 4.2. 1. For $s = t = 1$, the above metric coincides with the one found by F. Tricerri in [13] on the Inoue surfaces of type S_M.

2. We stress that, unlike $\sqrt{-1}\partial\overline{\partial}H$, the above potential H *is not* acted on by homotheties. Moreover, no potential with this automorphy property can exist on Oeljeklaus-Toma manifolds, as this would impose the deck group to be isomorphic to $\mathbb{Z}$, [10].

Remark 4.3. A very important subclass of LCK manifolds is defined in terms of the Lee form. Namely, if (X, g) is an LCK manifold with Lee form θ, then (X, g) is called a *Vaisman manifold* if

$$\nabla\theta = 0$$

where ∇ is the Levi-Civita connection of the metric g. Typical examples are the diagonal Hopf manifolds (see [4] for Hopf surfaces or [11] for higher dimensions); other examples appear on surfaces, [1]. Compact Vaisman manifolds have very good geometric properties and are intimately related to Sasakian manifolds.

It is easily seen that the LCK metric in [8] is not Vaisman. Moreover:

Proposition 4.4. ([6]) *Oeljeklaus-Toma manifolds cannot carry any Vaisman metric.*

This is again consistent with the result in [1] that no Inoue surface can carry Vaisman metrics. Kasuya's proof uses the homogeneous presentation of the Oeljeklaus-Toma manifold and a characterization of the existence of Vaisman metrics on certain types of solvmanifolds in terms of cohomology of Lie algebras.

4.2. The Vaisman conjecture

The Vaisman conjecture. In [14], it was asserted that any compact LCK manifold X should have at least one odd Betti number of odd degree:

$$b_{2k+1}(X) = 1 (\bmod 2)$$

for some k. The conjecture was a long-standing one, until the paper of [8] appeared. The counter-example given there is as follows. Take any number field with $s = 2, t = 1$ and any admissible subgroup $U \subset \mathcal{O}_K^{*,+}$. Then the manifold $X(K,U)$ will carry an LCK metric, by the above 4.1. On the other hand, $X(K,U)$ is of (complex) dimension $s+t = 3$ and its first Betti number is $b_1(X) = s = 2$ by 3.3, a). Consequently, one also has $b_5(X) = b_1(X) = 2$ from Poincaré duality. As $X(K,U)$ carries a global, nowhere vanishing 1-form (recall the forms defined in (3.1)), its Euler-Poincaré characteristic vanishes, so $b_3(X)$ is also even. We see $X(K,U)$ is indeed a counter-example to Vaisman's conjecture.

4.3. Submanifolds of Oeljelkaus-Toma manifolds

As noticed already, in the case $s = t = 1$, the Oeljeklaus-Toma manifolds are Inoue surfaces of type S_M. This particular kind of surfaces are remarkable, as they carry no closed analytic curve. It is thus a natural question to ask about submanifolds, or more general, closed analytic subspaces of Oeljeklaus-Toma manifolds. Of course, for convenient choices of the number field K and for the admissible subgroup U, the corresponding manifold $X(K,U)$ will contain proper submanifolds. For instance, if K is a proper extension of another number field L and if $U \subset \mathcal{O}_L^*$, then $X(L,U) \subset X(K,U)$, see [8] for details. It is thus reasonable to restrict our attention to the cases with "nice geometry", more exactly to the case $t = 1$, where the existence of LCK metrics holds. In this case, one has:

Theorem 4.5. ([9]) *Let K be a number field with $t = 1$ and let $X = X(K,U)$ be an associated Oeljeklaus-Toma manifold. If $Y \subset X$ is a closed connected reduced analytic subspace, then either $Y = X$ or Y is a point. In other words, X carries no proper closed analytic subspaces, i.e. it is a* simple manifold*, in the sense of Campana. In particular, LCK Oeljeklaus-Toma manifolds do not admit non-constant meromorphic functions.*

The proof relies on two deep facts. One is of purely geometrico-differential nature: the LCK metric leads to a "highly-positive" $(1,1)-$form, derived from the Lee form of the metric. The positivity of this form implies that a certain, very naturally defined foliation Σ on X has a very intriguing property: if a closed connected analytical subspace Y of X contains a point z sitting on the leaf Σ_x, then the whole Σ_z is contained in Y. Now, if $Y \subset X$ is a proper analytic subspace (i.e. $\dim(Y) > 0$) one shows that the closure of the leaves is the whole X; but to achieve this, one has to use a deep result in algebraic number theory, namely the so-called "strong adelic approximation theorem".

In the very general case (hence without restricting to $t = 1$, i.e. to LCK geometry), one can show

Theorem 4.6. ([15]) *Let X be an Oeljeklaus-Toma manifold. Then X carries no closed 1-dimensional analytic subspaces.*

Recently the same author obtained an extension of this theorem, to

Theorem 4.7. ([16]) *Let X be an Oeljeklaus-Toma manifold. If $S \subset X$ is a smooth compact surface, then S is a Inoue surface.*

An interesting (and apparently rather difficult) question imposes by itself:

Question 4.8. Is it true that if $X = X(K,U)$ is an Oeljeklaus-Toma manifold and if $X' \subset X$ is a connected, closed, reduced, analytical space, then X' is of the form $X' = X(K',U)$ with $K' \subset K$ and $U \subset U$ (i.e X' is obtained by the procedure described at the beginning of the section)?

Note that an affirmative answer would imply all theorems above, as fields with $t_K = 1$ complex embeddings have no proper subfields K' with $t_{K'} > 0$, thus we would get Theorem 4.5, and also Theorem 4.6, since in quadratic imaginary fields the rank of the group of units is zero and Theorem 4.7, as OT-surfaces are Inoue surfaces, according to Remark 3.1.

4.4. LCK rank of Oeljeklaus-Toma manifolds

The deep interplay between geometry and number theory, emphasized in the skecth of proof of 4.5 is actually much more extended. We illustrate this in the following.

Recall that one of the possible definitions of an LCK metric on a manifold X involves the homothety factors described by relation (4.1). Note that if $\gamma \in \pi_1(X)$ is a deck-transformation with $\chi(\gamma) = 1$, then actually γ is an isometry of the Kähler metric Ω. Hence, a natural question occurs: "how many" of the elements $\gamma \in \pi_1(X)$ are "honest homotheties", i.e. with $\chi(\gamma) \neq 1$? Put in a more rigourous setup:

Question 4.9. Determine how large can be the rank of the group

$$\{\chi(\gamma)\,;\, \gamma \in \pi_1(X)\}. \tag{4.2}$$

Of course, for LCK, non-GCK manifolds, this LCK rank is bounded from below by 1 (as at least one of the γ's must not be an isometry) and from above by the first Betti number of X. Until the Oeljeklaus-Toma manifolds appeared, in all examples known, the rank above actually had only these two extremal values: either 1, or $b_1(X)$. Some Oeljeklaus-Toma manifolds are -so far- the only known examples when this rank is non-trivial; more precisely, we have:

Theorem 4.10. ([12]) *Let K be a number field with $t = 1$ and $X = X(K, U)$ be an Oeljeklaus-Toma manifold. Then, the rank of the above group (defined in (4.2)) is different form 1 and $b_1(X)$ if and only if K is a quadratic extension of a (totally real) number field. In this last case, the rank equals $\frac{b_1(X)}{2}$, and this possibility occurs for Oeljeklaus-Toma manifolds of arbitrary high dimensions.*

The basic idea behind the proof is as follows. For any $u \in U$ (seen as an element in $\pi_1(X)$), the automorphy factor $\chi(u)$ is actually $|\sigma(u)|$, where σ is the only (up to complex conjugation) complex embedding of K. Now, if the rank is different from $b_1(X)$, then at least one $u \in U$ must have $|\sigma(u)| = 1$. This forces u to be a *reciprocal unit,* i.e. its minimal polynomial over $\mathbb{Q}$ to be a reciprocal one. But if u is a reciprocal unit, then the field $K' = \mathbb{Q}(u + \frac{1}{u})$ is a subfield of K, of relative degree 2, and it can be easily shown that K' must actually be totally real. Eventually, to produce infinitely many examples of Oeljeklaus-Toma manifolds with non-trivial rank, one reverses the process. One starts with a totally real number field K' (for instance with cyclotomic fields) and extend it to a field $K \supset K'$ with $[K : K'] = 2$, taking care to ramify precisely one real embedding of K'.

4.5. Oeljeklaus-Toma manifolds with $t > 1$

As already noticed, the main ingredient (apart from the number-theoretical ones) in most of the results above is the existence of LCK metrics. So far, existence of such metrics is known to hold on Oeljeklaus-Toma manifolds $X(K, U)$ for which the number field K has precisely $t = 1$ complex embeddings. It is thus natural to ask whether this condition can be dropped.

Actually, as already noticed in [8], in the other "extreme case", i.e. if K is a number field with $s = 1$ real embedings (and $t > 1$ complex ones) then for any choice of the admissible group of units U, the resulting Oeljeklaus-Toma manifold $X(K, U)$ has no LCK metric.

There are (so far) at least two results showing that probably, if K is a number field with $t > 1$ complex embedings, then no Oeljeklaus-Toma manifold $X(K, U)$ carries an LCK metric.

The first one was already recalled (4.4): an Oeljeklaus-Toma manifold cannot carry Vaisman metrics. But this does not rule out the possibility of existence of non-Vaisman, LCK metrics. However, when there are "too many" complex embeddings, this is not true. More exactly, we have:

Theorem 4.11. ([17]) *let K be a number field with $t > 2s$. then for any admissible group of units U, the Oeljeklaus-Toma manifold $X(K, U)$ carries no LCK metric.*

The proof relies again on the interplay between differential geometry and number theory. Namely, first one shows that if an LCK metric exists on $X(K, U)$ then, by looking at the automorphy factors $\chi(u)$ of any unit $u \in U$ one gets that $|\sigma(u)|$ is the same for *any* complex embedding σ of K. But then, one exploits a nice fact about algebraic integers with "many" Galois conjugates of the same absolute value: their minimal polynomial f must actually be of the form $f(X) = g(X^t)$, and from here one easily derives a contradiction.

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