WEYL STRUCTURES ON QUATERNIONIC MANIFOLDS. A STATE OF THE ART.

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This is a survey on quaternion Hermitian Weyl (locally conformally quaternion Kähler) and hyperhermitian Weyl (locally conformally hyperkähler) manifolds. These geometries appear by requesting the compatibility of some quaternion Hermitian or hyperhermitian structure with a Weyl structure. The motivation for such a study is two-fold: it comes from the constantly growing interest in Weyl (and Einstein-Weyl) geometry and, on the other hand, from the necessity of understanding the existing classes of quaternion Hermitian manifolds.

Various geometries are involved in the following discussion. The first sections give the minimal background on Weyl geometry, quaternion Hermitian geometry and 3-Sasakian geometry. The reader is supposed familiar with Hermitian (Kähler and, if possible, locally conformally Kähler) and metric contact (mainly Sasakian) geometry.

All manifolds and geometric objects on them are supposed differentiable of class $\mathcal{C}^\infty$.

1. Weyl structures

We present here the necessary background concerning Weyl structures on conformal manifolds. We refer to [16], [18], [21] or to the most recent survey [14] for more details and physical interpretation (motivation) for Weyl and Einstein-Weyl geometry.

Let M be a n-dimensional, paracompact, smooth manifold, $n \geq 2$. A $\mathrm{CO}(n) \simeq \mathrm{O}(n) \times \mathbb{R}_+$ structure on M is equivalent with the giving of a conformal class c of Riemannian metrics. The pair (M, c) is a *conformal manifold*.

For each metric $g \in c$ one can consider the Levi-Civita connection ∇^g, but this will not be compatible with the conformal class. Instead, we shall work with $\mathrm{CO}(n)$-connections. Precisely:

Definition 1.1. A *Weyl connection* D on a conformal manifold (M, c) is a torsion-free connection which preserves the conformal class c. We say that D defines a Weyl structure on (M, c) and (M, c, D) is a Weyl manifold.

Preserving the conformal class means that for any $g \in c$, there exists a 1-form θ_g (called the Higgs field) such that

$$Dg = \theta_g \otimes g.$$

This formula is conformally invariant in the following sense:

$$\text{if } h = e^f g, \quad f \in \mathcal{C}^\infty(M), \text{ then } \theta_h = \theta_g - df. \tag{1.1}$$

Conversely, if one starts with a fixed Riemannian metric g on M and a fixed 1-form θ (with $T = \theta^\sharp$), the connection

$$D = \nabla^g - \frac{1}{2}\{\theta \otimes Id + Id \otimes \theta - g \otimes T\}$$

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is a Weyl connection, preserving the conformal class of g. Clearly, (g, θ) and $(e^f, \theta - df)$ define the same Weyl structure.

On a Weyl manifold (M, c, D), Weyl introduced the *distance curvature function*, a 2-form defined by $\Theta = d\theta_g$. By (1.1), the definition does not depend on $g \in c$. If $\Theta = 0$, the cohomology class $[\theta_g] \in H^1_{dR}(M)$ is independent on $g \in c$. A Weyl structure with $\Theta = 0$ is called *closed.*

All these geometric objects can be interpreted as sections in tensor bundles of the bundle of scalars of weight 1, associated to the bundle of linear frames of M *via* the representation $GL(n, \mathbb{R}) \ni A \mapsto |\det A|^{1/n}$. *E.g.* c is a section of $S^2T^*M \otimes L^2$, θ is a connection form in L whose curvature form is exactly the distance curvature function etc. This also motivates the terminology. We refer to [18] for a systematic treatment of this viewpoint.

A fundamental result on Weyl structures is the following "co-closedeness lemma":

Theorem 1.1. [17] *Let (M, c) be a compact, oriented, conformal manifold of dimension > 2. For any Weyl structure D preserving c, there exists a unique (up to homothety) $g_0 \in c$ such that the associated Higgs field θ_{g_0} is g_0-coclosed.*

The metric g_0 provided by the theorem is called the *Gauduchon metric* of the Weyl structure.

In Weyl geometry, the good notion of Einstein manifold makes use of the Ricci tensor associated to the Weyl connection:

$$Ric^D = \frac{1}{2}\sum_{i=1}^{n}\{g(R^D(X, e_i)Y, e_i) - g(R^D(X, e_i)e_i, Y)\}$$

where $g \in c$ and $\{e_i\}$ is a local g-orthonormal frame. The scalar curvature of D is then defined as the conformal trace of Ric^D. For each choice of a $g \in c$, $Scal^D$ is represented by $Scal^D_g = \mathrm{trace}_g Ric^D$. The relations between Ric^D and Ric^{∇^g} and, correspondingly, between their scalar curvatures are:

$$Ric^D = Ric^{\nabla^g} + \delta^g\theta \cdot g - (n-2)\{\nabla^g\theta + \|\theta\|^2_g \cdot g - \theta \otimes \theta\}. \tag{1.2}$$

$$Scal^D_g = Scal^{\nabla^g} + 2(n-1)\delta^g\theta - (n-1)(n-2)\|\theta\|^2_g. \tag{1.3}$$

Definition 1.2. A Weyl structure is *Einstein-Weyl* if the symmetric part of the Ricci tensor Ric^D of the Weyl connection is proportional to one (hence any) metric of c.

For an Einstein-Weyl structure, one has

$$Ric^D = \frac{1}{n}Scal^D_g \cdot g - \frac{n-2}{2}d\theta \tag{1.4}$$

for any $g \in c$.

Note that for an Einstein-Weyl structure, the scalar curvature $Scal^D$ need not be constant (this means D-parallel w.r.t. to D as a section of L^{-2}). But, if the Weyl connection is precisely the Levi-Civita connection of a metric in c (in this case the Weyl structure is called *exact*), then $Scal^D$ is constant.

Observe that for any Einstein-Weyl structure and any $g \in c$ one has the formula

$$\begin{aligned}\frac{1}{n}dScal^D_g = \frac{2n}{n-2}d\delta^g\theta - 2(\delta^g\theta)\theta - 2\delta^g\nabla^g\theta + \delta^g d\theta - \\ - 2\nabla^g_T\theta - (n-3)d\|\theta\|^2_g.\end{aligned} \tag{1.5}$$

This follows from (1.4), (1.2) and (1.3).

If g is the Gauduchon metric, $\delta^g\theta = 0$ and (1.5) reduces to

$$\frac{1}{n}dScal^D_g + 2\delta^g\nabla^g\theta - \delta^g d\theta + +2\nabla^g_T\theta + (n-3)d\|\theta\|^2_g = 0.$$

Contracting here with θ yields

$$D\theta = \frac{1}{2}d\theta \tag{1.6}$$

This, together with the relation between D and ∇^g prove the first statement of the following extremely important result (the second statement will be proved in a more particular situation):

Theorem 1.2. [18] *Let D be an Einstein-Weyl structure on a compact, oriented manifold (M, c) of dimension > 2. Let g be the Gauduchon metric in c associated to D and θ the corresponding Higgs field. If the Weyl structure is closed, but not exact, then*
1) θ is ∇^g parallel: $\nabla^g\theta = 0$ (in particular, also g-harmonic).
2) $Ric^D = 0$.

Odd dimensional spheres and products of spheres $S^1 \times S^{2n+1}$ admit Einstein-Weyl structures (note that $S^1 \times S^2$ and $S^1 \times S^3$ can bear no Einstein metric, cf. [22]). Further examples, with $Ric^D = 0$, will be the compact quaternion Hermitian Weyl and hyperhermitian Weyl manifolds.

2. Quaternionic Hermitian manifolds

This section is devoted to the introduction of quaternion Hermitian geometry. The standard references are [40], [4], [43], [3].

Definition 2.1. Let (M, g) be a $4n$-dimensional Riemannian manifold. Suppose $End(TM)$ has a rank 3 subbundle H with transition functions in SO(3), locally generated by orthogonal almost complex structures I_α, $\alpha = 1, 2, 3$ satisfying the quaternionic relations. Precisely:

$$I_\alpha^2 = -Id,\ I_\alpha I_\beta = \varepsilon_{\alpha\beta\gamma} I_\gamma,\ g(I_\alpha\cdot, I_\alpha\cdot) = g(\cdot,\cdot)\ \alpha,\beta,\gamma = 1,2,3 \tag{2.1}$$

where $\varepsilon_{\alpha\beta\gamma}$ is 1 (resp. -1) when $(\alpha\beta\gamma)$ is an even (resp. odd) permutation of (123) (such a basis of H is called admissible.) The triple (M, g, H) is called a *quaternionic Hermitian manifold* whose *quaternionic bundle* is H.

Any (local) or global section of H is called *compatible*, but in general, H has no global section. A striking example is $\mathbb{H}P^n$, the quaternionic projective space. The three canonical almost complex structures of $\mathbb{H}^{n+1}$ induced by multiplication with the imaginary quaternionic units descend to only local almost complex structures on $\mathbb{H}P^n$ generating the bundle H. The metric is the one projected by the flat one on $\mathbb{H}^{n+1}$, *i.e.* the Fubini-Study metric written in quaternionic coordinates. Note that $\mathbb{H}P^1$ is diffeomorphic with S^4, hence cannot bear any almost complex structure. Consequently, no greater dimensional $\mathbb{H}P^n$ can have an almost complex structure neither, because this would be induced on any quaternionic projective line $\mathbb{H}P^1$, contradiction.

This shows that the case when H is trivial is of a special importance and motivates

Definition 2.2. A quaternionic Hermitian manifold with trivial quaternionic bundle is called a *hyperhermitian manifold*.

In this terminology, an admissible basis of a quaternion Hermitian manifold is a local almost hyperhermitian structure.

For a hyperhermitian manifold we shall always fix a (global) basis of H satisfying the quaternionic relations, so we shall regard it as a manifold endowed with three Hermitian structures (g, I_α) related by the identities (2.1) The simplest example is $\mathbb{H}^n$. But we shall encounter other many examples.

The analogy with Hermitian geometry suggests imposing conditions of Kähler type. Let ∇^g be the Levi-Civita connection of the metric g.

Definition 2.3. A quaternionic Hermitian manifold (M, g, H) of dimension at least 8 is *quaternion Kähler* if ∇^g parallelizes H, *i.e.* $\nabla^g I_\alpha = a_\alpha^\beta \otimes I_\beta$ (with sqew-symmetric matrix of one forms (a_α^β)).

A hyperhermitian manifold is called *hyperkähler* if $\nabla^g I_\alpha = 0$ for $\alpha = 1, 2, 3$.

Remark 2.1. This definition of quaternion Kähler manifold is redundant in dimension 4. As S. Marchiafava proved (see [29]) that any four-dimensional isometric submanifold of a quaternion Kähler manifold whose tangent bundle is invariant to each element of H is Einstein and self-dual, one takes this as a definition. We won't be concerned with dimension 4 in this report.

Note that, unless in the complex case, here the parallelism of H does not imply the integrability of the single almost complex structures.

Example 2.1. $\mathbb{H}^n$ with its flat metric is hyperkähler. By a result of A. Beauville, the K^3 surfaces also, see [4], Chapter 14. The irreducible, symmetric quaternion Kähler were classified by J. Wolf. Apart $\mathbb{H}P^n$, the compact ones are: the Grassmannian of oriented 4-planes in $\mathbb{R}^m$, the Grassmannian of complex 2-planes of $\mathbb{C}^m$ and five other exceptional spaces (see [4], *loc. cit.*)

From the holonomy viewpoint, equivalent definitions are obtained as follows: A Riemannian manifold is quaternion Kähler (resp. hyperkähler) iff its holonomy is contained in $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1) = \mathrm{Sp}(n)\times\mathrm{Sp}(1)/\mathbb{Z}_2$ (resp. $\mathrm{Sp}(n)$).

On a quaternion quaternionic Hermitian manifold, the usual Kähler forms are only local: on any trivializing open set U, one has the 2-forms $\omega_\alpha(\cdot,\cdot) = g(I_a\cdot,\cdot)$. But the 4-form $\omega = \sum_{\alpha=1}^3 \omega_\alpha\wedge\omega_\alpha$ is global (because the transition functions of H are in $\mathrm{SO}(3)$), nondegenerate and, if the manifold is quaternion Kähler, parallel. Hence, it gives a nontrivial 4-cohomology class, precisely $[\omega] = 8\pi^2 p_1(H) \in H^4(M,\mathbb{R})$ ([27]).

To get a converse, let $\mathcal{H}$ be the algebraic ideal generated by H in Λ^2T^*M (by identifying, as usual, a local almost complex structure with the associated Kähler 2-form). It is a differential ideal if for any admissible basis of H, one has $d\omega_\alpha = \sum_{\beta=1}^3 \eta_{\alpha\beta}\wedge\omega_\beta$ for some local 1-forms $\eta_{\alpha\beta}$. Then we have:

Theorem 2.1. [43] *A quaternion Hermitian manifold of dimension at least* 12 *with closed* 4*-form* ω *is quaternion Kähler.*

A quaternion Hermitian manifold of dimension 8 *is quaternion Kähler iff* ω *is closed and* $\mathcal{H}$ *is a differential ideal.*

Swann's proof uses representation theory. A more direct one can be found in [2].

Remark 2.2. It is important to note that the condition of being a differential ideal is conformally invariant and, moreover, invariant to different choices of admissible basis.

For an almost almost quaternionic Hermitian manifold (M,g,H), we define its structure tensor by

$$T^H = \frac{1}{12}\sum_{\alpha=1}^{3}[I_\alpha, I_\alpha].$$

Clearly, T^H is zero if one can choose, locally, admissible basis formed by integrable almost complex structures. The *Obata connection* ∇^H is then the unique connection which preserves H and has torsion equal to T^H. It defines the fundamental 1-form η by the relation

$$\eta(x) = \frac{1}{(8(n+1)}\mathrm{trace}\{g^{-1}\nabla^H_X g\}.$$

A direct (but lengthy) computation proves:

Lemma 2.1. [2] *Let* (M,g,H) *be a quaternion Hermitian manifold such that* $\mathcal{H}$ *is a differential ideal. For any admissible basis of* H*, the following formulae for* T^H *and* ∇^H *hold good:*

$$\begin{aligned} T^H_X Y =& \frac{1}{60}\sum_{\alpha=1} 3\{[(5\varphi_\alpha+\rho_a l)(I_\alpha X]I_\alpha Y - [(5\varphi_\alpha+\rho_a l)(I_\alpha Y]I_\alpha X + \\ &+4\omega_\alpha X, Y)g^{-1}(\rho_\alpha\circ I_\alpha)\}, \end{aligned} \tag{2.2}$$

$$
\begin{aligned}
(\nabla^H_Z g)(X,Y) =&\frac{1}{12}\{2\varepsilon(Z)g(X,Y)+\varepsilon(X)g(Y,Z)+\varepsilon(Y)g(X,Z)-\\
&-\sum_{\alpha=1}^{3}\varepsilon(I_\alpha X)g(Y,I_\alpha Z)-\sum_{\alpha=1}^{3}\varepsilon(I_\alpha Y)g(X,I_\alpha Z)+\\
&+4\eta(Z)g(X,Y)\}
\end{aligned}
\tag{2.3}
$$

where

$$
\begin{aligned}
\varphi_\alpha =&2\eta_{[\beta\gamma]}\circ I_\alpha-\eta_{[\gamma\alpha]}\circ I_\beta-\eta_{[\alpha\beta]}\circ I_\gamma,\\
\rho_\alpha =&-6\eta_{\alpha\alpha}+2\eta-3\eta_{(\alpha\beta)}\circ I_\gamma+3\eta_{(\gamma\alpha)}\circ I_\beta\\
\varepsilon =&\sum_{\alpha=1}^{3}\eta_{[\alpha\beta]}\circ I_\gamma
\end{aligned}
$$

the subscript () *(resp.* []*) indicating symmetrization (resp. sqew-symmetrization).*

A most important geometric property of quaternion Kähler manifolds, partly motivating the actual interest in their study is:

Theorem 2.2. [5] *A quaternionic Kähler manifold is Einstein.*

We briefly sketch, following [4], p. 403, S. Ishihara's proof. We fix a local admissible basis. Direct computations lead to the formulae:

$$[R^g(X,Y),I_\alpha]=\sum_{\beta=1}^{3}\eta_{\alpha\beta}I_\beta$$

with a sqew-symmetric matrix of 2-forms $(\eta_{\alpha\beta})$ which can be expressed in terms of Ricci tensor as follows:

$$\eta_{\alpha\beta}(X,Y)=\frac{2}{n+2}Ric^g(I_\gamma X,Y),\quad \dim M=4n.$$

From these one gets:

$$
\begin{aligned}
&g(R^g(X,I_1X)Z,I_1Z)+g(R^g(X,I_1X)I_2Z,I_3Z)+\\
+&g(R^g(I_2X,I_3X)Z,I_1Z)+g(R^g(I_2X,I_3X)I_2Z,I_3Z)=\\
=&\frac{4}{n+4}Ric^g(X,X)\|Z\|^2=\frac{4}{n+4}Ric^g(Z,Z)\|X\|^2
\end{aligned}
$$

for any X and Z, hence $Ric^g(X,X)=\lambda g(X,X)$ and (M,g) is Einstein.

On the other hand, hyperkähler manifolds have holonomy included in $\mathrm{Sp}(n)\subset\mathrm{SU}(2n)$, hence they are Ricci-flat, in particular Einstein (see [4]).

Although apparently hyperkähler manifolds form a subclass of quaternion Kähler ones, this is not quite true. Besides the holonomy argument, the following result motivates the dichotomy:

Theorem 2.3. [5] *A quaternion Kähler manifold is Ricci-flat iff its reduced holonomy group is contained in* $\mathrm{Sp}(n)$*. And if it is not Ricci-flat, then it is de Rham irreducible.*

From these results it is clear that when discussing quaternion Kähler manifolds, one is mainly interested in the non-zero scalar curvature.

Ricci flat quaternion Kähler manifolds are called *locally hyperkähler*. Similarly, P. Piccinni discussed in [37], [38] the class of *locally quaternion Kähler manifolds*, having the *reduced* holonomy group contained in $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$ and proved:

Proposition 2.1. [37] *Any complete locally quaternion Kähler manifold with positive scalar curvature is compact, locally symmetric and admits a finite covering by a quaternion Kähler Wolf symmetric space.*

As the local sections of H are generally non-integrable, one cannot use the methods of complex geometry directly on quaternion Kähler manifolds. However, one can construct an associated bundle whose total space is Hermitian. Let $p : Z(M) \to M$ be the unit sphere subbundle of H. Its fibre $Z(M)_m$ is the set of all almost complex structures on $T_m M$. This is called the *twistor bundle* of M. Using the Levi-Civita connection ∇^g, one splits the tangent bundle of $Z(M)$ in horizontal and vertical parts. Then an almost complex structure $\mathcal{J}$ can be defined on $Z(M)$ as follows: each $z \in Z(M)$ represents a complex structure on $T_{p(z)}M$; as the horizontal subspace in z is naturally identified with $T_{p(z)}M$, the action of $\mathcal{J}$ on horizontal vectors will be the tautological one, coinciding with the action of z. The vertical subspace in z is isomorphic with the tangent space of the fibre S^2. Hence we let $\mathcal{J}$ act on vertical vectors as the canonical complex structure of S^2. Happily, $\mathcal{J}$ is integrable. Moreover:

Theorem 2.4. [40] *Let (M, g, H) be a quaternion Kähler manifold with positive scalar curvature. Then $(Z(M), \mathcal{J})$ admits a Kähler - Einstein metric with positive Ricci curvature with respect to which* p *becomes a Riemannian submersion.*

3. Local and global 3-Sasakian manifolds

We now describe the odd dimensional analogue, within the frame of contact geometry, of hyperkähler manifolds, as well as a local version of it. We send the interested reader to the excellent recent survey [9], where also a rather exhaustive list of references is given and to [33] for the local version.

Definition 3.1. A $4n+3$ dimensional Riemannian manifold (N, h) such that the cône metric $dr^2 + r^2 h$ on $\mathbb{R}_+ \times N$ is hyperkähler is called a 3-*Sasakian* manifold.

This is equivalent to the existence of three mutually orthogonal unit Killing vector fields ξ_1, ξ_2, ξ_3, each one defining a Sasakian structure (*i.e.*: $\varphi_\alpha := \nabla^h \xi_\alpha$ satisfies the differential equation $\nabla^h \varphi_\alpha = Id \otimes \xi_\alpha^\flat - h \otimes \xi_\alpha$) and related by:

$$[\xi_1, \xi_2] = 2\xi_3, [\xi_2, \xi_3] = 2\xi_1, [\xi_3, \xi_1] = 2\xi_2.$$

3-Sasakian manifolds are necessarily Einstein ([24]) with positive scalar curvature and their Einstein constant is $4n+2$.

Starting with a 3-Sasakian manifold N, one has to consider the foliation generated by the three structure vector fields ξ_α. It is easy to compute the curvature of the leaves: it is precisely one. Hence, the leaves are spherical space forms. If the foliation is quasi-regular (it is enough to have compact leaves), then the quotient space is a quaternion Kähler orbifold M of positive sectional curvature (see [10] for a thorough discussion about the geometry and topology of orbifolds and their applications in contact geometry). As all the geometric constructions we are interested in can be carried out in the category of orbifolds, one considers now the twistor space $Z(M)$. The triangle is closed by observing that, fixing one of the contact structures of N, one has an S^1-bundle $N \to Z(M)$ whose Chern class is, up to torsion, the one of an induced Hopf bundle (this is a particular case of a Boothby-Wang fibration, cf. [6]). Moreover, all three orbifold fibrations involved in this commutative triangle are Riemannian submersions.

Conversely, given a positive quaternion Kähler orbifold (M, g, H), one constructs its Kähler-Einstein twistor space (it will be an orbifold) and an SO(3)-principal bundle over M. The total space N will then be a 3-Sasakian orbifold which, as above, fibers in S^1 over $Z(M)$ closing the diagram. One of the deepest results in this theory was the determination of conditions under which N is indeed a manifold (cf. [13]).

A local version of 3-Sasakian structure will be also useful in the sequel:

Definition 3.2. [33] A Riemannian manifold (N, h) is said to be a *locally* 3-*Sasakian manifold* if a rank 3 vector subbundle $\mathcal{K} \subset TN$ is given, locally spanned by an orthonormal triple ξ_1, ξ_2, ξ_3 of Killing vector fields satisfying:

(i) $[\xi_\alpha, \xi_\beta] = 2\xi_\gamma$ for $(\alpha, \beta, \gamma) = (1, 2, 3)$ and circular permutations.

(ii) Any two such triples ξ_1, ξ_2, ξ_3 and ξ_1', ξ_2', ξ_3' are related on the intersections $U \cap U'$ of their definition open sets by matrices of functions with values in SO(3).

(iii) If $\varphi_\alpha = \nabla^h \xi_\alpha$, $(\alpha = 1, 2, 3)$, then $(\nabla^h_Y \varphi_\alpha)\ Z = \xi^\flat_\alpha(Z)Y - h(Y,Z)\xi_\alpha$, for any local vector fields Y, Z.

Clearly, if $\mathcal{K}$ can be globally trivialized with Killing vector fields as above, (N, h) is 3-Sasakian. It is easily seen that locally 3-Sasakian manifolds share the local properties with the (global) 3-Sasakian spaces: they are Einstein with positive scalar curvature; hence, by Myers' theorem we have

Proposition 3.1. *Complete locally and globally* 3*-Sasakian manifolds are compact.*

But a specific property of the local case is:

Proposition 3.2. [33] *The bundle* $\mathcal{K}$ *of a locally* 3*-Sasakian manifold is flat.*

Proof. Let (ξ_1, ξ_2, ξ_3), (ξ'_1, ξ'_2, ξ'_3) be two local orthonormal triples of Killing fields trivializing $\mathcal{K}$ on U, U'. Then, on $U \cap U' \neq \emptyset$ we have $\xi'_\lambda = f^\sigma_\lambda \xi_\sigma$. We shall show that f^σ_λ are constant. Compute first the bracket

$$2\xi'_\nu = [\xi'\lambda, \xi'_\mu] = \{f^\rho_\lambda \xi_\rho(f^\sigma_\mu) - f^\rho_\mu \xi_\rho(f^\sigma_\lambda)\}\xi_\sigma + f^\rho_\lambda f^\sigma_\mu [\xi_\rho, \xi_\sigma].$$

From $(f^\mu_\lambda) \in \mathrm{SO}(3)$ and $[\xi_\rho, \xi_\sigma] = 2\xi_\tau$ ($(\rho, \sigma, \tau) = (1, 2, 3)$ and cyclic permutations), we can derive:

$$f^\rho_\lambda f^\sigma_\mu [\xi_\rho, \xi_\sigma] = 2\{f^\rho_\lambda f^\sigma_\mu - f^\sigma_\lambda f^\rho_\mu\}\xi_\tau = 2\xi'_\nu.$$

Hence

$$f^\rho_\lambda \xi_\rho(f^\sigma_\mu) - f^\rho_\mu \xi_\rho(f^\sigma_\lambda) = 0.$$

Thus, for any $\lambda, \mu, \sigma = 1, 2, 3$: $\xi'_\lambda(f^\mu_\sigma) - \xi'_\mu(f^\sigma_\lambda) = 0$. It follows:

$$\xi_\lambda(f^\mu_\sigma) - \xi_\mu(f^\lambda_\sigma) = 0. \tag{3.1}$$

Now we use the Killing condition applied to $\xi'_\lambda = f^\mu_\lambda \xi_\mu$:

$$Y(f^\mu_\lambda)h(\xi_\mu, Z) + Z(f^\mu_\lambda)h(\xi_m u, Y) = 0, \quad Y, Z \in \mathcal{X}(M)$$

which yields, on one hand $Z(f^\mu_\lambda) = 0$ for any $Z \perp$ *span*$\{\xi_1, \xi_2, \xi_3\}$ and, on the other hand

$$\xi_\rho(f^\sigma_\lambda) + \xi_\sigma(f^\rho_\lambda) = 0. \tag{3.2}$$

This and (3.1) imply $\xi_\sigma(f^\rho_\lambda) = 0$ and the proof is complete.

□

The vector bundle $\mathcal{K}$ generates a 3-dimensional foliation that, for simplicity, we equally denote $\mathcal{K}$. It can be shown that $\mathcal{K}$ is Riemannian. As in the global case, if the leaves of $\mathcal{K}$ are compact, the leaf space $M = N/\mathcal{K}$ is a compact orbifold. The metric h projects to a metric g on P making the natural projection π a Riemannian submersion with totally geodesic fibers. The locally defined endomorphisms φ_λ can be projected on M producing locally defined almost complex structures: $J_\alpha X_{\pi(x)} = \pi_*(\varphi_\alpha(\tilde{X}_x))$, where $\tilde{X}$ is the horizontal lift of X w.r.t. the submersion. As $\varphi_\alpha \circ \varphi_\beta = -\varphi_\gamma + \xi_\alpha \otimes \xi^\flat_\beta$, P can be covered with open sets endowed with local almost hyperhermitian structures $\{J_\alpha\}$. As the transition functions of $\mathcal{K}$ are in SO(3), so are the transition functions of the bundle $\mathcal{F}$ locally spanned by the φ_α. Hence, two different almost hyperhermitian structures are related on their common domain by transition functions in SO(3). This means that the bundle H they generate is quaternionic. Using the O'Neill formulae, it is now seen, as in the global case, that (M, g, H) is a quaternion Kähler orbifold. Summing up we can state:

Proposition 3.3. [33] *Let* $(N, h, \mathcal{K})$ *be a locally* 3*-Sasakian manifold such that* $\mathcal{K}$ *has compact leaves. Then the leaf space* $M = N/\mathcal{K}$ *is a quaternion Kähler orbifold with positive scalar curvature and the natural projection* $\pi : N \to M$ *is a Riemannian, totally geodesic submersion which fibers are (generally inhomogeneous)* 3*-dimensional spherical space forms.*

Remark 3.1. P. Piccinni proved in [37] that some global 3-Sasakian manifolds also project over local quaternion Kähler manifolds with positive scalar curvature.

A further study of the (supposed compact) leaves of $\mathcal{K}$ will show a very specific property of locally 3-Sasakian manifolds. To this end, we recall, following [41], some aspects of the classification of 3-dimensional spherical space forms S^3/G, with G a finite group of isometries of S^3, hence a finite subgroup of SO(4). The finite subgroups of S^3 are known: they are cyclic groups of any order or binary dihedral, tetrahedral, octahedral, icosahedral and, of course, the identity. In all these cases, S^3/G is a homogeneous 3-dimensional space form carrying an induced (global) 3-Sasakian structure, see [11]. The other finite subgroups of SO(4), not contained in but acting freely on S^3, are characterized by being conjugated in SO(4) to a subgroup of $\Gamma_1 = \mathrm{U}(1) \cdot \mathrm{Sp}(1)$ or $\Gamma_2 = \mathrm{Sp}(1) \cdot \mathrm{U}(1)$. Observe that the right (resp. left) isomorphism between $\mathbb{H}$ and $\mathbb{C}^2$ induces an isomorphism between Γ_1 (resp. Γ_2) and U(2). Hence, any finite subgroup Γ of Γ_1 or Γ_2 will preserve two structures of S^3: the locally 3-Sasakian structure induced by the hyperhermitian structure of $\mathbb{C}^2$ and a global Sasakian structure induced by some complex Hermitian structure of $\mathbb{C}^2$ belonging to the given hyperhermitian one. Moreover, altering Γ by conjugation in SO(4) does not affect the above preserved structures; only the global Sasakian structure will come from a hermitian structure of $\mathbb{R}^4$ conjugate with the standard one. Altogether, we obtain:

Proposition 3.4. [33] *On any locally 3-Sasakian manifold, the compact leaves of $\mathcal{K}$ are locally 3-Sasakian 3-dimensional space-forms carrying a global almost Sasakian structure.*

We end with another consequence of Proposition 3.4:

Corollary 3.1. [33] *Let $\tilde{\mathcal{K}} \to \tilde{N}$ be the pull-back of the bundle $\mathcal{K} \to N$ to the universal Riemannian covering space of a locally 3-Sasakian manifold. Then $\tilde{\mathcal{K}}$ is globally trivialized by a global 3-Sasakian structure on $\tilde{N}$.*

Proof. By Proposition 3.2, the bundle $\tilde{\mathcal{K}} \to \tilde{N}$ is trivial. However, this is not enough to deduce that the trivialization can be realized with Killing fields generating a su(2) algebra. *E.g.* the inhomogeneous 3-dimensional spherical space forms are parallelizable but locally, not globally 3-Sasakian. To overcome this difficulty, start with the induced locally 3-Sasakian structure of $\tilde{N}$. Let X_1 be the global Sasakian structure of $\tilde{N}$ provided by Proposition 3.4 and consider an open set $\tilde{U}$ on which $\tilde{\mathcal{K}}$ is trivialized by a local 3-Sasakian structure incuding X_1.

The manifold $\tilde{N}$ is simply connected and Einstein, hence analytic (see [4], Theorem 5.26). By a result of Nomizu (cf. [31]) each local Killing vector field on $\tilde{N}$ can be extended uniquely to the whole $\tilde{N}$. We thus extend the above three local Killing fields. Clearly, the extension Y_1 of X_1 coincides with X_1. The extension Y_2 of X_2 is thus orthogonal to Y_1 and belongs to $\tilde{\mathcal{K}}$ in every point of $\tilde{N}$. It follows from Proposition 3.2 that Y_2 is a global Sasakian structure. Now $Y_3 = \frac{1}{2}[Y_1, Y_2]$ completes the desired global 3-Sasakian structure. □

4. Quaternion Hermitian Weyl and hyperhermitian Weyl manifolds

We now arrive to the structures giving the title of this survey. We consider $4n$-dimensional quaternion Hermitian manifolds and let the metric vary in its conformal class. In this setting, the natural connection to work with is no more the Levi-Civita connection, but a Weyl connection which has to be compatible with the quaternionic structure too.

4.1. **Definitions. First properties.** Let (M^{4n}, c, H), $n \geq 2$ be a conformal manifold endowed with a quaternionic bundle H such that (M, g, H) is quaternion Hermitian for each $g \in c$.

Definition 4.1. (M^{4n}, H, c, D) is said *quaternion-Hermitian-Weyl* if:
1) (M, c, D) is a Weyl manifold;
2) (M, g, H) is quaternion-Hermitian for any $g \in c$;
3) $DH = 0$, *i.e.* $DI_\alpha = a_\alpha^\beta \otimes I_\beta$ with sqew-symmetric matrix of one-forms (a_α^β) for any admissible basis of H.

(M^{4n}, c, H, D) is said *hyperhermitian Weyl* if it satisfies condition 1) and:

2') (M, g, H) is hyperhermitian for any $g \in c$;
3') $DI = 0$ for any section of H.

The above definition is clearly inspired by the complex case, where the theory of Hermitian-Weyl (locally conformal Kählerian in other terminology) is widely studied (see [15] for a recent survey). Indeed, the following equivalent definition is available:

Proposition 4.1. [34] (M^{4n}, c, H, D) *is quaternion-Hermitian-Weyl (resp. hyperhermitian Weyl) if and only if* (M, g, H) *is locally conformally quaternion Kähler (resp. locally conformally hyperkähler) (i.e.* $g_{|U_i} = e^{f_i} g'_i$*, where the* g'_i *are quaternion Kähler (resp. hyperkähler) over open neighbourhoods* $\{U_i\}$ *covering* M*) for each* $g \in c$*.*

Proof. Let (M^{4n}, c, H, D) be quaternion-Hermitian Weyl. Fix a metric $g \in c$ and choose an open set U on which H is trivialized by an admissible basis I_1, I_2, I_3. Then $Dg = \theta_g \otimes g$ together with condition 2) of the definition imply $D\omega_\alpha = \theta \otimes \omega_\alpha + a_\alpha^\beta \otimes \omega_\beta$, hence

$$d\omega_\alpha = \theta \wedge \omega_\alpha + a_\alpha^\beta \wedge \omega_\beta. \tag{4.1}$$

This implies that $\mathcal{H}$ is a differential ideal and, on the other hand, the derivative of the fundamental four-form is $d\omega = \theta_g \wedge \omega$. Differentiating here we get $0 = d^2\omega = d\theta_g \wedge \omega$. As ω is nondegenerate, this means $d\theta_g = 0$. Consequently, locally, on some open sets U_i, $\theta_g = df_i$ for some differentiable functions defined on U_i. It is now easy to see that for each $g'_i = e^{-f_i} g_{|U_i}$, the associated 4-form is closed, hence, taking into account Proposition 2.1 and Remark 2.2, the local metrics g'_i are quaternion Kähler.

Conversely, starting with the local quaternion Kähler metrics $g'_i = e^{-f_i} g_{|U_i}$, define $(\theta_g)_{|U_i} = df_i$. It can be seen that these local one forms glue together to a global, closed one-form and $d\omega = \theta_g \wedge \omega$. Then construct the Weyl connection associated to g and θ_g:

$$D = \nabla^g - \frac{1}{2}\{\theta \otimes Id + Id \otimes \theta - g \otimes \theta_g^\sharp\}. \tag{4.2}$$

A straightforward computation shows that D has the requested properties.

The proofs for the global case are completely similar. □

Corollary 4.1. *A quaternion Hermitian manifold* (M, g, H) *is quaternion Hermitian Weyl if and only if there exist a* 1*-form* θ *(necessarily closed) such that the fundamental* 4*-form* ω *satisfy the integrability condition* $d\omega = \theta \wedge \omega$*. In particular,* (M, g, H) *is quaternion Kähler if and only if* $\theta = 0$*.*

The form θ is the Higgs field associated to the Weyl manifold (M, c, D). But in this context, we shall prefer to call it *the Lee form* (see [15] for a motivation).

As on a simply connected manifold any closed form is exact we derive:

Corollary 4.2. *A quaternion Hermitian Weyl (hyperhermitian Weyl) manifold which is not globally conformal quaternion Kähler (hyperkähler) cannot be simply connected.*

The universal Riemannian covering space of a quaternion Hermitian Weyl (hyperhermitian Weyl) manifold is globally conformal quaternion Kähler (hyperkähler).

Example 4.1. We give here just one example of compact hyperhermitian Weyl manifold and leave the description of other examples for the end of the paper, following the structure of quaternion Hermitian Weyl and hyperhermitian Weyl manifolds.

The standard quaternionic Hopf manifold is $H^n_{\mathbb{H}} = \mathbb{H} - \{0\}/\Gamma_2$, where Γ_2 is the cyclic group generated by the quaternionic automorphism $(q_1, ..., q_n) \mapsto (2q_1, ..., 2q_n)$. The hypercomplex structure of $\mathbb{H}^n$ is easily seen to descend to $H^n_{\mathbb{H}}$. Moreover, the globally conformal quaternion Kähler metric $(\sum_i q_i \overline{q}_i)^{-1} \sum_i dq_i \otimes d\overline{q}_i$ on $\mathbb{H}^n - \{0\}$ is invariant to the action of Γ_2, hence induces a locally conformally hyperkähler metric on the Hopf manifold. Note that, as in the complex case, $H^n_{\mathbb{H}}$ is diffeomorphic with a product of spheres $S^1 \times S^{4n-1}$. Consequently, its first Betti number is 1 and it cannot accept any hyperkähler metric.

Before going over, let us note the following result:

Proposition 4.2. [33] *A quaternion Hermitian manifold (M, g, H) admits a unique quaternion Hermitian Weyl structure.*

Proof. We have to prove that there exists a unique torsion free connection preserving both H and $[g]$. Indeed, if D_1, D_2 are such, let θ_1, θ_2 be the associated Lee forms. Then the fundamental 4-form ω satisfies

$$d\omega = \theta_1 \wedge \omega = \theta_2 \wedge \omega. \tag{4.3}$$

Using the operator $L : \Lambda^1 T^*M \to \Lambda^5 T^*M$, $L\alpha = \alpha \wedge \omega$, (4.3) yields $L(\theta_1 - \theta_2) = 0$. But L is injective, because it is related to its formal adjoint Λ by $\Lambda L = (n-1)Id$. Hence $\theta_1 = \theta_2$. Finally, formula (4.2) proves that $D_1 = D_2$. $\square$

Remark 4.1. [37] For hyperhermitian Weyl manifolds, this uniqueness property is implied by the characterization of the Obata connection as the unique torsion-free hypercomplex connection. It must then coincide with our Weyl connection D. In general, the set of torsion-free quaternionic connections has an affine structure modelled on the space of 1-forms. However, only one torsion-free connection can preserve a given conformal class of hyperhermitian metrics. This follows from the fact that the exterior multiplication with the fundamental four-form of the metric maps injectively $\Lambda^1(T^*M)$ into $\Lambda^5(T^*M)$.

Note that the connection $D|_{U_i}$ is in fact the Levi Civita connection of the local quaternion Kähler metric g'_i. As quaternion-Kähler manifolds are Einstein, we obtain the following fundamental result:

Proposition 4.3. [34] *Quaternion Hermitian Weyl manifolds are Einstein Weyl.*

Hence, as $d\theta = 0$, *i.e.* the Weyl structure (M, c, D) is closed and not exact, because the D is the Levi-Civita connection of *local* metrics (the Weyl structure is only locally exact), the quoted Theorem 1.2 of P. Gauduchon implies:

Proposition 4.4. [34] *On any compact quaternion-Weyl (hyperhermitian Weyl) manifold which is not globally conformal quaternion Kähler (hyperkähler) there exists a representative $g \in c$ (the Gauduchon metric) such that the associated Lee form θ_g be ∇^g-parallel.*

In the sequel, the parallel Lee form of the Gauduchon metric will always be supposed of unit length.

Corollary 4.3. *Let g be the above metric with parallel Lee form on a compact hyperhermitian Weyl manifold and $\{I_\alpha\}$ an adapted hyperhermitian structure. Then (g, I_α) are Vaisman structures on M (cf.* [15]*).*

Proposition 4.5. [32] *On a compact quaternion Weyl manifold which is not globally conformal quaternion Kähler, the local quaternion Kähler metrics g'_i are Ricci-flat.*

Proof. This result follows directly from Theorem 1.2, 2), but we prefer to give here a direct proof, adapted to our situation.

On each U_i, the relation between the scalar curvatures Scal'_i and $Scal$ of g'_i and g is (cf. [4], p. 59):

$$\mathrm{Scal}'_i = e^{-f_i}\left\{\mathrm{Scal}_{|U_i} - \frac{(4n-1)(2n-1)}{2}\right\}.$$

Hence Scal'_i is constant. If Scal'_i is not identically zero, differentiation of the above identity yields:

$$\theta_{|U_i} = d\log\left\{\mathrm{Scal}_{|U_i} - \frac{(4n-1)(2n-1)}{2}\right\}.$$

As both θ and Scal are global objects on M, it follows that θ is exact, contradiction. But if $\mathrm{Scal}'_i = 0$ on some U_i, then $Scal = \mathrm{Scal}_{|U_i} = \frac{(4n-1)(2n-1)}{2}$, constant on M. This proves that $\mathrm{Scal}'_i = 0$ on each U_i. $\square$

Remark 4.2. The above result says that quaternion-Hermitian Weyl manifolds are locally conformally locally hyperkähler. In particular, the open subsets U_i can always be taken simply connected and endowed with admissible basis made of by integrable, parallel almost complex structures. But this does not mean that M would be a locally conformal Kähler manifold, because a global Kähler structure might not exist.

Another characterization, using the differential ideal $\mathcal{H}$ is the following (recall that the differential ideal condition is conformally invariant, so one can speak about the differential ideal of a conformal manifold):

Theorem 4.1. [1] *A quaternionic conformal manifold (M, c, H) of dimension at least 12 is quaternion-Hermitian Weyl if and only if $\mathcal{H}$ is a differential ideal.*

The following result is essential in the author's proof, also motivating the restriction on the dimension:

Lemma 4.1. [1] *Let (M, g, H) be an almost hyperhermitian manifold with $\dim M \geq 12$. Suppose $\sum_{\alpha=1}^{3} \phi_\alpha \wedge \omega_\alpha = 0$ for some 2-forms ϕ_α. Then there exists the sqew-symmetric matrix of real functions $f_{\alpha\rho}$ such that $\phi_\alpha = \sum_{\rho\neq\alpha} f_{\alpha\rho}\omega_\rho$.*

Proof. Let F_α be the $1-1$ tensor fields metrically equivalent with the 2-forms ϕ_α. The identity in the statement can be rewritten as:

$$\begin{aligned}\sum_{\rho=1}^{3}\{ -\phi_\rho(X,Y)I_\rho Z + \phi_\rho(X,Z)I_\rho Y + \omega_\rho(Y,Z)F_\rho X - \\ - \phi_\rho(Y,Z)I_\rho X + \omega_\rho(Z,X)F_\rho Y + \omega_\rho(X,Y)F_\rho Z\} = 0.\end{aligned} \tag{4.4}$$

Let now X be unitary, fixed. In the orthogonal complement of $\mathbb{H}X = \{X, I_1X, I_2X, I_3X\}$ we choose a unitary Z and let $Y = I_\alpha Z$. With these choices, the above identity reads:

$$F_\alpha(X) = \sum_{\rho=1}^{3}\{\phi_\rho(X, I_\rho Z)I_\rho Z - \phi_\rho(X,Z)I_\rho I_\alpha Z\} + \sum_{\rho=1}^{3}\phi_\rho(I_\rho Z, Z)I_\rho X.$$

Here we use the assumption $n \geq 3$ to obtain:

$$F_\alpha(X) = \sum_{\rho=1}^{3}\phi_\rho(I_\rho Z, Z)I_\rho X,$$

hence ϕ_α have the form $\phi_\alpha = \sum_{\rho\neq\alpha} f_{\alpha\rho}\omega_\rho$ which, introduced in the equation (4.4), gives:

$$\begin{aligned}\sum_{\alpha=1}^{3} f_{\alpha\alpha}\{-\omega_\alpha(X,Y)I_\alpha Z + \omega_\alpha(X,Z)I_\alpha Y - \omega_\alpha(Y,Z)I_\alpha X\} + \\ + \sum_{\rho\neq\alpha}(f_{\alpha\rho} + f_{\rho\alpha})\{\omega_\alpha(X,Y)I_\rho Z + \omega_\alpha(X,Z)I_\rho Y + \omega_\alpha(Y,Z)I_\rho X\} = 0.\end{aligned}$$

Again using $n \geq 3$, we may choose Y and Z orthogonal to $\mathbb{H}X$ and get:

$$-f_{\alpha\alpha}\omega_\alpha(Y,Z) - \sum_{\rho\neq\alpha}(f_{\alpha\rho} + f_{\rho\alpha})\omega_\alpha(Y,Z) = 0.$$

Now it remains to take $Z = I_\alpha Y$ to derive the sqew-symmetry of $(f_{\alpha\rho})$. □

Proof. (of Theorem 4.1). Fix $g \in c$ and an admissible basis for H. Starting from equations (2.2), (2.3) and $d\omega_\alpha = \sum_{\beta=1}^{3}\eta_{\alpha\beta}\wedge\omega_\beta$, one can derive the following formula:

$$d\omega_\alpha = \eta_\gamma \wedge \omega_\beta - \omega_\beta \wedge \omega_\gamma + \frac{1}{3}\eta \wedge \omega_a l, \tag{4.5}$$

where $2\eta := \eta_{\beta\gamma} - \eta_{\gamma\beta}$. After differentiating (4.5) we get:

$$\frac{1}{3}d\eta \wedge \omega_\alpha + (d\eta_\gamma + \eta_\alpha \wedge \eta_\beta) \wedge \omega_\beta - (d\eta_\beta + \eta_\gamma \wedge \eta_\alpha) \wedge \omega_\gamma = 0.$$

The previous Lemma applies and provides:

$$\begin{aligned}\frac{1}{3}\eta =& f_{\alpha\beta}\omega_\beta + f_{\beta\gamma}\omega_\gamma\\ d\eta_\gamma + \eta_\alpha \wedge \eta_\beta =& f_{\beta\alpha}\omega_\alpha + f_{\beta\gamma}\omega_\gamma\\ -d\eta_\beta + \eta_\gamma \wedge \eta_\alpha =& f_{\gamma\alpha}\omega_\alpha + f_{\gamma\beta}\omega_\beta\end{aligned}$$

This yields $d\eta = 0$ and $d\eta_\alpha + \eta_\beta \wedge \eta_\gamma = f\omega_\alpha$ with f not depending on α. Hence, locally $f = d\sigma$ and we have

$$d\omega_\alpha = \eta_\gamma \wedge \omega_\beta - \eta_\beta \wedge \omega_\gamma - \frac{1}{3}d\sigma \wedge \omega_\alpha,$$

an equation similar to (4.1). The rest and the converse are obvious. □

Remark 4.3. It is still unknown if this result is true in dimension 8 too.

Remark 4.4. For quaternion Hermitian manifolds, various adapted canonical connections were introduced by V. Oproiu, M. Obata and others. A unified treatement can be found in some recent papers of D. Alekseevski, E. Bonan, S. Marchiafava (see *e.g.* [3] and the references therein). In particular, in [30], one finds a characterization of hyperhermitian Weyl manifolds in terms of canonical connections and structure tensors of the subordinated quaternionic Hermitian structure.

We end this section with a characterizations of quaternion Kähler manifolds among (non compact) quaternion Hermitian Weyl manifolds by means of submanifolds (compare with [44] for the complex case):

Proposition 4.6. [32] *A quaternion Hermitian Weyl manifold* (M, g, H) *of dimension at least* 8 *is quaternion Kähler if and only if through each point of it passes a totally geodesic submanifold of real dimension* $4h \geq 8$ *which is quaternion Kähler with respect to the structure induced by* (g, H)*.*

Proof. On a given submanifold of M, locally one can induce the metric g and the quaternion Kähler one g'_i. Correspondingly, there are two second fundamental forms b and b'_i. As g and g'_i are conformally related on U_i, the relation between b and b'_i is

$$b'_i = b + \frac{1}{2}g \otimes T^\nu,$$

where T^ν is the part of T normal to the submanifold. Now let $x \in M$ and let $j : Q \to M$ be a quaternion Kähler submanifold through x as stated. We have $j^*d\omega = 0$. From $d\omega = \theta \wedge \omega$ we then derive $j^*\theta \wedge j^*\omega = 0$. But rank $j^*\omega = 4h \geq 8$, hence $j^*\theta = 0$ meaning that T is normal to Q: $T = T^\nu$. On the other hand, the same relation $j^*\theta = 0$ shows that $Q \cap U_i$ is a quaternion Kähler submanifold of the quaternion Kähler manifold $(U_i, H_{|U_i}, g'_i)$. As quaternion submanifolds of quaternion Kähler manifolds are totally geodesic, $Q \cap U_i$ is totally geodesic in U_i with respect to g'_i. It follows $2b = -g \otimes T$ on $Q \cap U_i$. But b is zero from the assumption (Q is totally geodesic with respect to g). This yields $T = 0$ on $Q \cap U_i$, in particular $T_x = 0$. Since x was arbitrary in M, $T = 0$ on M proving that (M, g, H) is quaternion Kähler.

For the converse, just take $Q = M$. □

We end this general presentation with a recent result which makes quaternion Hermitian Weyl manifolds interesting for physics. We first recall (sending to [20] and [23] for details and further references) the notion of *quaternionic Kähler* (resp. *hyperkähler*) *manifold with torsion*, briefly QKT (resp. HKT) manifolds. Let (M, g, H) be a quaternionic Hermitian (resp. hyperhermitian) manifold. It is called QKT (resp. HKT) manifold if it admits a metric quaternionic (resp. hypercomplex) connection ∇ with totally skew symmetric torsion tensor which is, moreover, of type $(1,2) + (2,1)$ w.r.t. each local section I_α, that is it satisfies:

$$T(X, Y, Z) = T(I_\alpha X, I_\alpha Y, Z) + T(I_\alpha X, Y, I_\alpha Z) + T(X, I_\alpha Y, I_\alpha Z),$$

where $T(X,Y,Z) = g(Tor^{\nabla}(X,Y),Z)$ and $Tor^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. The holonomy of such a connection is contained in $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$. These structures appear naturally on the target space of $(4,0)$ supersymmetric two-dimensional sigma model with Wess-Zumino term and seem to be of growing interest for physicists. Let us introduce the 1-forms:

$$t_\alpha(X) = -\frac{1}{2}\sum_{i=1}^{4n} T(X, e_i, I_\alpha e_i), \quad \alpha = 1,2,3.$$

Then the 1-form $t = I_\alpha t_\alpha$ is independent on the choice of I_α. We can now state:

Proposition 4.7. [23] *Every quaternion Hermitian Weyl (resp. hyperhermitian Weyl) manifold admits a QKT (resp. HKT) structure.*

Conversely, a $4n$ dimensional $(n > 1)$ QKT manifold (M, g, H, ∇) is quaternion Hermitian Weyl if and only if:

$$T = \frac{1}{2n+1}\sum_\alpha t_\alpha \wedge \omega_\alpha \ \text{ and } \ dt = 0.$$

4.2. **The canonical foliations.** From now on (M, c, H, D) will be compact, non globally conformal quaternion Kähler. According to Proposition 4.4, we let $g \in c$ be the Gauduchon metric whose Lee form $\theta := \theta_g$ is parallel w.r.t. the Levi-Civita connection $\nabla := \nabla^g$. Hence we look at the quaternion Hermitian manifold (M, g, H). We also suppose $\theta \neq 0$ meaning that M is not quaternion Kähler, see Corollary 4.1. We recall that, being parallel, we can suppose θ normalised, *i.e.* $\mid \theta \mid = 1$. We denote $T := \theta^\sharp$ and let $T_\alpha = I_\alpha T$ and $\theta_\alpha = \theta \circ I_\alpha$.

The following proposition gathers the computational formulae we need:

Proposition 4.8. [32] *Let (M, g, H) be a compact quaternion Hermitian Weyl manifold and $\{I_1, I_2, I_2\}$ a local admissible basis of H with I_α integrable and parallel (as in remark 4.2). The following formulae hold good:*

$$\mathcal{L}_T I_\alpha = 0, \quad \mathcal{L}_T g = 0, \quad \mathcal{L}_T \omega_\alpha = 0, \quad \mathcal{L}_T \omega = 0 \tag{4.6}$$

$$\nabla I_\alpha = \frac{1}{2}\{Id \otimes \theta_\alpha - I_\alpha \otimes \theta - \omega_\alpha \otimes T + g \otimes T_\alpha\} \tag{4.7}$$

$$\mathcal{L}_{T_\alpha} I_\alpha = 0, \quad \mathcal{L}_{T_\alpha} I_\beta = I_\gamma, \quad \mathcal{L}_{T_\alpha} g = 0 \tag{4.8}$$

$$[T, T_\alpha] = 0, \quad [T_\alpha, T_\beta] = T_\gamma \tag{4.9}$$

$$\nabla \theta_\alpha = \frac{1}{2}\{\theta \otimes \theta_\alpha - \theta_\alpha \otimes \theta - \omega_\alpha\} \tag{4.10}$$

$$d\theta_\alpha = -\omega_\alpha + \theta \wedge \theta_\alpha \tag{4.11}$$

$$\mathcal{L}_{T_\alpha}\omega_\alpha = 0, \quad \mathcal{L}_{T_\alpha}\omega_\beta = \omega_\beta, \quad \mathcal{L}_{T_\alpha}\omega = 0 \tag{4.12}$$

where $\mathcal{L}$ is the operator of Lie derivative.

The proof is by direct computation and mimics the corresponding one for Vaisman manifolds, see [15]. In particular, from (4.6) and (4.8), we obtain according to [36]:

Corollary 4.4. *The vector fields T and T_α are infinitesimal automorphisms of the quaternion Hermitian structure.*

There are two interesting foliations on any compact quaternion Hermitian Weyl manifold:

- the $(4n-1)$-dimensional $\mathcal{F}$, spanned by the kernel of θ and
- the 4-dimensional $\mathcal{D}$, locally generated by T, T_1, T_2, T_3.

Here are their properties:

Proposition 4.9. [33], *On a compact quaternion Hermitian Weyl manifold, $\mathcal{F}$ is a Riemannian, totally geodesic foliation. Its leaves have an induced locally 3-Sasakian structure.*

Proof. The first statement is a consequence of (4.6). As for the second one, the bundle $\mathcal{K}$ is locally generated by the (rescaled to be unitary) local vector fields T_α. Indeed, they are Killing by the last equation of (4.8); the first condition of definition 3.2 is given by (4.9); the transition functions of $\mathcal{K}$ are in SO(3) because the transition functions of H are so; finally, condition 3) of the definition is implied by (4.10). □

Corollary 4.5. [32] *On a compact hyperhermitian Weyl manifold, $\mathcal{F}$ is a Riemannian, totally geodesic foliation whose leaves have an induced (global) 3-Sasakian structure.*

Proposition 4.10. [32] *On a compact quaternion Hermitian Weyl manifold, the foliation $\mathcal{D}$ is Riemannian, totally geodesic. Its leaves are conformally flat 4-manifolds $(\mathbb{H}-\{0\})/G$, with G a discrete subgroup of $\mathrm{GL}(1,\mathbb{H})\cdot\mathrm{Sp}(1)$ inducing an integrable (in the sense of G-structures) quaternionic structure.*

Proof. Let X be a leaf of $\mathcal{D}$ and let the superscript $'$ refer to restrictions of objects from M to X. A local orthonormal basis of tangent vectors for X is provided by $\{T', T_1', T_2', T_3'\}$. As X is totally geodesic, $\nabla'\theta' = 0$ and a direct computation of the curvature tensor of the Weyl connection R^D on this basis proves $R^D = 0$ on X. Hence X is conformally flat and the curvature tensor of the Levi-Civita connection is

$$\begin{aligned} R'(U,Y)Z =& \theta'(U)\theta'(Z)Y - \theta'(Y)\theta'(Z)U - \theta'(U)g'(Y,Z)T' + \\ &+\theta'(Y)g'(U,Z)T' + g'(Y,Z)U - g'(U,Z)Y. \end{aligned} \tag{4.13}$$

It follows that the Ricci tensor $Ric' = g' - \theta'\otimes\theta'$ is g'-parallel and, on the other hand, the sectional curvature is non-negative and strictly positive on any plane of the form $\{T'_\alpha, T'_\beta\}$. Now recall that the universal Riemannian covering spaces of conformally flat Riemannian manifolds with parallel Ricci tensor were classified in [28]. By the above discussion and the reducibility of X (due to $\nabla' T' = 0$), the only class fitting from Lafontaine's classification is that with universal cover $\mathbb{R}^4 - \{0\}$ equipped with the conformally flat metric written in quaternionic coordinate $(h\overline{h})^{-1}dh\otimes d\overline{h}$. We still have to determine the allowed deck groups.

Happily, Riemannian manifolds with such universal cover were studied in [18] and, in arbitrary dimension, in [47]. Here it is proved that equation (4.13) forces the deck group of the covering to contain only conformal transformations of the form (in real coordinates) $\tilde{x}^i = \rho a^i_j x^j$ where $\rho > 0$ and $(a^i_j) \in \mathrm{SO}(4)$. This leads to the following form of G:

$$G = \{h t_0^k\ ;\ h \in G_0, k \in \mathbb{Z}\} \tag{4.14}$$

where t_0 is a conformal transformation of maximal module $0 < \rho < 1$ and G' is one of the finite subgroups of U(2) listed in [25]. Finally, as $\mathrm{CO}^+(4) \simeq \mathrm{GL}(1,\mathbb{H})\cdot\mathrm{Sp}(1)$, X has an induced integrable quaternionic structure. □

Corollary 4.6. [32] *On a compact hyperhermitian Weyl manifold, the foliation $\mathcal{D}$ is Riemannian, totally geodesic. Its leaves, if compact, are complex Hopf surfaces (non-primary, in general) admitting an integrable hypercomplex structure.*

Proof. Only the second statement has to be proved. It is clear that the leaves inherit a hyperhermitian Weyl, non hyperkähler (because $\theta \neq 0$) structure. The compact hyperhermitian surfaces are classified in [8] and the only class having the stated property is that of Hopf surfaces. □

As above, here *integrable hypercomplex structure* is intended in the sense of G-structures, i.e. of the existence of a local quaternionic coordinate such that the differential of the change of coordinate belongs to $\mathbb{H}^*$. For further use we recall the following:

Theorem 4.2. (cf. [26]) *A complex Hopf surface S admits an integrable hypercomplex structure if and only if $S = (\mathbb{H}-\{0\})/\Gamma$ where the discrete group Γ is conjugate in $\mathrm{GL}(2,\mathbb{C})$ to any of the following subgroups $G \subset \mathbb{H}^* \subset \mathrm{GL}(2,\mathbb{C})$:*

(i) $G = \mathbb{Z}_m \times \Gamma_c$ *with* $\mathbb{Z}_m$ *and* Γ_c *both cyclic generated by left multiplication by* $a_m = e^{2\pi i/m}$, $m \geq 1$, *and* $c \in \mathbb{C}^*$.

(ii) $G = L \times \Gamma_c$, *where* $c \in \mathbb{R}^*$ *and* L *is one of the following:* D_{4m}, *the dihedral group,* $m \geq 2$, *generated by the quaternion* j *and* $\rho_m = e^{\pi i/m}$*;* T_{24}, *the tetrahedral group generated by* ζ^2 *and* $1/\sqrt{2}(\zeta^3 + \zeta^3 j)$, $\zeta = e^{\pi i/4}$*;* O_{48}, *the octahedral group generated by* ζ *and* $1/\sqrt{2}(\zeta^3 + \zeta^3 j)$*;* I_{120}, *the icosahedral group generated by* $\epsilon^3,\ j,\ 1/\sqrt{5}[\epsilon^4 - \epsilon + (\epsilon^2 - \epsilon^3)j],\ \epsilon = e^{2\pi i/5}$.

(iii) G *generated by* $\mathbb{Z}_m$ *and* $cj, m \geq 3,\ c \in \mathbb{R}^*$.

(iv) G *generated by* D_{4m} *and* $c\rho_{2n},\ c \in \mathbb{R}^*$ *or by* T_{24} *and* $c\zeta,\ c \in \mathbb{R}^*$.

Contrary to $\mathcal{D}$, the distribution $\mathcal{D}^\perp$ is not integrable. In fact, it plays the part of the contact distribution from contact geometry:

Proposition 4.11. [32] *On any compact quaternion Hermitian Weyl or hyperhermitian Weyl manifold, the distribution* $\mathcal{D}^\perp$ *is not integrable. Moreover, its integral manifolds are totally real and have maximal dimension* $n-1$.

Proof. Note that a submanifold N is an integral manifold of $\mathcal{D}^\perp$ if and only if θ and θ_α vanish on N. In this case, also $d\theta_\alpha$ vanishes on N. Then (4.11) implies that $I_\alpha X$ is normal to N for any X tangent to N, *i.e.* N is totally real. The statement about the dimension of N is now obvious. □

Example 4.2. Examples of hyperhermitian Weyl manifolds having as leaves of $\mathcal{D}$ any of the groups of the surfaces in the above list can be obtained as follows: start with the standard hypercomplex Hopf manifold $S^1 \times S^{4n-1} = \mathbb{H}^n - \{0\}/\Gamma_2$ (see example 4.1). Consider now the diagonal action of any G in Kato's list on $\mathbb{H}^n$. The action is induced on the fibers of the projection $S^\times S^{4n-1} \to \mathbb{H}P^{n-1}$, hence on the primary standard Hopf surface $S^1 \times S^3$ obtaining the desired examples.

4.3. **Structure theorems.** In this section we use the properties of the foliations described above to clarify the structure of compact quaternion Hermitian Weyl and hyperhermitian Weyl manifolds whose foliations have compact leaves, in relation with the other geometries involved: Kähler, quaternion Kähler, locally and globally 3-Sasakian.

4.3.1. *The link with (locally)* 3*-Sasakian geometry.*

Theorem 4.3. [33] *The class of compact quaternion Hermitian Weyl manifolds* M *which are not quaternion Kähler and whose Lee field is quasi-regular, (i.e. each point of* M *has a cubic neighbourhood in which the orbit of* T *enters a finite number of times), coincides with the class of flat principal* S^1*-bundles over compact locally* 3*-Sasakian orbifolds* $N = M/T$.

Proof. Let first (M, g, H) be a compact quaternion Hermitian Weyl manifold as in the statement. The orbits of T are closed, hence after rescaling, one may suppose they are circles S^1 acting on M by isometries because T is Killing. The quotient space $N = M/T$ is an orbifold (a manifold if T is regular) and, with respect to the induced metric h, the natural projection π becomes a Riemannian submersion. Hence, for any leaf N' of $\mathcal{F}$, $\pi_{|N'} : N' \to N$ is a Riemannian covering map. As, according to Proposition 4.9, the leaves of $\mathcal{F}$ have a locally 3-Sasakian structure, (N, h) is locally 3-Sasakian.

Conversely, consider a flat principal S^1-bundle $\pi : M \to N$ over a compact locally 3-Sasakian manifold (N, h) with local Killing field ξ_α. Choose a closed 1-form θ on M defining the flat connection of the bundle π and define the metric $g := \pi^* h + \theta \otimes \theta$. Also, define an almost quaternionic bundle H on M by defining its local basis as:

$$\begin{aligned} I_\alpha &= -\varphi_\alpha - \xi_\alpha^\flat \otimes T, \quad \text{on horizontal fields} \\ I_\alpha T &= \xi_\alpha \end{aligned} \tag{4.15}$$

where $\varphi_\alpha = \nabla^h \xi_\alpha$ and $T = \theta^\sharp$. It is straightforward to check, as in the complex case (see [15], chapter 6) that (M, g, H) is quaternion Hermitian Weyl with Lee form θ. □

Corollary 4.7. [32] *The class of compact hyperhermitian Weyl manifolds, not hyperkähler and having a quasi-regular (resp. regular)* T *coincide with the class of flat principal* S^1*-bundles over compact* 3*-Sasakian orbifolds (resp. manifolds).*

4.3.2. *The link with quaternion Kähler geometry.* We now describe the leaf space of the foliation $\mathcal{D}$, when it exists.

Theorem 4.4. [32], [34] *Let* (M, g, H) *be a compact quaternion Hermitian Weyl (resp. hyperhermitian Weyl) manifold, non quaternion Kähler (resp. non hyperkähler) whose foliation* $\mathcal{D}$ *has compact leaves. Then the leaves space* $P = M/\mathcal{D}$ *is a compact quaternion Kähler orbifold with positive scalar curvature, the projection is a Riemannian, totally geodesic submersion and a fibre bundle map with fibres as described in Proposition 4.10 (resp. 4.6).*

Proof. In the local case of quaternion Hermitian Weyl M, we have to explain how to project the structure of M over P. The key point is that locally, H has admissible basis formed by ∇-parallel (hence integrable) complex structures. Then formulae (4.6), (4.8) show that H is projectable. The foliation being Riemannian, g is also projectable. The compatibility of the projected quaternion bundle with the projected metric is clear. To show that the projected structure is quaternion Kähler, let ω_P be the 4-form of the projected structure. As the projection is a totally geodesic Riemannian submersion, ω_P coincides with the restriction of ω to basic vector fields on M. Hence, it is enough to show that $\nabla\omega = 0$ on basic vector fields. But $\nabla\omega = \sum_\alpha \nabla\omega_\alpha \wedge \omega_\alpha + \omega_\alpha \wedge \nabla\omega_\alpha$ and the result follows from equation (4.7). The scalar curvature of (P, g) is easily computed using O'Neill formulae.

The global case of a hyperhermitian Weyl M now follows. ∎

Remark 4.5. The above fibration can never be trivial, according to Proposition 4.11.

Let now M be hyperhermitian Weyl, $\mathcal{T}$ be the foliation generated by the vector field T and $\mathcal{V}$ the 2-dimensional foliation generated by T and JT, where J is a fixed compatible global complex structure belonging to H. Theorem 4.4, together with the structure of 3-Sasakian manifolds described in section 3, furnish the following structure theorem:

Theorem 4.5. [32], [33] *Let* (M, g, H) *be a compact hyperhermitian Weyl manifold, non hyperkähler, such that the foliations* $\mathcal{D}$*,* $\mathcal{V}$*,* $\mathcal{T}$ *and* $\mathcal{K}$ *have compact leaves. There exists the following commutative diagram of fibre bundles and Riemannian submersions in the category of orbifolds:*

$$\begin{tikzcd}
Z \arrow[rr, dash, "S^1"] & & N \\
& M \arrow[ul, "T^1_{\mathbb{C}}"'] \arrow[ur, phantom, "S^1"] \arrow[d] & S^3/G \\
& P \arrow[uul, "S^2"] &
\end{tikzcd}$$

Here N *is globally* 3*-Sasakian. The fibres of* $M \to P$ *are Kato's integrable hypercomplex Hopf surfaces* $(S^1 \times S^3)/G$*, non necessarily primary and non necessarily all homeomorphic if* M *is hyperhermitian Weyl. The* S^1*-bundle* $P \to Z$ *is a Boothby-Wang fibration.*

Note that all arrows appearing in the diagram are canonical, except for $M \to Z$, which depends on the choice of the compatible global complex structure on M. However, different choices of this complex structure produce analytically equivalent complex manifolds Z.

Remark 4.6. The diagram 4.5 holds also if $dim(M) = 8$. In this case P is still Einstein by the above discussion. The integrability of the complex structure on its twistor space implies it is also self-dual (cf. [4]). Then just recall that a 4-dimensional N is usually defined to be quaternionic Kähler if it is Einstein and self-dual.

Remark 4.7. For the hyperhermitian Weyl manifold $M = S^1 \times S^{4n-1}$, diagram 4.5 becomes the well-known:

$$\begin{tikzcd}
\mathbb{C}P^{n-1} \arrow[dr, bend right, "S^2"'] & & S^{4n-1} \arrow[ll, dash, "S^1"'] \arrow[dl, phantom, "S^3"] \\
& S^1\times S^{4n-1} \arrow[ul, "T^1_{\mathbb{C}}"'] \arrow[ur, phantom, "S^1"] \arrow[d] & \\
& \mathbb{H}P^{n-1} &
\end{tikzcd}$$

which was the model for the general one. Also, examples of quaternion Hermitian Weyl manifolds will be obtained by considering appropriate quotients of the manifolds in the vertices of this diagram.

Remark 4.8. It is proved in [11] that in every dimension $4k-5, k \geq 3$ there are infinitely many distinct homotopy types of complete inhomogeneous 3-Sasakian manifolds. Thus, by simply making the product with S^1, we obtain infinitely many non-homotopically equivalent examples of compact hyperhermitian Weyl manifolds.

4.3.3. *Some topological consequences of diagram 4.9.* A first consequence of the diagram 4.5 concerns cohomology. Note first that the property $\nabla\theta = 0$ implies the vanishing of the Euler characteristic of M. Then, applying twice the Gysin sequence in the upper triangle one finds the relations between the Betti numbers of M and Z :

$$b_i(M) = b_i(Z) + b_{i-1}(Z) - b_{i-2}(Z) - b_{i-3}(Z) \quad (0 \leq i \leq 2n-1),$$
$$b_{2n}(M) = 2\left[b_{2n-1}(Z) - b_{2n-3}(Z)\right].$$

On the other hand, since P has positive scalar curvature, both P and its twistor space Z have zero odd Betti numbers, cf. [4]. The Gysin sequence of the fibration $Z \to P$ then yields:

$$b_{2p}(Z) = b_{2p}(P) + b_{2p-2}(P)$$

Together with the previous found relations this implies:

Theorem 4.6. [32], [33] *Let M be a compact hyperhermitian Weyl manifold satisfying the assumptions of Theorem 4.5. Then the following relations hold good:*

$$b_{2p}(M) = b_{2p+1}(M) = b_{2p}(P) - b_{2p-4}(P) \quad (0 \leq 2p \leq 2n-2),$$
$$b_{2n}(M) = 0,$$
$$\sum_{k=1}^{n-1} k(n-k+1)(n-2k+1)b_{2k}(M) = 0.$$

(Poincaré duality gives the correspondent of the first two equalities for $2n+2 \leq 2p \leq 4n$). In particular $b_1(M) = 1$. Moreover, if n is even, M cannot carry any quaternion Kähler metric.

The last identity is obtained, by applying S. Salamon's constraints on compact positive quaternion Kähler manifolds to the same diagram (cf [19]).

Remark 4.9. We obtain in particular $b_{2p-4}(P) \leq b_{2p}(P)$ for $0 \leq 2p \leq 2n-2$. Since any compact quaternion Kähler P with positive scalar curvature can be realized as the quaternion Kähler base of a compact quaternion Hermitian Weyl manifold M, this implies, in the positive scalar curvature case, the Kraines - Bonan inequalities for Betti numbers of compact quaternion Kähler manifolds (cf. [4]).

$b_1(M) = 1$ is a much stronger restriction on the topology of compact quaternion Hermitian Weyl manifolds in the larger class of compact complex Vaisman (generalized Hopf) manifolds. For the latter, the only restriction is b_1 odd and the induced Hopf bundles over compact Riemann surfaces of genus g provide examples of Vaisman (generalized Hopf) manifolds with $b_1 = 2g+1$ for any g , cf. [45].

The properties $b_1 = 1$ and $b_{2n} = 0$ have the following consequences:

Corollary 4.8. *Let (M, I_1, I_2, I_3) be a compact hypercomplex manifold that admits a locally and non globally conformal hyperKähler metric. Then none of the compatible complex structures $J = a_1I_1 + a_2I_2 + a_3I_3$, $a_1^2 + a_2^2 + a_3^2 = 1$, can support a Kähler metric. In particular, (M, I_1, I_2, I_3) does not admit any hyperKähler metric.*

Let M be a $4n$-dimensional $\mathcal{C}^\infty$ manifold that admits a locally and non globally conformal hyperKähler structure (I_1, I_2, I_3, g). Then, for n even, M cannot admit any quaternion Kähler structure and, for n odd, any quaternion Kähler structure of positive scalar curvature.

4.3.4. *Homogeneous compact hyperhermitian Weyl manifolds.* In the complex case, a complete classification of compact homogeneous Vaisman manifolds is still lacking. By contrast, for compact homogeneous hyperhermitian Weyl manifolds a precise classification may be obtained.

Definition 4.2. A hyperhermitian Weyl manifold $(M, [g], H, D)$ is homogeneous if there exists a Lie group which acts transitively and effectively on the left on M by hypercomplex isometries.

The homogeneity implies the regularity of the canonical foliations:

Theorem 4.7. [32] *On a compact homogeneous hyperhermitian Weyl manifold the foliations $\mathcal{D}$, $\mathcal{V}$ and $\mathcal{B}$ are regular and in the diagram 4.5, N, Z, P are homogeneous manifolds, compatible with the respective structures.*

Proof. Fix $J \in H$ be a compatible complex structure on M. Then (M, g, J) is a homogeneous Vaisman manifold and by Theorem 3.2 in [46] we have the regularity of both the foliations $\mathcal{V}_J$ and $\mathcal{B}$. Therefore M projects on homogeneous manifolds Z_J and N. In particular the projections of $I_\alpha B$ on N are regular Killing vector fields. Then Lemma 11.2 in [42] assures that the 3-dimensional foliation spanned by the projections of I_1B, I_2B, I_3B is regular. This, in turn, implies that P is a homogeneous manifold, thus $\mathcal{D}$ is regular on M. □

On the other hand, a compact homogeneous 3-Sasakian manifolds have been classified in [11]. We use this classification together with Corollary 4.7 to derive:

Proposition 4.12. [32] *The class of compact homogeneous hyperhermitian Weyl manifolds coincides with that of flat principal S^1-bundles over one of the 3-Sasakian homogeneous manifolds: S^{4n-1}, $\mathbb{R}P^{4n-1}$ the flag manifolds $\mathrm{SU}(m)/\mathrm{S}(\mathrm{U}(m-2)\times \mathrm{U}(1)), m \geq 3$, $\mathrm{SO}(k)/(\mathrm{SO}(k-4)\times \mathrm{Sp}(1)), k \geq 7$, the exceptional spaces $G_2/\mathrm{Sp}(1)$, $F_4/\mathrm{Sp}(3)$, $E_6/\mathrm{SU}(6)$, $E_7/\mathrm{Spin}(12)$, E_8/E_7.*

The flat principal S^1-bundles over P are characterized by having zero or torsion Chern class $c_1 \in H^2(P; \mathbb{Z})$ and classified by it. The integral cohomology group H^2 of the 3-Sasakian homogeneous manifolds can be computed by looking at the long homotopy exact sequence

$$\dots \to \pi_2(K) \to \pi_2(G) \to \pi_2(G/K) \to \pi_1(K) \to \pi_1(G) \to \dots$$

for the 3-Sasakian homogeneous manifolds G/K listed above. Since $\pi_2(G) = 0$ for any compact Lie group G, one obtains the following isomorphisms (cf. [12]):

$$H^2\,(\frac{\mathrm{SU}(m)}{\mathrm{S}(\mathrm{U}(m-2)\times\mathrm{U}(1))}) \cong \mathbb{Z}, \qquad H^2(\mathbb{R}P^{4n-1}) \cong \mathbb{Z}_2$$

and $H^2(G/K) = 0$ for all the other 3-Sasakian homogeneous manifolds. Hence:

Corollary 4.9. [32] *Let M be a compact homogeneous hyperhermitian Weyl manifold. Then M is one of the following:*

(i) A product $(G/K) \times S^1$, where G/K can be any of the 3-Sasakian homogeneous manifolds in the list:
S^{4n-1}, $\mathbb{R}P^{4n-1}$, $\mathrm{SU}(m)/\mathrm{S}(\mathrm{U}(m-2)\times\mathrm{U}(1)), m \geq 3$, $\mathrm{SO}(k)/(\mathrm{SO}(k-4)\times\mathrm{Sp}(1)), k \geq 7$, $G_2/\mathrm{Sp}(1)$, $F_4/\mathrm{Sp}(3)$, $E_6/\mathrm{SU}(6)$, $E_7/\mathrm{Spin}(12)$, E_8/E_7.

(ii) The Möbius band, i.e. the unique non trivial principal S^1-bundle over $\mathbb{R}P^{4n-1}$.

For example in dimension 8 one obtains only the following spaces: $S^7\times S^1$, $\mathbb{R}P^7\times S^1$, $\{\mathrm{SU}(3)/\mathrm{S}(\mathrm{U}(1)\times \mathrm{U}(1))\}\times S^1$ and the Möbius band over $\mathbb{R}P^7$. The first exceptional example appears in dimension 12: the trivial bundle $\{G^2/\mathrm{Sp}(1)\}\times S^1$ whose 3-Sasakian base is diffeomorphic to the Stiefel manifold $V_2(\mathbb{R}^7)$ of the orthonormal 2-frames in $\mathbb{R}^7$.

4.3.5. *A hyperhermitian Weyl finite covering of a quaternion Hermitian Weyl manifold.* In general, quaternion Kähler manifolds are not finitely covered by non simply connected hyperkähler ones. But in the locally conformal Kähler case we have:

Theorem 4.8. [33] *Let M be a compact quaternion Hermitian Weyl manifold which is not quaternion Kähler. If the leaves of $\mathcal{T}$ are compact, then M admits a finite covering space carrying a structure of a hyperhermitian Weyl manifold.*

Proof. Let first T be a regular vector field. Accordingly, $N = M/T$ is compact locally 3-Sasakian *manifold*, Einstein with positive scalar curvature. From Myers theorem, its Riemannian universal cover $\tilde{N}$ is compact and $\pi_1(N)$ is finite. Hence, the pull-back $\tilde{\mathcal{K}} \to \tilde{N}$ (see Corollary 3.1) is trivial and $\tilde{N}$ is globally 3-Sasakian. Let now $\tilde{M} \to \tilde{N}$ be the pull-back of the S^1-bundle $M \to N$: being a flat principal circle bundle over a 3-Sasakian manifold, Corollary 4.7 provides a hyperhermitian Weyl structure on $\tilde{M}$. By construction, this one projects on the quaternion Hermitian Weyl structure of M.

In the weaker assumption that $\mathcal{T}$ has only compact leaves (it is a quasi-regular foliation), the leaves space N is a compact orbifold with same Riemannian properties as above. Its universal orbifold covering $\tilde{N}^{orb}$ is a complete Riemannian orbifold with positive Ricci curvature. According to Corollary 21 in [7], the diameter of $\tilde{N}^{orb}$ is finite. Hence $\tilde{N}^{orb}$ is compact and $\pi_1^{orb}(N)$ is finite. Now the pull-back of $\mathcal{K} \to N$ to $\tilde{N}^{orb}$ is again trivial and, as in the manifold case, one shows that $\tilde{N}^{orb}$ is a globally 3-Sasakian orbifold. The proof then continues as above. Note that the total space $\tilde{M}$ is again a *manifold*. ☐

4.4. **Examples.** Using the structure theorems, we can now describe a large class of examples of quaternion Hermitian Weyl manifolds.

Recall first that a real 4-dimensional Hopf manifold is an integrable quaternion Hopf manifold, *i.e.* a quotient $(\mathbb{H} - \{0\})/G = (\mathbb{R}^4 - \{0\})/G$, where G is a discrete subgroup of $\mathrm{CO}(4) \sim \mathrm{GL}(1,\mathbb{H})\cdot\mathrm{Sp}(1)$. The metric $(h\bar{h})^{-1}dh\otimes d\bar{h}$, globally conformal with the flat one on $\mathbb{H}$, is invariant w.r.t. the action of G. This proves:

Proposition 4.13. [32] *Any real 4-dimensional Hopf manifold is a compact quaternion Hermitian Weyl manifold.*

We generalize this construction to higher dimensions by considering the quaternion Hopf manifold $M = (\mathbb{H}^n - \{0\})/G$, with G of the form (4.14), acting diagonally on the quaternionic coordiantes $(h^1, ..., h^n)$.

The metric on M will now be the projection of $(\sum_i h^i \overline{h}^i)^{-1} \sum_i dh^i \otimes d\overline{h}^i$ and is denoted with g. Moreover, we shall assume the resulting 4-dimensional foliation $\mathcal{D}$ to have compact leaves. We may state:

Proposition 4.14. [32] *The quaternion Hopf manifold $M = (\mathbb{H}^n - \{0\})/G$ endowed with the metric g is a compact quaternion Hermitian Weyl manifold. The leaves of the foliation $\mathcal{D}$ are integrable quaternion Hopf 4-manifolds. The leaf space $P = M/\mathcal{D}$ is a quaternion Kähler orbifold quotient of $\mathbb{H}P^{n-1}$ whose set of singular points is, generally, $\mathbb{R}P^{n-1} \subset \mathbb{H}P^{n-1}$. Moreover:*

If G is one of the groups in Kato's list (see Theorem 4.2), then M is hyperhermitian Weyl, The leaves of $\mathcal{D}$ are integrable Hopf surfaces and P is $\mathbb{H}P^{n-1}$.

The result follows from the fact that the group G, being a discrete subgroup of $\mathrm{GL}(n, \mathbb{H}) \cdot \mathrm{Sp}(1)$, preserves the quaternionic structure of the universal covering of M. The structure of the leaves was discussed in Proposition 4.10. Note that $\mathrm{GL}(n, \mathbb{H})$ acts on the left and $\mathrm{Sp}(1)$ acts on the right on the quaternionic coordinates, hence the induced action of G on $\mathbb{H}P^{n-1}$ fixes the points which can be represented in real coordinates. If G belongs to Kato's list, then it is a subgroup of $\mathrm{GL}(n, \mathbb{H})$ and preserves the hyperhermitian structure of the covering, inducing the same structure on the leaves.

Example 4.3. [34], [32] For $n = 2$, let G be the cyclic group generated by $(h^0, h^1) \mapsto (2e^{2\pi i/3} h^0, 2e^{4\pi i/3} h^1)$ and $M = (\mathbb{H}^2 - \{0\})/G$. Here the leaf space $P = M/\mathcal{D}$ is a $\mathbb{Z}_3$ quotient $\mathbb{H}P^1$. The leaves of $\mathcal{D}$ are standard Hopf surfaces $S^1 \times S^3$ over the regular points of the orbifold P and are non-primary Hopf surfaces $(S^1 \times S^3)/\mathbb{Z}_3$ over the two singular points of homogeneous coordinates $[1 : 0]$ and $[0 : 1]$ of P.

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