# Harmonicity and spectral theory on Sasakian manifolds

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**Abstract**

We study in this paper harmonic maps and harmonic morphism on Sasakian manifolds. We also give some results on the spectral theory of a harmonic map for which the target manifold is a Sasakian space form.

## 1 Introduction

The theory of harmonic maps between Riemannian manifolds endoweed with some special structures has its origin in a paper of Lichnerowicz (see [12]) in which he considered holomorphic maps between compact Kähler manifolds. He proved that such a map is not only a harmonic map but also a minimizer of the energy functional in its homotopy class.

Ianuş and Pastore developed a theory of harmonic maps between manifolds endowed with almost contact metric structures (see [9]). Following the ideas of Rawnsley, they introduced a notion analogous to holomorphy. Using similar tools Chinea studied submersions between almost contact metric manifolds (see [2]). In Section 3 we prove an analougue result to that obtained by Lichnerowicz but in the case when the target manifold is Sasakian.

The Laplace-Beltrami operator of a compact Riemannian $(M, g)$ can be viewed as the Jacobi operator of a constant harmonic map from $(M, g)$ into a unit circle This is a good reason to study the spectral geometry of the Jacobi operator of a harmonic map. This was done for Kähler manifolds with constant holomorphic curvature (see [14]) and for Sasakian manifolds with constant $\varphi$-sectional curvature (see [11]). We will prove in Section 4 that the spectrum of the Jacobi operator associated to a harmonic map determines the geometric properties as those of harmonic morphisms in the casewhen the target manifold is a sasakian space-form.

Harmonic morphism are maps which pull back germs of real valued harmonic functions on the target manifold to germs of harmonic functions on the domain. In Section 5 We prove a characterization theorem for harmonic morphisms defined on Sasakian manifolds to a Hermitian manifold. A similar type of result was obtained by Gudmundsson and Wood but in the complex case (see [8]).

[*]Partially supported by PN-II-ID-PCE-2011-3-0118

[†]2010 AMS Mathematics Subject Classification: 53C55, 53C43, 58E20

[‡]Keywords and phrases: Sasakian manifolds, holomorphic maps, harmonic map, harmonic morphisms, spectral theory

## 2 Prelminaries

In this section, we recall some well known facts concerning harmonic maps on Riemannin manifolds and almost contact metric manifolds.
Let $F : (M, g) \longrightarrow (N, h)$ be a smooth map between two Riemannian manifolds of dimensions $m$ and $n$ respectively. The energy density of $F$ is a smooth function $e(F) : M \longrightarrow [0, \infty)$ given by

$$e(F)_p = \frac{1}{2} Tr_g(F^* h)(p) = \frac{1}{2} \sum_{i=1}^{m} h(F_{*p} u_i, F_{*p} u_i),$$

for any $p \in M$ and any orthonormal basis $\{u_1, \ldots, u_m\}$ of $T_p M$. If $M$ is a compact Riemannian manifold, the energy $E(F)$ of F is the integral of its energy density:

$$E(F) = \int_M e(F) \upsilon_g,$$

where $\upsilon_g$ is the volume measure associated with the metric g on M. A map $F \in C^\infty(M, N)$ is said to be harmonic if it is a critical point of the energy functional $E$ on the set of all maps between $(M, g)$ and $(N, h)$. Now, let $(M, g)$ be a compact Riemannian manifold. If we look at the Euler-Lagrange equations for the corresponding variational problem, a map $F : M \longrightarrow N$ is a harmonic if and only if $\tau(F) \equiv 0$, where $\tau(F)$ is tension field which is defined by

$$\tau(F) = Tr_g \widetilde{\nabla} dF,$$

where $\widetilde{\nabla}$ is the connection induced by the Levi-Civita connection on $M$ and the $F$-pullback connection of the Levi Civita connection on $N$.

We recall now some definitions and basic formulas on almost contact metric manifolds (see [1] for more details).
An odd dimensional Riemannian manifold $M^{2n+1}$ is said to be an almost contact manifold if there exist on $M$ a $(1, 1)$-tensor field $\varphi$, a vector field $\xi$ and a 1-form $\eta$ such that

$$\varphi^2 = -I + \eta \otimes \xi \quad and \quad \eta(\xi) = 1.$$

In an almost contact manifold we also have $\varphi(\xi)$=0 and $\eta o \varphi = 0$.
On any almost contact manifold, we can define a compatible metric that is a metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on M. In this case the manifold will be called almost contact metric manifold. An almost contact metric manifold is said to be a contact metric manifold if $d\eta$=$\Omega$, where $\Omega$ is the fundamental 2 form defined by $\Omega$(X, Y)=g(X, $\varphi$Y) for X, Y $\in \Gamma(TM)$. In analogy with the integrability condition on almost complex manifolds, the almost contact metric structure of M is said to be normal if

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of $\varphi$, given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A normal contact metric manifold is called a Sasakian manifold.

In 1990 Chinea and Gonzalez (see [[3]]) gave a classification for almost contact metris manifolds through the study of the covariant derivative of the fundamental 2-form of those manifolds. An almost contact metric manifold belongs to the class $\mathcal{C}_5 \oplus \mathcal{C}_6$ if and only if

$$(\nabla_X \Omega)(Y, Z) = \frac{1}{2n}\{[g(X, Z)\eta(Y) - g(X, Y)\eta(Z)]\delta\Omega(\xi)+$$

$$+[-g(X, \varphi Y)\eta(Z) + g(X, \varphi(Z))\eta(Y)]\delta\eta\}$$

where $\nabla$ is the Levi-Civita connection on $M$. If $\beta = \frac{1}{2n}\delta\eta = 0$ we get the class $\mathcal{C}_6$ which includes the class of Sasakian manifolds for $\alpha = \frac{1}{2n}\delta\Omega(\xi) = 1$.

## 3 Harmonic maps on Sasakian manifolds

Ianuş and Pastore developed a theory of harmonic maps between manifolds endowed with almost contact metric structures (see [[9]]). Following the ideas of Rawnsley, they introduced a notion analogous to holomorphy. In this section we prove an analougue result to that obtained by Lichnerowicz but in the case when teh target manifold is Sasakian.

We will prove for the begining an usefull Lemma:

**Lemma 3.1.** *Let $F : N \to M$ be a $(J, \varphi)$-holomorphic map from an almost Hermitian manifold $N(J, h)$ to a manifold $M(\varphi, \xi, \eta, g)$ which belongs to the class $\mathcal{C}_5 \oplus \mathcal{C}_6$. Then $\tau(F) \in \Gamma(D)$ if and only if $M$ belongs to the class $\mathcal{C}_6$.*

*Proof.* Let $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ be an orthonormal local frame in $TN$. As $F$ is $(\varphi, J)$-holomorphic, we have

$$\begin{aligned} g(\tau(F), \xi) &= g((\tilde{\nabla}_{e_i} F_*)e_i + (\tilde{\nabla}_{Je_i} F_*)Je_i, \xi) = \\ &= g(\nabla_{F_*e_i} F_*e_i, \xi) + g(\nabla_{\varphi F_*e_i} \varphi F_*e_i, \xi) - \\ &- g(F_*(\bar{\nabla}_{e_i} e_i + \bar{\nabla}_{Je_i} Je_i), \xi) \end{aligned} \tag{1}$$

where $\nabla$ si $\bar{\nabla}$ are the Levi Civita connections on $M$ and $N$ respectively.
We have to notice that $F_*e_i$ and $F_*Je_i$ belongs to the distribution $\mathcal{D}$. Indeed,

$$\eta(F_*e_i) = -\eta(F_*J^2e_i) = -\eta(\varphi F_*Je_i) = 0.$$

Using the above remark, as $\nabla$ is the Levi-Civita connection, we have

$$\begin{aligned} g(\tau(F), \xi) &= -g(F_*e_i, \nabla_{F_*e_i}\xi) - g(\varphi F_*e_i, \nabla_{\varphi F_*e_i}\xi) - \\ &- g(F_*(\bar{\nabla}_{e_i} e_i + \bar{\nabla}_{Je_i} Je_i), \xi). \end{aligned}$$

Computing now the divergence of $J$, we get

$$\begin{aligned} divJ &= (\bar{\nabla}_{e_i} J)e_i + (\bar{\nabla}_{Je_i} J)Je_i = \\ &= \bar{\nabla}_{e_i} Je_i - J(\bar{\nabla}_{e_i} e_i) - \bar{\nabla}_{Je_i} e_i - J(\bar{\nabla}_{Je_i} Je_i) = \\ &= [e_i, Je_i] - J(\bar{\nabla}_{e_i} e_i + \bar{\nabla}_{Je_i} Je_i). \end{aligned}$$

and thus

$$\bar{\nabla}_{e_i} e_i + \bar{\nabla}_{Je_i} Je_i = J(divJ - [e_i, Je_i]) \tag{2}$$

Now, from the relations (1) and (3), as $\varphi \circ F_* = F_* \circ J$, the last term in (1) vanishes and we get

$$g(\tau(F), \xi) = -g(F_* e_i, \nabla_{F_* e_i} \xi) - g(\varphi F_* e_i, \nabla_{\varphi F_* e_i} \xi) \tag{3}$$

As $F_* e_i, \varphi F_* e_i \in \Gamma(D)$ and because $M$ is a manifold belonging to the class $\mathcal{C}_5 \oplus \mathcal{C}_6$, we have

$$\nabla_{F_* e_i} \xi = -\alpha \varphi F_* e_i + \beta F_* e_i$$

$$\nabla_{\varphi F_* e_i} \xi = \alpha F_* e_i + \beta \varphi F_* e_i.$$

If we use the above relations in (3) and using the skewsymmetry of the fundamental 2-form $\Omega$, we get

$$g(\tau(F), \xi) = -2\beta \sum_{i=1}^{n} g(F_* e_i, F_* e_i).$$

We have just obtained $\tau(F) \in \Gamma(D)$ if and only if $\beta = 0$, that is $M$ is a manifold belonging to the class $\mathcal{C}_6$. $\square$

We are able to prove now the following

**Theorem 3.1.** *Let $N(J, h)$ be a Kähler manifold and $M(\varphi, \xi, \eta, g)$ be a manifold belonging to the class $\mathcal{C}_5 \oplus \mathcal{C}_6$. Then any $(J, \varphi)-$holomorphic map $F : N \to M$ is harmonic if and only if $M$ belongs to the $\mathcal{C}_6$.*

*Proof.* We recall that for any $(J, \varphi)-$holomorphic map $F$, the tension field can be computed by

$$\varphi(\tau(F)) = F_*(divJ) - Tr_h \beta$$

where $\beta(X, Y) = (\tilde{\nabla}_X \varphi) F_* Y$.

Let $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ be a local Hermitian frame in $TN$. As $N$ is a Kähler manifold, we have

$$divJ = \sum_{i=1}^{n} \{(\nabla_{e_i} J) e_i + (\nabla_{Je_i} J) Je_i\} = 0$$

and thus the above formula become

$$\varphi(\tau(F)) = -Tr_h \beta$$

If we look at the trace of the bilinear form $\beta$ we have:

$$\begin{aligned} Tr_h \beta &= \sum_{i=1}^{n} \{(\tilde{\nabla}_{e_i} \varphi) F_* e_i + (\tilde{\nabla}_{Je_i} \varphi) F_* Je_i\} = \\ &= \sum_{i=1}^{n} \{(\nabla_{F_* e_i} \varphi) F_* e_i + (\nabla_{\varphi F_* e_i} \varphi) \varphi F_* e_i\}. \end{aligned}$$

As $M$ belongs to the class $\mathcal{C}_5 \oplus \mathcal{C}_6$ we get

$$Tr_h \beta = 2\alpha \sum_{i=1}^{n} g(F_* e_i, F_* e_i) \xi$$

and thus

$$\varphi(\tau(F)) = -2\alpha \sum_{i=1}^{n} g(F_* e_i, F_* e_i)\xi.$$

Finally we obtain

$$\tau(F) = \eta(\tau(F))\xi$$

and thus $F$ is harmonic if and only if $\tau(F) \in \Gamma(D)$.

Using now the Lemma 3.1 we obtain that $F$ is harmonic if and only if $\beta = 0$ that is $M$ belongs to the class $\mathcal{C}_6$ $\square$

In particular, as the class $\mathcal{C}_6$ includes the Sasakian manifolds we get

**Corollary 3.1.** *Let $N(J, h)$ be a Kähler manifold and $M(\varphi, \xi, \eta, g)$ be a Sasakian manifold. Then any $(J, \varphi)$-holomorphic map $F : N \to M$ is harmonic.*

# 4 Spectral geometry on Sasakian manifolds

The spectral geometry of the Laplace-Beltrami operator is a field full of interesting and deep results. For example, we know that if the spectrum $\mathrm{Spec}(\Delta)$ of the Laplace-Beltrami operator of a compact Riemannian manifold $(M, g)$ is the same with those of the standard sphere $(S^n, can)$, for any $n < 7$, then $(M, g)$ is izometric with $(S^n, can)$.

As the Laplace-Beltrami operator of a compact Riemannian $(M, g)$ can be viewed as the Jacobi operator of a constant harmonic map from $(M, g)$ into a unit circle, a good subject is to study the spectral geometry of the Jacobi operator of a harmonic map. This was done for Kähler manifolds with constant holomorphic curvature (see [14]) and for Sasakian manifolds with constant $\varphi$-sectional curvature (see [11]). We will give new results concerning the last topic.

We recall that the Jacobi operator $J_F$ of a harmonic map $F : (M, g) \to (N, h)$ is defined by

$$J_F = \overline{\Delta}_F V - \mathcal{R}_F V \tag{4}$$

for any $V \in \Gamma(E)$, where $\overline{\Delta}$ is the rough Laplacean associated to the induced connection $\tilde{\nabla}$ in the induced bundle $E = F^{-1}TN$ defined by $\tilde{\nabla}_X V = \nabla^N_{F_* X} V$ and

$$\mathcal{R}_F V = \sum_{i=1}^{m} R^N(V, F_* e_i) F_* e_i$$

for any $X \in \Gamma(TM)$ and any local orthonormal frame $\{e_1, \ldots, e_m\}$ in $TM$.

Consider the semigroup $e^{-tJ_F}$ defined by

$$e^{-tJ_F} V(x) = \int_M K(t, x, y, J_F) d\vartheta_g(y)$$

where $K(t, x, y, J_F) \in Hom(E_y, E_x)$ is the kernel function ($x, y \in M, E_x$ is the fiber over $x$). Then we have the asymptotic expansion for the $L^2$-trace

$$Tr(e^{-tJ_F}) = \sum_{i=1}^{\infty} e^{-t\lambda_i} \simeq (4\pi t)^{-\frac{m}{2}} \sum_{n=0}^{\infty} t^n a_n(J_F), t \to 0_+ \tag{5}$$

where each $a_n(J_F)$ is the spectral invariant of $J_F$, which depends only on the discret spectrum:

$$Spec(J_F) = \{\lambda_1 \le \lambda_2 \le \dots \lambda_n \le \dots \to \infty\}$$

Using the results of Gilkey ([7]) for the Jacobi operator $J_F$, Urakawa obtained (see [14]):

**Theorem 4.1.** *For any harmonic map $F : (M,h) \to (N,g)$ from a $m$-dimensional compact connected manifold without border we have:*

$$a_0(J_F) = nvol(M,h)$$

$$a_1(J_F) = \frac{n}{6}\int_M \tau_h d\vartheta_h + \int_M Tr(\mathcal{R}_F) d\vartheta_h$$

$$a_2(J_F) = \frac{n}{360}\int_M [5\tau_h^2 - 2\|\rho_h\|^2 + 2\|R_h\|^2] d\vartheta_h +$$

$$+\frac{1}{360}\int_M [-30\|R^{\tilde{\nabla}}\|^2 + 60\tau_h Tr(\mathcal{R}_F) + 180 Tr(\mathcal{R}_F^2)] d\vartheta_h$$

*where $R^{\tilde{\nabla}}$ is the curvature tensor of the connection $\tilde{\nabla}$ in $E$, defined by $R^{\tilde{\nabla}} = F^* R^N$, and $R_h, \rho_h, \tau_h$ are the curvature tensor, the Ricci tensor and scalar curvature on $M$.*

Let $M(\varphi, \eta, \xi, g)$ be a Sasakian manifold. A plane section $T_pM$ is called a $\varphi$-section if there exists a vector $X \in T_pM$ orthogonal to $\xi$ such that $\{X, \varphi X\}$ is a orthonormal basis of this plane section. The sectional curvature $K(X, \varphi X) = H(X)$ it is called the $\varphi$-sectional curvature (see [1]).

In analogy with the case of holomorphic sectional curvatures of Kähler manifolds, the $\varphi$-sectional curvatures determine the curvature of a Sasakian manifold.

It is known that, if on each point of a Sasakian manifold of dimension $\ge 5$ the $\varphi$-sectional curvature is independent with respect to the $\varphi$-section on this point, then this is constant and the curvature tensor is given by:

$$R(X,Y)Z = \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y -$$
$$-\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi +$$
$$+\Omega(Z,Y)\varphi X - \Omega(Z,X)\varphi Y + 2\Omega(X,Y)\varphi Z\}$$

where $c$ is the constant $\varphi$-sectional curvature.

The sphere $S^{2n+1}$ with the canonical contact structure induced on the unit sphere on $\mathbb{C}^{n+1}$ is an example of Sasakian manifold with the constant $\varphi$-sectional curvature $c = 1$.

Another example is $\mathbb{R}^{2n+1}$ with the coordinates $(x^i, y^i, z), 1 \le i \le n$, which admits the Sasakian structure

$$\eta = \frac{1}{4}(dz - \sum_{i=1}^n y^i dx^i), \quad g = \frac{1}{4}(\eta \otimes \eta + \sum_{i=1}^n ((dx^i)^2 + (dy^i)^2)).$$

With this metric, $\mathbb{R}^{2n+1}$ is a Sasakian manifold with the constant $\varphi$-sectional curvature $c = -3$.

Let $F : (M,h) \to (N,g)$ be a harmonic map from a $m$-dimensional compact connected Riemannian manifold without border to a $(2n+1)$-dimensional Sasakian manifold with constant $\varphi$-sectional curvature $c$.

In the following we will use the notations: $\alpha = \frac{c+3}{4}, \beta = \frac{c-1}{4}$

$$\|F^*\eta\|^2 = \sum_{i=1}^{m} \eta(F^*e_i)^2 \quad \|F^*\Omega\|^2 = \sum_{i,j=1}^{m} g(F_*e_i, \varphi F_*e_j)^2$$

$$\|F^*g\|^2 = \sum_{i,j=1}^{m} g(F_*e_i, F_*e_j)^2$$

$$F^*(\eta \times \eta \times g) = \sum_{i,j=1}^{m} \eta(F_*e_i)\eta(F_*e_j)g(F_*e_i, F_*e_j)$$

where $\{e_1, \ldots, e_m\}$ is a local orthonormal frame on $TM$.
From the definition of $\mathcal{R}_F$ we have

$$Tr(\mathcal{R}_F) = \sum_{i=1}^{m}\sum_{a=1}^{2n+1} g(R(v_a, F_*e_i)F_*e_i, v_a) =$$

$$= 4(\alpha n + \beta)e(F) - 2\beta(n+1)\|F^*\eta\|^2.$$

Using the above relations we obtain

$$Tr(\mathcal{R}_F^2) = \sum_{i,j=1}^{m}\sum_{a=1}^{2n+1} g(R(v_a, F_*e_i)F_*e_i, R(v_a, F_*e_j)F_*e_j) =$$

$$= (\alpha^2 + 9\beta^2)\|F^*g\|^2 - 4(\alpha\beta + 4\beta^2)F^*(\eta \times \eta \times g) + 2\beta^2(n+7)\|F^*\eta\|^4 +$$
$$+ 4[\beta^2 + \alpha^2(2n-1) + 4\alpha\beta]e(F)^2 - 8(2\beta^2 + \alpha\beta n)e(F)\|F^*\eta\|^2 - 6\alpha\beta\|F^*\Omega\|^2.$$

Now we are able to obtain:

$$\|R^{\tilde{\nabla}}\|^2 = \sum_{i,j=1}^{m}\sum_{a,b=1}^{2n+1} g(R(F_*e_i, F_*e_j)v_a, v_b)g(R(F_*e_i, F_*e_j)v_a, v_b) =$$

$$= 8(\alpha^2 + \beta^2)e(F)^2 - 16\alpha\beta e(F)\|F^*\eta\|^2 - 2(\alpha^2 + \beta^2)\|F^*g\|^2 +$$
$$+ 4[2\beta^2(n+1) + 3\alpha\beta]\|F^*\Omega\|^2 + 8\alpha\beta F^*(\eta \times \eta \times g)$$

Finally, by replacing these relations in the formulas in the Theorem **??** we obtain:

**Theorem 4.2.** *Let $F : (M,h) \to (N,g)$ be a harmonic map from a $m$-dimensional compact connected Riemannian manifold without border to a $(2n+1)$-dimensional Sasakian manifold with constant $\varphi$-sectional curvature $c$. Then*

$$a_0(J_F) = (2n+1)vol(M,h)$$

$$a_1(J_F) = \frac{2n+1}{6}\int_M \tau_h d\vartheta_h + 4(\alpha n + \beta)\int_M e(F)d\vartheta_h - 2\beta(n+1)\int_M \|F^*\eta\|^2 d\vartheta_h$$

$$a_2(J_F) = \frac{2n+1}{360}\int_M (5\tau_h^2 - 2\|\rho_h\|^2 + 2\|R_h\|^2)d\vartheta_h +$$

$$+\frac{1}{3}\int_M [2(\alpha n+\beta)e(F)-\beta(n+1)\|F^*\eta\|^2)\tau_h d\vartheta_h+$$

$$+\frac{4}{3}[\alpha^2(3n-2)+6\alpha\beta+\beta^2]\int_M e(F)^2 d\vartheta_h+\beta^2(n+7)\int_M \|F^*\eta\|^4\vartheta_h-$$

$$-\frac{2}{3}[\beta^2(1+n)+6\alpha\beta]\int_M \|F^*\Omega\|^2 d\vartheta_h-\frac{8}{3}(\alpha\beta+3\beta^2)\int_M F^*(\eta\times\eta\times g)d\vartheta_h+$$

$$+\frac{4}{3}[\alpha\beta(1-3n)-6\beta^2]\int_M e(F)\|F^*\eta\|^2 d\vartheta_h+\frac{2}{3}(\alpha^2+7\beta^2)\int_M \|F^*g\|^2 d\vartheta_h.$$

Now we consider the case when $F: M\to N$ is a $(J,\varphi)$-holomorphic from a Kähler manifold with dimension $m=2p$, to a $(2n+1)$-dimensional Sasakian manifold with constant $\varphi$-sectional curvature $c$. As we have seen, this is a harmonic map.

Let us remark that for any $X\in\Gamma(TM)$ we have

$$\eta(F_*X)=-\eta(F_*J^2X)=-\eta(\varphi F_*JX)=0$$

and thus we get

$$\|F^*\eta\|^2=0,\qquad F^*(\eta\times\eta\times g)=0.$$

On the other hand, $\|F^*\Omega\|^2=\|F^*g\|^2$. Indeed, let $\{e_i\}=\{X_k,JX_k\}$ be a local Hermitian frame in $TM$, $1\le k\le p$. Then we have

$$\|F^*\Omega\|^2=\sum_{i=1}^{m}\sum_{k=1}^{p}\{g(F_*e_i,\varphi F_*X_k)^2+g(F_*e_i,\varphi F_*JX_k)^2\}=$$

$$=\sum_{i=1}^{m}\sum_{k=1}^{p}\{g(F_*e_i,F_*JX_k)^2+g(F_*e_i,F_*X_k)^2\}=$$

$$=\sum_{i,j=1}^{m}g(F_*e_i,F_*e_j)^2=\|F^*g\|^2.$$

We have obtained

**Proposition 4.1.** *Let $F:(M,h)\to(N,g)$ be a $(J,\varphi)$-holomorphic map from a compact connected Kähler manifold to a $(2n+1)$-dimensional Sasakian manifold with constant $\varphi$-sectional curvature $c$. Then*

$$a_0(J_F)=(2n+1)vol(M,h)$$

$$a_1(J_F)=\frac{2n+1}{6}\int_M \tau_h d\vartheta_h+4(\alpha n+\beta)\int_M e(F)d\vartheta_h$$

$$a_2(J_F)=\frac{2n+1}{360}\int_M(5\tau_h^2-2\|\rho_h\|^2+2\|R_h\|^2)d\vartheta_h+$$

$$+\frac{2}{3}(\alpha n+\beta)\int_M e(F)\tau_h d\vartheta_h+\frac{4}{3}(3\alpha^2 n-2\alpha^2+6\alpha\beta+\beta^2)\int_M e(F)^2 d\vartheta_h+$$

$$+\frac{2}{3}(\alpha^2+6\beta^2-6\alpha\beta+n\beta^2)\int_M \|F^*g\|^2 d\vartheta_h.$$

As a first application, using the above proposition, we get

**Corollary 4.1.** *Let $F, F' : M \to N$ two $(J, \varphi)$-holomorphic maps from a compact Kähler manifold with constant curvature $\tau_h$, to a Sasakian manifold with constant $\varphi$-sectional curvature $c = 1$, such that $SpecJ_F = SpecJ_{F'}$. Then*

$$E(F) = E(F')$$

$$(6n-4)\int_M e(F)^2 d\vartheta_h + \int_M \|F^* g\|^2 d\vartheta_h = (6n-4)\int_M e(F')^2 d\vartheta_h + \int_M \|F'^* g\|^2 d\vartheta_h.$$

Another application of the Proposition 4.1 is the following

**Corollary 4.2.** *Let $F, F' : M \to N$ be two $(J, \varphi)$-holomorphic maps which are horizontally conformal from a compact Kähler manifold with constant curvature $\tau_h$, to a Sasakian manifold with constant $\varphi$-sectional curvature $c = 1$, such that $SpecJ_F = SpecJ_{F'}$. Suppose that $SpecJ_F = SpecJ_{F'}$. Then, if $F$ is a Riemannian submersion, $F'$ is also a Riemannin submersion.*

*Proof.* Let $\{e_i\}_{i=1}^m$ be a orthonormal local frame on $TM$. As $F$ is a Riemannian submersion, we have $F^* g = h$ and thus

$$e(F) = \frac{1}{2}\sum_{i=1}^m g(F_* e_i, F_* e_i) = \frac{1}{2}\sum_{i=1}^m h(e_i, e_i) = \frac{m}{2}.$$

On the other hand,

$$\|F^* g\|^2 = \sum_{i,j=1}^m g(F_* e_i, F_* e_j)^2 = \sum_{i,j=1}^m h(e_i, e_j)^2 = m.$$

As $F'$ is horizontally conformal, we have $F^* g = \lambda^2 h$, ($\lambda$ is the dilation function of $F'$), and thus

$$e(F') = \frac{1}{2}\sum_{i=1}^m g(F_* e_i, F_* e_i) = \frac{\lambda^2 m}{2}$$

$$\|F^* g\|^2 = \sum_{i,j=1}^m \lambda^4 h(e_i, e_j)^2 = \lambda^4 m$$

Using now the formulas obtained in Proposition 4.1 we get

$$\int_M \lambda^2 d\vartheta_h = \int_M 1 d\vartheta_h \qquad \int_M \lambda^4 d\vartheta_h = \int_M 1 d\vartheta_h.$$

Finally from the Cauchy inequality, we obtain $\lambda^2 = 1$ and thus $F'$ is also a Riemannian submersion. $\square$

# 5 Harmonic morphisms on Sasakian manifolds

Harmonic morphism are maps which pull back germs of real valued harmonic functions on the target manifold to germs of harmonic functions on the domain, that is, a smooth map $F : (M, g) \to (N, h)$ is a harmonic morphism if for any harmonic function $f : U \to \mathbb{R}$, defined on an open subset U of N such that $\pi^{-1}(U)$ is non-empty, the composition $f \circ F : \pi^{-1}(U) \to \mathbb{R}$ is a harmonic function. The following characterization of harmonic morphisms is due to Fuglede and Ishihara: A smooth map F is a harmonic morphism if and only if F is horizontally conformal harmonic map (see [6] and [10]). Now we look for harmonic morphisms defined on Sasakian manifolds.

**Proposition 5.1.** *Let $M(\varphi, \xi, \eta, g)$ be an almost contact metric manifold, $N(J, h)$ be an almost Hermitian manifold and $F : M \to N$ a horizontally conformal $(\varphi, J)$-holomorphic map. Then the following conditions are equivalent:*
*(i) $divJ = 0$*
*(ii) $Tr_g\beta = 0$*
*where $\beta(X, Y) = (\tilde{\nabla}_X J)F_*Y$ for any $X, Y \in \Gamma(TM)$.*

*Proof.* Suppose that $M$ is $(2m + 1)$-dimensional and $N$ has the real dimension $2n$. Let $\{Y_1, \ldots, Y_n, JY_1, \ldots, JY_n\}$ a local Hermitian frame on $TN$.

We denote by $\{Y_1^*, \ldots, Y_n^*\}$ the unique horizontal lifts of the fields $\{Y_1, \ldots, Y_n\}$ that is, $Y_i^*$ are horizontal fields and $F_*(Y_i^*) = Y_i$.
For the normalization reasons, we define the fields $X_i = \lambda Y_i^*$, where $\lambda$ is the dilation function of $F$. As $F$ is a horizontally conformal map, we have

$$g(X_i, X_j) = \lambda^2 g(Y_i^*, Y_j^*) = h(F_*Y_i^*, F_*Y_j^*) = h(Y_i, Y_j) = \delta_{ij}$$

and thus $\{X_1, \ldots, X_n\}$ is an orthonormal system. Extend this system to a local $\varphi$-adaptated frame

$$\{X_1, \ldots, X_n, X_{n+1}, \ldots, X_m, \varphi X_1, \ldots, \varphi X_n, \varphi X_{n+1}, \ldots, \varphi X_m, \xi\}.$$

It is clear that $\{X_1, \ldots, X_n, \varphi X_1, \ldots, \varphi X_n, \xi\}$ is a frame on the horizontal space, and $\{X_{n+1}, \ldots, X_m, \varphi X_{n+1}, \ldots, \varphi X_m, \xi\}$ are vertical fields. Then, the trace of the biliear form $\beta$ is

$$Tr_g\beta = \sum_{i=1}^{m}(\tilde{\nabla}_{X_i}J)F_*X_i + \sum_{i=1}^{m}(\tilde{\nabla}_{\varphi X_i}J)F_*\varphi X_i + (\tilde{\nabla}_\xi J)F_*\xi.$$

As $F$ is $(\varphi, J)$-holomorphic, it is not difficult to see that $F_*\xi = 0$. On the other hand, from the construction, $\{X_{n+1}, \ldots, X_m, \varphi X_{n+1}, \ldots, \varphi X_m\}$ is a system of vertical fields, and thus

$$Tr_g\beta = \sum_{i=1}^{n}(\tilde{\nabla}_{X_i}J)F_*X_i + \sum_{i=1}^{n}(\tilde{\nabla}_{\varphi X_i}J)F_*\varphi X_i =$$

$$= \sum_{i=1}^{n}(\tilde{\nabla}_{X_i}J)F_*X_i + \sum_{i=1}^{n}(\tilde{\nabla}\varphi X_i J)JF_*X_i.$$

Now, using the above contruction, we have:

$$Tr_g\beta = \sum_{i=1}^{n}\{\nabla_{F_*X_i}JF_*X_i - J(\nabla_{F_*X_i}F_*X_i) + \nabla_{JF_*X_i}J^2F_*X_i - J(\nabla_{JF_*X_i}JF_*X_i)\} =$$

$$\lambda\sum_{i=1}^{n}\{\nabla_{F_*Y_i^*}\lambda JF_*Y_i^* - J(\nabla_{F_*Y_i^*}\lambda F_*Y_i^*) + \nabla_{JF_*Y_i}\lambda J^2F_*Y_i^* - J(\nabla_{JF_*Y_i^*}\lambda JF_*Y_i^*)\} =$$

$$= \lambda \sum_{i=1}^{n} \{\nabla_{Y_i} \lambda J Y_i - J(\nabla_{Y_i} \lambda Y_i) + \nabla_{JY_i} \lambda J^2 Y_i - J(\nabla_{JY_i} \lambda J Y_i)\} =$$

$$= \lambda^2 \sum_{i=1}^{n} \{\nabla_{Y_i} J Y_i - J(\nabla_{Y_i} Y_i) +$$

$$+ \nabla_{JY_i} J^2 Y_i + J(\nabla_{JY_i} J Y_i)\} =$$

$$= \lambda^2 \sum_{i=1}^{n} \{(\nabla_{Y_i} J) Y_i + (\nabla_{JY_i} J) J Y_i\} =$$

$$= \lambda^2 div J$$

and thus

$$Tr_g \beta = \lambda^2 div J.$$

Now the equivalence of the two statements is proved. □

We will prove now a characterization theorem for harmonic morphisms defined on an almost contact manifolds to a Hermitian manifold. A similar type of result was obtained by Gudmundsson and Wood but in the complex case (see [8]).

**Theorem 5.1.** *Let $F : M \to N$ be a horizontally conformal $(\varphi, J)$-holomorphic map from an almost contact metric manifold $M(\varphi, \xi, \eta, g)$ to a Hermitian manifold $N(J, h)$. Then any two of the following conditions imply the third*
*(i) $div J = 0$*
*(ii) $F_*(div\varphi) = 0$*
*(iii) $F$ is harmonic and thus harmonic morphism.*

*Proof.* We recall that if $F : M \to N$ is a $(\varphi, J)$-holomorphic map from an almost contact metric manifold $M(\varphi, \xi, \eta, g)$ to a Hermitian manifold $N(J, h)$ then the tension fielf of $F$ satisfiyes the relation

$$J(\tau(F)) = F_*(div\varphi) - Tr_g \beta$$

If we suppose that $div J = 0$, as $F$ is horizontally conformal $(\varphi, J)$-holomorphic, using the above proposition, we get $Tr_g \beta = 0$. If moreover, $F_*(div\varphi) = 0$ using the above formula for the tension field of $F$ we get $\tau(F) = 0$ that is $F$ is harmonic. As $F$ is horizontally conformal, using Fuclede's theorem we obtain that $F$ is a harmonic morphism. In this way we have just proved that the first two conditions imply the third. In a similar maner we prove the other implications

□

Using the above theorem, we are able to prove the following:

**Theorem 5.2.** *Let $F : M \to N$ be a horizontally conformal $(\varphi, J)$-holomorphic map from a Sasaki manifold $M(\varphi, \xi, \eta, g)$ to a Hermitian manifold $N(J, h)$. Then $F$ is a harmonic morphism if and only if $N$ is a semi-Kähler manifold.*

*Proof.* Let $\{X_1, \ldots, X_m, \varphi X_1, \ldots, \varphi X_m, \xi\}$ be a local orthonarmal $\varphi$-adaptated frame on $TM$. Then the divergence of $\varphi$ is given by

$$div\varphi = \sum_{i=1}^{m} (\nabla_{X_i} \varphi) X_i + \sum_{i=1}^{m} (\nabla_{\varphi X_i} \varphi) \varphi X_i + (\nabla_\xi \varphi) \xi.$$

As $M$ is a Sasakian manifold, we have

$$(\nabla_{X_i}\varphi)X_i = \alpha\xi, \qquad (\nabla_{\varphi X_i}\varphi)\varphi X_i = \alpha\xi$$

$$(\nabla_\xi\varphi)\xi = 0$$

and thus

$$div\varphi = 2n\alpha\xi.$$

But as $F$ is a $(\varphi, J)$-holomorphic map, we know that $F_*\xi = 0$ and thus $F_*(div\varphi) = 0$.

Using now the above theorem, as $F$ is a horizontally conformal $(\varphi, J)$-holomorphic map, we obtain that $F$ is a harmonic morphism if and only if $divJ = 0$, that is $N$ is a semi-Kähler manifold. $\square$

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