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Optimal inequalities for the normalized δ-Casorati curvatures of submanifolds in Kenmotsu space forms

Abstract: In this paper, we establish two sharp inequalities for the normalized δ-Casorati curvatures of submanifolds in a Kenmotsu space form, tangent to the structure vector field of the ambient space. Moreover, we show that in both cases, the equality at all points characterizes the totally geodesic submanifolds.

Keywords: Casorati curvature; Riemannian submanifold; almost contact manifold; Kenmotsu space form

MSC: 53C40; 53C25

1 Introduction

The Casorati curvature of a surface in Euclidean 3-space $\mathbb{E}^3$ was introduced in [6] as the normalized sum of the squared principal curvatures. This curvature was preferred by Casorati over the traditional Gauss curvature because it seems to correspond better with the *common* intuition of curvature due to the following reason: the Gauss curvature may vanish for surfaces that look intuitively *curved*, while the Casorati curvature of a surface only vanishes at planar points [11, 15]. The original idea of Casorati was extended in the general setting of Riemannian geometry as follows. The Casorati curvature of a submanifold of a Riemannian manifold, usually denoted by $\mathcal{C}$, is an extrinsic invariant defined

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as the normalized square of the length of the second fundamental form of the submanifold [10]. In particular, if M is a hypersurface of an $(n+1)$-dimensional Riemannian manifold $\overline{M}$, then $\mathcal{C} = \frac{1}{n}\left(k_1^2 + \ldots + k_n^2\right)$, whereby $k_1, \ldots, k_n$ are the principal normal curvatures of the hypersurface. We note that an interesting geometrical interpretation of the Casorati curvatures for general submanifolds was given in [13]. Moreover, in computer vision, the Casorati curvature occurs as the *bending energy* [16] (see also [1, 17, 24, 33, 34] for geometrical meaning, importance and various applications of the Casorati curvatures).

On the other hand, the contact geometry turns out to be a very fruitful branch of the differential geometry, with many applications not only in mathematics, but also in geometrical optics, mechanics of dynamical systems with time dependent Hamiltonian, thermodynamics and geometric quantization [28]. The Kenmotsu geometry, one of the newest chapter of the contact geometry, was born in 1972 in a paper of Katsuei Kenmotsu [14], who proposed to study the properties of the warped product of the complex space with the real line, a very natural problem since this product is one of the three classes in S. Tanno's classification of connected almost contact Riemannian manifolds with automorphism group of maximum dimension [30].

In this paper we prove two optimal inequalities that relate the normalized scalar curvature with the Casorati curvature for a submanifold in a Kenmotsu space form. The proof is based on an optimization procedure involving a quadratic polynomial in the components of the second fundamental form (see [10, 11, 18, 20, 21]). We note that an alternative proof for this kind of inequalities can be done sometimes using T. Oprea's optimization method on submanifolds [25], namely analyzing a suitable constrained extremum problem (see also [19, 23, 26, 27, 29, 36, 37]).

2 Preliminaries

This section gives several basic definitions and notations for our framework based mainly on [9, 28].

Let M be an $(m+1)$-dimensional Riemannian submanifold of a $(2m+1)$-dimensional Riemannian manifold $(\overline{M}, \overline{g})$. Then we denote by g the metric tensor induced on M. Let $K(\pi)$ be the sectional curvature of M associated with a plane section $\pi \subset T_pM,\ p \in M$. If $\{e_1, ..., e_{m+1}\}$ is an orthonormal basis of the tangent space T_pM and $\{e_{m+2}, ..., e_{2m+1}\}$ is an orthonormal basis of the normal space

$T_p^\perp M$, then the scalar curvature τ of M at p is given by

$$\tau(p) = \sum_{1\le i<j\le m+1} K(e_i \wedge e_j)$$

and the normalized scalar curvature ρ of M is defined as

$$\rho = \frac{2\tau}{m(m+1)}.$$

We denote by H the mean curvature vector of M in $\overline{M}$, that is

$$H(p) = \frac{1}{m+1}\sum_{i=1}^{m+1} h(e_i, e_i),$$

where h is the second fundamental form of M in $\overline{M}$, and we also set

$$h_{ij}^{\alpha} = g(h(e_i, e_j), e_\alpha),$$

for all $i, j \in \{1, ..., m+1\},\ \alpha \in \{m+2, ..., 2m+1\}$.

The squared norm of h over dimension $m+1$ is denoted by $\mathcal{C}$ and is called the Casorati curvature of the submanifold M. Therefore we have

$$\mathcal{C} = \frac{1}{m+1}\sum_{\alpha=m+2}^{2m+1}\sum_{i,j=1}^{m+1}\left(h_{ij}^{\alpha}\right)^2.$$

Suppose now that L is an s-dimensional subspace of T_pM, $s \ge 2$ and let $\{e_1, ..., e_s\}$ be an orthonormal basis of L. Then the Casorati curvature $\mathcal{C}(L)$ of the subspace L is defined as

$$\mathcal{C}(L) = \frac{1}{s}\sum_{\alpha=m+2}^{2m+1}\sum_{i,j=1}^{s}\left(h_{ij}^{\alpha}\right)^2.$$

The normalized δ-Casorati curvature $\delta_c(m)$ and $\hat{\delta}_c(m)$ are given by

$$[\delta_c(m)]_x = \frac{1}{2}\mathcal{C}_x + \frac{m+2}{2(m+1)}\inf\{\mathcal{C}(L) \mid L : \text{a hyperplane of } T_xM\},$$

and

$$[\hat{\delta}_c(m)]_x = 2\mathcal{C}_x - \frac{2m+1}{2(m+1)}\sup\{\mathcal{C}(L) \mid L : \text{a hyperplane of } T_xM\}.$$

We recall now that a submanifold of a Riemannian manifold is called *totally geodesic* (resp. *minimal*) if its second fundamental form (resp. mean curvature) vanishes identically. We also recall that if the second fundamental form and the mean curvature of M in $\overline{M}$ satisfy for any vector fields X, Y tangent to M the relation

$$\overline{g}(h(X,Y),H) = fg(X,Y),$$

for some function f on M, then M is said to be a *pseudo-umbilical* submanifold.

Assume now that $(\overline{M},\overline{g})$ is a $(2m+1)$-dimensional almost contact metric manifold with structure tensors (ϕ,ξ,η), where ϕ is a $(1,1)$-tensor field, ξ a vector field, η a 1-form on $\overline{M}$. These structure tensors satisfy [3]

$$\begin{aligned} &\eta(\xi)=1, \quad \varphi^2=-I+\eta\otimes\xi, \quad \varphi\xi=\eta\circ\varphi=0 \\ &g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y) \\ &\eta(X)=g(X,\xi) \end{aligned} \tag{2.1}$$

for any vector fields X and Y on $\overline{M}$. An almost contact metric manifold is called *Kenmotsu* if the Riemannian connection $\overline{\nabla}$ of $\overline{g}$ satisfies [14]

$$\left(\overline{\nabla}_X\phi\right)(Y)=\overline{g}(\phi X,Y)\xi-\eta(Y)\phi X, \quad \overline{\nabla}_X\xi=X-\eta(X)\xi. \tag{2.2}$$

Let M be an $(m+1)$-dimensional Riemannian submanifold with the induced metric g of $\overline{M}$. Then, for any vector field X tangent to M and N normal to M, we have the decompositions:

$$\phi X=PX+FX, \quad \phi N=tN+fN, \tag{2.3}$$

where PX (FX) denotes the tangential (normal, respectively) component of ϕX, and tN (fN) denotes the tangential (normal, respectively) component of ϕN. Given a local orthonormal frame $\{e_1,e_2,\cdots,e_{m+1}\}$ of M, we can define the squared norms of P and F by

$$|P|^2=\sum_{i=1}^{m+1}|Pe_i|^2, \quad |F|^2=\sum_{i=1}^{m+1}|Fe_i|^2. \tag{2.4}$$

Next, we suppose that $(\overline{M},\phi,\xi,\eta,\overline{g})$ is a Kenmotsu manifold of dimension ≥ 5. Then $\overline{M}$ is said to be *a pointwise Kenmotsu space form*, usually denoted by $\overline{M}(c)$, if its ϕ-sectional curvature of a ϕ-holomorphic plane $\{X,\phi X\}$, where $X\in T_x\overline{M}$, is depending only on the point $x\in\overline{M}$, but not on the ϕ-holomorphic plane at x. A connected Kenmotsu pointwise space form whose ϕ-sectional curvature does not depend on the point is called *Kenmotsu space form* and we also denote by $\overline{M}(c)$. An elementary computation shows that a Kenmotsu manifold has constant ϕ-sectional curvature c at a point if and only if the curvature vector $\overline{R}$

is given by [14]:

$$\begin{aligned}\overline{R}(X,Y)Z = {} & \frac{c-3}{4}\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ & - \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi \\ & - g(\phi X,Z)\phi Y + g(\phi Y,Z)\phi X + 2g(X,\phi Y)\phi Z\}.\end{aligned} \tag{2.5}$$

In our paper, we assume that the structure vector field ξ is tangent to the submanifold M. Then, for an orthonormal basis $\{e_1, e_2, \cdots, e_m, e_{m+1} = \xi\}$ on M, the scalar curvature τ of M is of the form

$$2\tau = \sum_{i \neq j}^{m} K(e_i \wedge e_j) + 2\sum_{i=1}^{m} K(e_i \wedge \xi) \tag{2.6}$$

Combining (2.4), (2.5), and (2.6), we obtain the relation between the scalar curvature and the length of the mean curvature vector of M:

$$\begin{aligned}2\tau = {} & (m+1)^2 |H|^2 - |h|^2 + \frac{c-3}{4} m(m+1) \\ & + \frac{3(c+1)}{4}|P|^2 - \frac{1}{2}m(c+1),\end{aligned} \tag{2.7}$$

where

$$|h|^2 = \sum_{i,j=1}^{m+1} g(h(e_i,e_j), h(e_i,e_j)).$$

3 The main result

Theorem 1. *Let M be an $(m+1)$-dimensional submanifold of a $(2m+1)$-dimensional (pointwise) Kenmotsu space form $\overline{M}(c)$ such that the structure vector field ξ is tangent to M. Then:*

(i) *The normalized δ-Casorati curvature $\delta_C(m)$ satisfies*

$$\rho \leq \delta_C(m) + \frac{c-3}{4} + \frac{3(c+1)}{4m(m+1)}|P|^2 - \frac{c+1}{2(m+1)}. \tag{3.1}$$

(ii) *The normalized δ-Casorati curvature $\widehat{\delta}_C(m)$ satisfies*

$$\rho \leq \widehat{\delta}_C(m) + \frac{c-3}{4} + \frac{3(c+1)}{4m(m+1)}|P|^2 - \frac{c+1}{2(m+1)}. \tag{3.2}$$

Moreover, the equality sign holds in the inequality (3.1) (resp. (3.2)) if and only if M is a totally geodesic submanifold of $\overline{M}(c)$.

Proof. (i) Because the structure vector ξ is tangent to M by the assumption, then from (2.2) and the Gauss formula we get

$$h(X,\xi)=0, \tag{3.3}$$

for any vector field X tangent to M.

Let us consider the function $\mathcal{P}:\mathbb{R}^{m(m+1)^2}\to\mathbb{R}$, associated with the following polynomial in the components $(h_{ij}^{\alpha})_{i,j=1,\dots,m+1;\alpha=m+2,\dots,2m+1}$ of the second fundamental form h of M in $\overline{M}(c)$:

$$\begin{aligned}\mathcal{P} &= \frac{1}{2}m(m+1)\mathcal{C}+\frac{1}{2}m(m+2)\mathcal{C}(L)-2\tau \\ &\quad+\frac{m(m+1)(c-3)}{4}+\frac{c+1}{4}\left(3|P|^2-2m\right),\end{aligned} \tag{3.4}$$

where L is a hyperplane of T_xM.

Without loss of generality, we can assume that L is spanned by $e_1,\cdots,e_m$. Then from (2.7), (3.3) and (3.4) we derive that

$$\begin{aligned}\mathcal{P} &= \sum_{\alpha=m+2}^{2m+1}\sum_{i=1}^{m}\left[(m+1)\left(h_{ii}^{\alpha}\right)^2\right] \\ &\quad+\sum_{\alpha=m+2}^{2m+1}\left[2(m+2)\sum_{1=i<j}^{m}\left(h_{ij}^{\alpha}\right)^2-2\sum_{1=i<j}^{m}h_{ii}^{\alpha}h_{jj}^{\alpha}\right].\end{aligned} \tag{3.5}$$

We can remark now that $\mathcal{P}$ is a quadratic polynomial in the components of the second fundamental form. From (3.5), it follows that the critical points

$$h^c=\left(h_{11}^{m+2},h_{12}^{m+2},\cdots,h_{mm}^{m+2},\cdots,h_{11}^{2m+1},\cdots,h_{mm}^{2m+1}\right)$$

of $\mathcal{P}$ are the solutions of the following system of linear homogeneous equations:

$$\begin{cases}\dfrac{\partial\mathcal{P}}{\partial h_{ii}^{\alpha}}=2(m+2)h_{ii}^{\alpha}-2\displaystyle\sum_{k=1}^{m}h_{kk}^{\alpha}=0\\[2ex] \dfrac{\partial\mathcal{P}}{\partial h_{ij}^{\alpha}}=4(m+2)h_{ij}^{\alpha}=0\end{cases} \tag{3.6}$$

with $i,j\in\{1,\cdots,m\}$, $i\neq j$ and $\alpha\in\{m+2,\cdots,2m+1\}$. Thus, every solution h^c has $h_{ij}^{\alpha}=0$ for $i,j\in\{1,\cdots,m\}$. Moreover, it is easy to see that the Hessian matrix of $\mathcal{P}$ has the form

$$\mathcal{H}(\mathcal{P})=\begin{pmatrix}H_1 & \mathbf{0}\\ \mathbf{0} & H_2\end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} 2m+2 & -2 & \dots & -2 \\ -2 & 2m+2 & \dots & -2 \\ \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & \dots & 2m+2 \end{pmatrix},$$

$\mathbf{0}$ denotes the null matrix of corresponding dimensions and H_2 is the following diagonal matrix

$$H_2 = \operatorname{diag}\left(4(m+2), 4(m+2), \dots, 4(m+2)\right).$$

Therefore, we find that $\mathcal{H}(\mathcal{P})$ has the following eigenvalues:

$$\lambda_{11} = 4,\ \lambda_{22} = \lambda_{33} = \dots = \lambda_{mm} = 2(m+2),$$

$$\lambda_{ij} = 4(m+2),\ \forall i,j \in \{1,\dots,m\},\ i \neq j.$$

Thus it follows that the Hessian is positive definite and we deduce that $\mathcal{P}$ reaches a minimum $\mathcal{P}(h^c) = 0$ for the solution h^c of the system (3.6). It follows $\mathcal{P} \geq 0$, and hence,

$$\begin{aligned} 2\tau \leq & \frac{1}{2}m(m+1)\mathcal{C} + \frac{1}{2}m(m+2)\mathcal{C}(L) \\ & + \frac{m(m+1)(c-3)}{4} + \frac{c+1}{4}\left(3|P|^2 - 2m\right). \end{aligned}$$

Therefore, we deduce that

$$\begin{aligned} \rho \leq & \frac{1}{2}\mathcal{C} + \frac{m+2}{2(m+1)}\mathcal{C}(L) + \frac{c-3}{4} \\ & + \frac{3(c+1)}{4m(m+1)}|P|^2 - \frac{c+1}{2(m+1)}, \end{aligned}$$

for every tangent hyperplane L of M. Taking the infimum over all tangent hyperplanes L, the conclusion trivially follows.
(ii) can be proved in a similar way, considering the following polynomial in the components of the second fundamental form:

$$\begin{aligned} \mathcal{Q} = & 2m(m+1)\mathcal{C} - \frac{1}{2}m(2m+1)\mathcal{C}(L) - 2\tau \\ & + \frac{m(m+1)(c-3)}{4} + \frac{c+1}{4}\left(3|P|^2 - 2m\right), \end{aligned} \tag{3.7}$$

where L is a hyperplane of T_pM.

In this case, we obtain

$$\begin{aligned}\mathcal{Q} &= \sum_{\alpha=m+2}^{2m+1}\sum_{i=1}^{m}\left[\frac{2m-1}{2}\left(h_{ii}^{\alpha}\right)^2\right] \\ &+ \sum_{\alpha=m+2}^{2m+1}\left[(2m+1)\sum_{1=i<j}^{m}\left(h_{ij}^{\alpha}\right)^2 - 2\sum_{1=i<j}^{m} h_{ii}^{\alpha}h_{jj}^{\alpha}\right]. \end{aligned} \tag{3.8}$$

From (3.8) it follows that the critical points

$$h^c = \left(h_{11}^{m+2}, h_{12}^{m+2}, \cdots, h_{mm}^{m+2}, \cdots, h_{11}^{2m+1}, \cdots, h_{mm}^{2m+1}\right)$$

of $\mathcal{Q}$ are the solutions of the following system of linear homogeneous equations:

$$\begin{cases} \dfrac{\partial \mathcal{Q}}{\partial h_{ii}^{\alpha}} = (2m+1)h_{ii}^{\alpha} - 2\displaystyle\sum_{k=1}^{m} h_{kk}^{\alpha} = 0 \\ \dfrac{\partial \mathcal{Q}}{\partial h_{ij}^{\alpha}} = 2(2m+1)h_{ij}^{\alpha} = 0 \end{cases} \tag{3.9}$$

with $i, j \in \{1, \cdots, m\}$, $i \neq j$ and $\alpha \in \{m+2, \cdots, 2m+1\}$. Thus, every solution h^c has $h_{ij}^{\alpha} = 0$ for $i, j \in \{1, \cdots, m\}$. Moreover, we can obtain easily that the Hessian matrix of $\mathcal{Q}$ has the following eigenvalues:

$$\lambda_{11} = 1,\ \lambda_{22} = \lambda_{33} = ... = \lambda_{mm} = 2m+1,$$

$$\lambda_{ij} = 4(m+2),\ \forall i, j \in \{1, ..., m\},\ i \neq j.$$

Therefore we derive that $\mathcal{H}(\mathcal{Q})$ is positive definite and we deduce that $\mathcal{Q}$ reaches a minimum $\mathcal{Q}(h^c) = 0$ for the solution h^c of the system (3.9). We conclude that $\mathcal{Q} \geq 0$, and hence,

$$\begin{aligned}\rho \leq& 2\mathcal{C} - \frac{2m+1}{2(m+1)}\mathcal{C}(L) + \frac{c-3}{4} \\ &+ \frac{3(c+1)}{4m(m+1)}|P|^2 - \frac{c+1}{2(m+1)},\end{aligned}$$

for every tangent hyperplane L of M. Finally, taking the supremum over all tangent hyperplanes L, we derive the conclusion.

□

Example 1. If $\overline{N}$ is a Kähler manifold equipped with the Kähler structure (J, G), then it is known that the warped product $\mathbb{R} \times_f \overline{N}$ admits a natural Kenmotsu structure (ϕ, ξ, η), where the differentiable map $f : \mathbb{R} \to \mathbb{R}$ is given by $f(t) = \exp t$ (see, e.g., [28]). We recall that if we denote by $\overline{g} = dt^2 + f^2 G$ the warped product metric, then the structure tensor fields ξ and η are defined by

$$\xi = \frac{d}{dt}, \ \eta(\cdot) = \overline{g}(\cdot, \xi).$$

On the other hand, the structure tensor field ϕ can be defined by putting

$$\phi_{(t,x)} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\phi}_{(t,x)} \end{pmatrix},$$

for any $(t,x) \in \mathbb{R} \times \overline{N}$, where

$$\widetilde{\phi}_{(t,x)} = (\exp(t\xi))_* J_x (\exp(-t\xi))_*.$$

In particular, if we consider that $\overline{N} = \mathbb{C}^n$ endowed with the usual Kähler structure (J, G), then it follows that $\overline{M} = \mathbb{R} \times_f \mathbb{C}^n$ is a Kenmotsu space form with constant ϕ-sectional curvature equal to -1. If N is a holomorphic submanifold of $\overline{N}$ (*i.e.* $J(T_pN) \subset T_pN$ for every point $p \in N$), then $M = \mathbb{R} \times_f N$ is an invariant submanifold of $\overline{M} = \mathbb{R} \times_f \mathbb{C}^n$ (*i.e.* $\phi(T_pM) \subset T_pM$ for every point $p \in M$), such that the structure vector field ξ is tangent to M.

But, on the other hand, it is known from [7] that an invariant submanifold of a Kenmotsu space form (tangent to the structure vector field ξ) is totally geodesic if and only if M is of the same constant ϕ-sectional curvature. Hence, we have that $M = \mathbb{R} \times_f \mathbb{C}$ is a 3-dimensional totally geodesic submanifold of the 5-dimensional Kenmotsu space form $\overline{M} = \mathbb{R} \times_f \mathbb{C}^2$, attaining equality in the inequalities (3.1) and (3.2).

Example 2. We consider the Kähler manifold $\mathbb{C}^2 = \mathbb{R}^4$ endowed with the standard Kähler structure (J, G) and let $\overline{M} = \mathbb{R} \times_f \mathbb{C}^2$ be the warped product between the real line $\mathbb{R}$ and $\mathbb{C}^2$, equipped with the natural Kenmotsu structure. We consider now the rotational surface of Vrănceanu [35] defined by the immersion $f : \mathbb{R} \times (0, 2\pi) \to \mathbb{R}^4$,

$$f(u,v) = \left(r(u)\cos u \cos v, r(u)\cos u \sin v, r(u)\sin u \cos v, r(u)\sin u \sin v\right),$$

where r is a real positive function. Then it follows that $N = \mathrm{Im} f$ is a totally real surface of $\mathbb{C}^2$ with respect to J, *i.e.* $J(T_pN) \subset T_p^{\perp}N$ for every point $p \in N$. Moreover, we note that the Vrănceanu rotation surface is not minimal, except the case when r is a solution of the following second order differential equation

$$r(u)r''(u) - 3\left(r'(u)\right)^2 - 2r(u)^2 = 0,$$

and this surface is not pseudo-umbilical, except the case when $r(u) = \alpha \exp(\beta u)$, where α and β are real constants [2]. Therefore we deduce that $M = \mathbb{R} \times_f N$ is in general a 3-dimensional non-totally geodesic submanifold of the 5-dimensional Kenmotsu space form $\overline{M} = \mathbb{R} \times_f \mathbb{C}^2$. Hence, in this case, both inequalities (3.1) and (3.2) are strict.

4 An application: the case of slant submanifolds

It is well known that the slant submanifolds of complex manifolds were introduced by B.-Y. Chen in [8] and later generalized for contact manifolds by A. Lotta in [22] (see also [4, 5] for main properties). A submanifold M of a Kenmotsu manifold $(\overline{M}, \phi, \xi, \eta, \overline{g})$ is said to be a *slant* submanifold if for each non-zero vector $X_p \in T_pM - \{\xi_p\}$, the angle $\theta(X)$ between $\phi(X)$ and T_pM is constant, i.e. it does not depend on the choice of $p \in M$ and $X \in T_pM - \{\xi_p\}$. We can easily see that invariant submanifolds are slant submanifolds with $\theta = 0$ and anti-invariant submanifolds are slant submanifolds with $\theta = \frac{\pi}{2}$. A slant submanifold of a Kenmotsu manifold is said to be *proper* (or θ-slant proper) if it is neither invariant nor anti-invariant [12].

Next we consider that M is an $(m+1)$-dimensional slant submanifold of a $(2m+1)$-dimensional Kenmotsu space form $\overline{M}(c)$ with slant angle θ such that the structure vector field ξ is tangent to M. Then, given a local orthonormal frame $\{e_1, e_2, \cdots, e_m\}$ of M, we remark that (see [32])

$$|P|^2 = m \cos^2 \theta. \tag{4.1}$$

Hence, from Theorem 1 and (4.1) we obtain the following result.

Corollary 1. *Let M be an $(m+1)$-dimensional θ-slant submanifold of a $(2m+1)$-dimensional (pointwise) Kenmotsu space form $\overline{M}(c)$, such that the structure vector field ξ is tangent to M. Then:*

(i) *The normalized δ-Casorati curvature $\delta_C(m)$ satisfies*

$$\rho \leq \delta_C(m) + \frac{c-3}{4} + \frac{3(c+1)}{4(m+1)} \cos^2 \theta - \frac{c+1}{2(m+1)}. \tag{4.2}$$

(ii) *The normalized δ-Casorati curvature $\widehat{\delta}_C(m)$ satisfies*

$$\rho \leq \widehat{\delta}_C(m) + \frac{c-3}{4} + \frac{3(c+1)}{4(m+1)} \cos^2 \theta - \frac{c+1}{2(m+1)}. \tag{4.3}$$

Moreover, the equality sign holds in the inequality (4.2) (resp. (4.3)) if and only if M is a totally geodesic submanifold of $\overline{M}(c)$.

Remark 1. For Corollary 1, we point out that there exist examples of totally geodesic and non-totally geodesic θ-slant submanifolds in Kenmotsu ambient space (see, e.g., [12, 31]). For instance, we have that $f_1 : \mathbb{R}^3 \to \mathbb{R}^5$, given by

$$f_1(t, u, v) = (t, u\cos\alpha, u\sin\alpha, v, 0),$$

defines for any constant $\alpha > 0$, a 3-dimensional totally geodesic slant submanifold $M_1 = \operatorname{Im} f_1$ of the 5-dimensional Kenmotsu space form $\overline{M} = \mathbb{R} \times_f \mathbb{C}^2$, with slant angle α, such that the structure vector field ξ of $\overline{M}$ is tangent to M_1. In this case, it is obvious that the submanifold M_1 attains equality in both inequalities (4.2) and (4.3).

Similarly, we obtain that $f_2 : \mathbb{R}^3 \to \mathbb{R}^5$, given by

$$f_2(t, u, v) = (t, u, k\cos v, v, k\sin v),$$

defines for any constant $k > 0$, a 3-dimensional non-totally geodesic slant submanifold $M_2 = \operatorname{Im} f_2$ of the 5-dimensional Kenmotsu space form $\overline{M} = \mathbb{R} \times_f \mathbb{C}^2$, with slant angle $\theta = \arccos \frac{k}{\sqrt{1+k^2}}$, such that the structure vector field ξ of $\overline{M}$ is tangent to M_2. Hence, we derive that for the submanifold M_2, the inequalities (4.2) and (4.3) are strict.

Remark 2. For Theorem 1 and Corollary 1 it is important to note that, although both normalized δ-Casorati curvatures $\delta_C(m)$ and $\widehat{\delta}_C(m)$ satisfy the same inequality, they are in general different, except the case when the submanifold is totally geodesic. For example, if we consider the 3-dimensional submanifold M_2 of the 5-dimensional Kenmotsu space form $\overline{M} = \mathbb{R} \times_f \mathbb{C}^2$ from the Remark 1, then by a straightforward computation we obtain that

$$\delta_c(2) = \frac{k^4}{6(1+k^2)^2} \exp(-2t)\sin^2 v$$

and

$$\hat{\delta}_c(2) = \frac{k^4}{4(1+k^2)^2} \exp(-2t)\sin^2 v.$$

Therefore we conclude that the normalized δ-Casorati curvatures are indeed different in this case.

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