ASYMPTOTIC PROPERTIES IN THE SHIFTED FAMILY OF A NUMERICAL SEMIGROUP WITH FEW GENERATORS

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Abstract. Let $a < b$ be positive integers and for any integer k consider the semigroup $H_k = \langle k, a+k, b+k \rangle$. If K is any field, we study the defining relations of the semigroup ring $K[H_k]$ and its tangent cone $\operatorname{gr}_{\mathfrak{m}} K[H_k]$, for $k \gg 0$. Recent results in [7], [11], [6] show that their Betti numbers are eventually periodic in k. We give a better threshold $k_{a,b}$ than the one already known for which this happens and we describe how the defining equations are periodically changing. We explicitly find all the shifts $k > k_{a,b}$ that produce complete intersections, completing a result in [7]. We write the minimal free resolution of $\operatorname{gr}_{\mathfrak{m}} K[H_k]$ and we show that its regularity is a quasilinear function for $k > k_{a,b}$.

Introduction

Let $n_1 < n_2 < \cdots < n_r$ be nonnegative integers and $H = \langle n_1, n_2, \ldots, n_r \rangle$ the monoid they generate. Let K be any field. The semigroup ring $K[H]$ is the K-algebra generated by the monomials t^{n_i} in the polynomial ring $K[t]$, for $i = 1, \ldots, r$. Consider the K-algebra map $\phi : K[x_1, \ldots, x_r] \to K[t]$ given by $\phi(x_i) = t^{n_i}$. Its image is $K[H]$. When $n_1, \ldots, n_r$ generate H minimally, we let $I_H = \ker \phi$ and we call it the defining ideal of $K[H]$.

Note that H has a unique minimal generating set. We shall refer to such monoids as numerical semigroups. Note that we do not require that the gcd of the elements in H be 1.

The tangent cone of $K[H]$ is the associated graded ring of $K[H]$ with respect to the filtration induced by the powers of the maximal ideal $\mathfrak{m} = (t^h | h \in H \setminus \{0\})K[H]$,

$$\operatorname{gr}_{\mathfrak{m}} K[H] = K[H]/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \cdots .$$

It is known that $\operatorname{gr}_{\mathfrak{m}} K[H] \cong K[x_1, \ldots, x_r]/I_H^*$, where for any ideal I in $K[x_1, \ldots, x_r]$ we let I^* be the ideal of the initial forms of the polynomials in I. Namely, $I^* = (f^* | f \in I, f \neq 0)$, where f^* denotes the component of least degree of f. The two algebras $K[H]$ and $\operatorname{gr}_{\mathfrak{m}} K[H]$ have attracted much attention and it has been of interest to relate algebraic properties like Cohen-Macaulay, Gorenstein or complete-intersection for either of them to arithmetic properties of the semigroup H, see [1], [9] and references therein.

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A recent surge of interest came with the idea of studying these properties for $H_k = \langle n_1 + k, n_2 + k, \dots, n_r + k\rangle$ as we let the integer $k \geq 0$ vary. We call $\{H_k\}_{k\geq 0}$ the shifted family of $(n_1, \dots, n_r)$. Note that H_k as defined above depends on the (not necessarily minimal) system of generators $n_1, \dots, n_r$ chosen for H. We do not want to complicate notation, and we hope it will be clear from the context what is the initial sequence of numbers that we shift.

Herzog and Srinivasan conjectured and Vu ([11]) has recently proved that there exists a shift k_V such that for $k > k_V$ the Betti numbers of I_{H_k} as a $K[x_1, \dots, x_r]$-module are periodic in k with period $n_r - n_1$. Soon after, Herzog and the author have shown in [6] that for $k \geq k_V$ the tangent cone $\mathrm{gr}_{\mathfrak{m}} K[H_k]$ is Cohen-Macaulay and moreover

$$\beta_i(I_k) = \beta_i(I_k^*) \text{ for all } i,$$

hence the eventual periodicity in k of the Betti numbers of $\mathrm{gr}_{\mathfrak{m}} K[H_k]$.

This paper started as an attempt to understand asymptotically the defining ideals of the algebras $K[H_k]$ and $\mathrm{gr}_{\mathfrak{m}} K[H_k]$ when we shift sequences with few terms. This would explain algebraically the periodic behaviour that was mentioned before.

When $r = 1$ and $H = \langle n_1\rangle$, $K[H] \cong \mathrm{gr}_{\mathfrak{m}} K[H] \cong K[x]$. When $r = 2$ and $H = \langle n_1, n_2\rangle$, one has that $K[H] \cong K[x, y]/(x^{n_2/d} - y^{n_1/d})$, where $d = \gcd(n_1, n_2)$. Hence $\mathrm{gr}_{\mathfrak{m}} K[H] \cong K[x, y]/(y^{n_1/d})$, a hypersurface ring.

For the first non trivial case, $r = 3$, it is known from Herzog's paper [4] how to canonically construct three polynomials f_1, f_2, f_3 that generate (not necessarily minimally) the defining ideal I_H, see Eq. (2) below.

Without loss of generality and in order to simplify notation we shift $0 < a < b$ and for any nonnegative integer k we let $H_k = \langle k, a + k, b + k\rangle$. Note that for $k > b$ we have $b + k < 2k$, hence the semigroup H_k is minimally generated by k, $k + a$ and $k + b$.

Let us introduce

$$k_{a,b} = \max\left\{ b\left(\frac{b-a}{D} - 1\right), b\frac{a}{D}\right\},$$

where we let $D = \gcd(a, b)$. Our original results are of two kinds: we highlight $k_{a,b}$ as a numerical improvement of the bound k_V known already to ensure a "tame" status and we also describe new qualitative properties which take place for $k > k_{a,b}$.

Next we outline the structure of the paper.

As a first result, Theorem 1.2 gives the precise expression of the middle polynomial f_2 above for I_{H_k} and $k > k_{a,b}$. It is homogeneous and it is the same for all $k > k_{a,b}$. As a first consequence, using previous characterizations by Herzog ([5]) and by Robbiano and Valla ([10]) for Cohen-Macaulay tangent cones of numerical semigroup rings, we conclude in Theorems 1.3 and 2.5 that for all shifts $k > k_{a,b}$, $\mathrm{gr}_{\mathfrak{m}} K[H_k]$ is Cohen-Macaulay and its Betti numbers are periodic in k with a period dividing b.

In Section 2 we find the shifts $k > k_{a,b}$ such that $\mathrm{gr}_{\mathfrak{m}} K[H_k]$ is a complete intersection ring, i.e., $I^*_{H_k}$ is generated by $r - 1 = 2$ equations. We also have that for such k the rings $K[H_k]$ and $\mathrm{gr}\, K[H_k]$ are complete intersections at the same time. This is not usually the case for arbitrary numerical semigroups.

In Theorem 2.1 we determine the principal period for the periodicity described above, correcting a statement in [7]. In that paper the authors focused on the periodic occurence of complete intersections in the shifted family $K[H_k]$ for $k \gg 0$ and any r. They claim that if $r > 3$ and there are infinitely many shifts k such that $K[H_k]$ is a complete intersection, then asymptotically this happens only if $n_1 + k$ is a multiple of $n_r - n_1$. Our Example 2.10 contradicts this assertion.

In Section 3 we prove that the canonical defining equations of $K[H_k]$ change in a controllable fashion for $k > k_{a,b}$. We have already seen that f_2 stays the same. In Proposition 3.2 we prove that when the shift k increases by b, in f_1 and f_3 only the exponents of x and z change, increasing by the same amount $e = \gcd(a, b)/\gcd(a, b, k)$.

In Section 4 we use Theorem 1.3 and the Hilbert-Burch Theorem to write down the minimal resolution of the tangent cone as an $K[x, y, z]$-module in terms of the canonical generators f_1, f_2, f_3. We conclude that for $k > k_{a,b}$ the regularity of $\operatorname{gr}_{\mathfrak{m}} K[H_k]$ is a quasilinear function on k, see Theorem 4.1.

In Section 5 we show that $k_{a,b}$, our threshold for well behaviour, is better (i.e. smaller) than other bounds from [11] and [7] proved or conjectured to work. In Table 1 we list several examples of shifted families for which we present for comparison the actual thresholds for the periodicity of the Betti numbers of $K[H_k]$ and $\operatorname{gr}_{\mathfrak{m}} K[H_k]$ respectively, as computed by SINGULAR ([2]), and the theoretical thresholds just mentioned already. We also conjecture explicit formulas for these actual thresholds, which seem to be very close to our estimate $k_{a,b}$.

1. The middle equation and Cohen-Macaulayness

Assume the numerical semigroup H is minimally generated by $n_1 < n_2 < n_3$. We first recall the construction of a generating set for the ideal I_H, according to [4]. For each n_i, one finds the least positive multiple $c_i n_i$ that lies in the semigroup generated by the other two generators. If, say

$$c_i n_i = \sum_{j \neq i} r_{ij} n_j, \quad i = 1, 2, 3, \tag{1}$$

for some nonnegative integers r_{ij} $(1 \leq i \neq j \leq 3)$, then the ideal I_H is generated by the polynomials

$$f_1 = x^{c_1} - y^{r_{12}} z^{r_{13}}, f_2 = y^{c_2} - x^{r_{21}} z^{r_{23}}, f_3 = z^{c_3} - x^{r_{31}} y^{r_{32}}. \tag{2}$$

We will refer to these polynomials as the *canonical generators of* I_H.

Note that some r_{ij} may be zero. If this is the case, then two of the polynomials f_1, f_2 and f_3 in (2) are the same up to a change of sign, and the other coefficients r_{st} are not necessarily uniquely determined. This coresponds to the situation when the semigroup ring $K[H]$ is a complete intersection. However, if all r_{ij} in (1) are positive, they are unique, $K[H]$ is a Cohen-Macaulay domain which is not a complete intersection, and

$$c_i = \sum_{j \neq i} r_{ji}, \text{ for all } i = 1, 2, 3. \tag{3}$$

We will prove in Theorem 1.2 that f_2 in (2) is the same for all large enough shifts k.

Definition 1.1. *For the integers $0 < a < b$ letting $D = \gcd(a, b)$ we set*

$$k_{a,b} = \max\left\{b\left(\frac{b-a}{D} - 1\right), b\frac{a}{D}\right\}. \tag{4}$$

It is easy to see that $k_{na,nb} = nk_{a,b}$ for any nonnegative integer n.

The following result is instrumental for the rest of the paper.

Theorem 1.2. *Consider the integers $0 < a < b$ and let $D = \gcd(a, b)$. For any nonnegative integer k, let $H_k = \langle k, a + k, b + k\rangle$. Then for $k > k_{a,b}$ the semigroup H_k is minimally generated by $k, a + k, b + k$, and the smallest positive multiple of $a + k$ in $\langle k, b + k\rangle$ is*

$$\frac{b}{D} \cdot (a + k) = \frac{b-a}{D} \cdot k + \frac{a}{D} \cdot (b + k). \tag{5}$$

Consequently, up to multiplication by a nonzero constant, the polynomial

$$f_2 = y^{\frac{b}{D}} - x^{\frac{b-a}{D}} z^{\frac{a}{D}} \tag{6}$$

is part of any minimal binomial generating set for the ideal I_{H_k}.

Proof. For k in the above range we have that $k > b$, equivalently $k + b < 2k$. Therefore H_k is minimally generated by $k < a + k < b + k$.

It is straightforward to check that (5) holds. We pick the smallest integer $c > 0$ such that there exist integers $u, v \geq 0$ with

$$c \cdot (a + k) = u \cdot k + v \cdot (b + k). \tag{7}$$

Clearly $c \leq b/D$. We claim that $u > 0$ and $v > 0$.

If $u = 0$, then $c(a + k) = v(b + k)$. Let $g = \gcd(a + k, b + k) = \gcd(a + k, b - a)$, hence $g \leq b - a$. Then $\frac{b+k}{g} | c$ and since $c \leq b/D$ we get that

$$\begin{aligned} b + k &\leq \frac{gb}{D} \leq \frac{(b-a)b}{D}, \\ k &\leq \frac{(b-a)b}{D} - b, \end{aligned}$$

which is a contradiction to our choice of k.

If $v = 0$, then $c(a + k) = uk$. Let $h = \gcd(a + k, k) = \gcd(a, k) \leq a$. Then $\frac{k}{h} | c$ and since $c \leq b/D$ we have that

$$k \leq bh/D \leq ab/D,$$

which is another contradiction.

Therefore $u, v > 0$. We subtract (7) from (5) and after clearing the denominators we have

$$\begin{gathered} (b - cD)(a + k) = (b - a - uD)k + (a - vD)(b + k), \\ ((b - cD) - (b - a - uD) - (a - vD))k = -(b - cD)a + (a - vD)b. \end{gathered}$$

After reducing the similar terms, the latter equation becomes

$$(u+v-c)k = ca - vb. \tag{8}$$

We claim that $c = u+v$.

Indeed, if $c > u+v$, then $c \geq v+2$ and $vb - ca > 0$ is divisible by k. Hence $k \leq vb - ca \leq (c-2)b - ca = c(b-a) - 2b \leq \frac{b}{D}(b-a) - 2b$, contradiction to our choice of k.

Similarly, if $c < u+v$, then $ca - vb > 0$ is divisible by k. Thus $k \leq ca - vb \leq \frac{b}{D}a - b$, another contradiction.

Therefore $c = u+v$. Using (8) we obtain $vb = ca = (u+v)a$, hence

$$\frac{u}{v} = \frac{b-a}{a}.$$

Using it with $c = u+v$ again, the latter gives $u = \frac{b-a}{b}c$ and $v = \frac{a}{b}c$. Since $D = \gcd(a,b)$, we have that b/D divides c and $c \leq b/D$, hence they are equal. This shows that c, u, v in a minimal relation (7) are those given in (5).

By the discussion at the beginning of this section we obtain that the polynomial f_2 in (6) is part of a minimal generating set for I_{H_k}. Up to a nonzero constant, in any minimal binomial generating set of I_{H_k} there exists a polynomial of the form $y^c - x^{r_{21}}z^{r_{23}}$. By the uniqueness of the solutions to equation (7) we also conclude that up to multiplication by a constant, f_2 is part of any minimal generating set for I_{H_k}.

The proof is now complete. $\square$

As a nice consequence of Theorem 1.2 we show that for all large enough shifts k, the associated graded ring $\mathrm{gr}_{\mathfrak{m}} K[H_k]$ is Cohen-Macaulay. We will use work of J. Herzog ([4]) and of L. Robbiano and G. Valla ([10]) who independently described the Cohen-Macaulay property of $\mathrm{gr}_{\mathfrak{m}} K[H]$ for a 3-generated semigroup H in terms of the c_i's and the r_{ij}'s introduced in (1).

J. Herzog and the author have recently proved that regardless of the number of generators of a numerical semigroup H, for all shifts $k \gg 0$ the tangent cone $\mathrm{gr}_{\mathfrak{m}} K[H_k]$ is Cohen-Macaulay, see [6, Theorem 1.4]. We recover this result, but with a better lower bound for k producing Cohen-Macaulay tangent cones.

Theorem 1.3. *(Herzog-Stamate* [6]*) Let $0 < a < b$ and denote $H_k = \langle k, a+k, b+k\rangle$. If $k > k_{a,b}$, the tangent cone $\mathrm{gr}_{\mathfrak{m}} K[H_k]$ is Cohen-Macaulay.*

Moreover, if $k > k_{a,b}$, then $\mathrm{gr}_{\mathfrak{m}} K[H_k]$ is a complete intersection if and only if $K[H_k]$ is a complete intersection.

Proof. As noted in Theorem 1.2, H_k is minimally generated by $k, a+k, b+k$. By the classification theorem of Herzog ([5, §3]) and Robbiano and Valla ([10, Table 3.4]), there are two cases to consider.

Case 1: If $K[H_k]$ is not a complete intersection, then $\mathrm{gr}_{\mathfrak{m}} K[H_k]$ is Cohen-Macaulay if and only if

$$c_2 \leq r_{21} + r_{23}. \tag{9}$$

By Theorem 1.2, if we let $D = \gcd(a, b)$, then $c_2 = b/D$, $r_{21} = (b-a)/D$, $r_{23} = a/D$, and the condition (9) is met. If $\mathrm{gr}_{\mathfrak{m}} K[H_k]$ were a complete intersection, then $K[H_k]$ would be a complete intersection, as well, which is not our case.

Case 2: If $K[H_k]$ is a complete intersection, then some $r_{ij} = 0$. Again, using Theorem 1.2 we have $r_{21}, r_{23} > 0$. In the terminology of [10, page 292], $K[H_k]$ is a complete intersection of type (b_{13}). For such a situation, the classification theorem says that $\mathrm{gr}_{\mathfrak{m}} K[H_k]$ is Cohen-Macaulay (and actually a complete intersection) if and only if the same inequality (9) holds. This is true in our setup by the discussion of the previous case. □

In [6] the *width* $\mathrm{wd}(H)$ of a numerical semigroup H is defined as the difference between the largest and the smallest elements of the minimal generating set of H. It is a conjecture of Herzog and the author that $\mu(I_H^*) \leq \binom{\mathrm{wd}(H)+1}{2}$, cf. [6, Conjecture 2.1]. We confirm this holds for large shifts in the shifted family of any 3-generated numerical semigroup.

As a by-product of the classification theorems in [10] and [5] we have that when the tangent cone is Cohen-Macaulay, the minimal generators of I_H are also a standard basis, i.e. their initial forms generate I_H^*. This gives us the following corollaries.

Corollary 1.4. *With notation as in Theorem 1.3, for all $k > k_{a,b}$ one has $\mu(I_{H_k}^*) \leq 3$.*

Corollary 1.5. *If $0 < a < b$, $H_k = \langle k, k+a, k+b \rangle$, and $D = \gcd(a, b)$, then for $k > k_{a,b}$ the polynomial $f = y^{\frac{b}{D}} - x^{\frac{b-a}{D}} z^{\frac{a}{D}}$ is part of any minimal homogenous generating set of $I_{H_k}^*$, up to multiplication by a nonzero constant.*

Proof. We showed that for such k, the ring $\mathrm{gr}_{\mathfrak{m}} K[H_k]$ is Cohen-Macaulay. As a consequence of Herzog's work in [5, §3] we have that $I_{H_k}^*$ is generated by the initial forms of the generators of I_{H_k}. As f is a homogeneous polynomial, using Theorem 1.2 the conclusion follows. □

Theorem 1.2 may be restated in an equivalent form which does not involve the shifting. What it says is that if a certain numerical condition holds between the minimal generators of a 3-generated numerical semigroup H, then we may determine one of the defining equations of the semigroup ring $K[H]$ and conclude that its tangent cone is Cohen-Macaulay.

Corollary 1.6. *Consider the positive integers $n_1 < n_2 < n_3$ and let $D = \gcd(n_2 - n_1, n_3 - n_1)$ and $H = \langle n_1, n_2, n_3 \rangle$. If*

$$n_1 > (n_3 - n_1) \cdot \max \left\{ \frac{n_3 - n_2}{D} - 1, \frac{n_2 - n_1}{D} \right\}, \tag{10}$$

then the smallest positive multiple of n_2 that lies in $\langle n_1, n_3 \rangle$ is

$$\frac{n_3 - n_1}{D} \cdot n_2 = \frac{n_3 - n_2}{D} \cdot n_1 + \frac{n_2 - n_1}{D} \cdot n_3,$$

and the polynomial

$$f_2 = y^{\frac{n_3 - n_1}{D}} - x^{\frac{n_3 - n_2}{D}} z^{\frac{n_2 - n_1}{D}}$$

is part of any minimal generating set of the ideals I_H and I_H^*. Moreover, the tangent cone $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Cohen-Macaulay.

Proof. The equation and the polynomial we are looking for are the same for the triplets n_1, n_2, n_3 and $n_1/\Delta, n_2/\Delta, n_3/\Delta$, where we let $\Delta = \gcd(n_1, n_2, n_3)$.

Condition (10) assures that we may apply Theorem 1.2 to the sequence $(0, \frac{n_2-n_1}{\Delta}, \frac{n_3-n_1}{\Delta})$ shifted up by n_1/Δ and the conclusion follows. □

2. Complete intersections and periodicity

In this section we fix the integers $0 < a < b$ and we let $H_k = \langle k, a+k, b+k\rangle$. We are interested to determine precisely for which (large enough) shifts k, if any, are the rings $K[H_k]$ and $\operatorname{gr}_{\mathfrak{m}} K[H_k]$ complete intersections (CI for short). By Theorem 1.3 it is enough to find out for which shifts k is the algebra $K[H_k]$ a complete intersection. This question was independently considered by A. V. Jayanthan and H. Srinivasan in [7]. The answer they provide in [7, Theorem 1.4] is partly incorrect, as our Example 2.10 will show.

We first recall how to verify the CI-property for our type of rings. In [4] Herzog proved that if the semigroup H is minimally generated by a, b, c with $\gcd(a, b, c) = 1$, then $K[H]$ is a complete intersection if and only if after eventually rearranging the generators we have that $\gcd(a, b) = d > 1$ and $c \in \langle \frac{a}{d}, \frac{b}{d}\rangle$. The latter condition means that there exist $u, v \in \mathbb{N}$ such that

$$c = u \cdot \frac{a}{d} + v \cdot \frac{b}{d}, \text{ or}$$

$$d \cdot c = u \cdot a + v \cdot b.$$

It is easy to see that this is in turn equivalent to

$$\langle c\rangle \cap \langle a, b\rangle = \gcd(a, b)\langle c\rangle. \tag{11}$$

Indeed, if $K[H]$ is a complete intersection and if there exist $\lambda, u', v' \in \mathbb{N}$ such that

$$\lambda \cdot c = u' \cdot a + v' \cdot b,$$

then d divides $\lambda \cdot c$. Since $\gcd(a, b, c) = \gcd(d, c) = 1$, we obtain that $d|\lambda$.

In the following, for a positive integer n and a prime p we denote

$$\nu_p(n) = \max\{i : p^i | n\}.$$

Theorem 2.1. *Consider the integers $0 < a < b$. Let*

$$T = \prod_{p \text{ prime}, \, \nu_p(a) < \nu_p(b)} p^{\nu_p(b)}.$$

If we denote $H_k = \langle k, a+k, b+k\rangle$, then for $k > k_{a,b}$ we have that $K[H_k]$ and $\operatorname{gr}_{\mathfrak{m}} K[H_k]$ are complete intersections if and only if k is a multiple of T.

Consequently, in the families of algebras $\{K[H_k]\}_{k > k_{a,b}}$ and $\{\operatorname{gr}_{\mathfrak{m}} K[H_k]\}_{k > k_{a,b}}$ the complete intersection property occurs periodically with principal period T.

Proof. By Theorem 1.3 it is enough to see when $K[H_k]$ is a complete intersection. Let $k > k_{a,b}$. We denote

$$\Delta = \gcd(k, a+k, b+k) = \gcd(k, a, b).$$

The semigroups H_k and $L_k = \langle \frac{k}{\Delta}, \frac{a+k}{\Delta}, \frac{b+k}{\Delta} \rangle$ are isomorphic, hence $K[H_k]$ is a complete intersection if and only if $K[L_k]$ is so.

It is easy to check that the inequality $k > k_{a,b}$ is equivalent to say that condition (10) is verified for the triplet $\frac{k}{\Delta}, \frac{a+k}{\Delta}, \frac{b+k}{\Delta}$. Hence by Corollary 1.6 we have that the smallest positive multiple of $\frac{a+k}{\Delta}$ in $\left\langle \frac{k}{\Delta}, \frac{b+k}{\Delta} \right\rangle$ may be obtained (only) as

$$\frac{\frac{b}{\Delta}}{\delta} \cdot \frac{a+k}{\Delta} = \frac{b-a}{\delta\Delta} \cdot \frac{k}{\Delta} + \frac{a}{\delta\Delta} \cdot \frac{b+k}{\Delta},$$

where we let $\delta = \gcd\left(\frac{b}{\Delta}, \frac{a}{\Delta}\right) = \frac{\gcd(a,b)}{\Delta}$.

If $K[L_k]$ is a complete intersection, by the discussion in Section 1 we get that some of the corresponding r_{ij}'s from the equations (1) is zero. Using (3) we derive that $r_{12} = 0$ or $r_{13} = 0$. If $r_{13} = 0$, in (1) we should have $c_1 \cdot \frac{k}{\Delta} = c_2 \cdot \frac{a+k}{\Delta}$, which is a contradiction to the uniqueness of the decomposition from (5). Therefore, if $K[L_k]$ is a complete intersection then $r_{12} = 0$ and the only way we may arrange its generators to fulfill an equation like (11) is

$$\left\langle \frac{a+k}{\Delta} \right\rangle \cap \left\langle \frac{k}{\Delta}, \frac{b+k}{\Delta} \right\rangle = \gcd\left(\frac{k}{\Delta}, \frac{b+k}{\Delta}\right) \left\langle \frac{a+k}{\Delta} \right\rangle. \tag{12}$$

Combining (2) and (12) gives that $K[L_k]$ is a complete intersection if and only if

$$\begin{aligned} \frac{b}{\delta\Delta} &= \gcd\left(\frac{k}{\Delta}, \frac{b+k}{\Delta}\right), \\ \frac{b}{\delta} &= \gcd(k, b+k), \\ \frac{b}{\gcd(k,b)} &= \delta = \frac{\gcd(a,b)}{\gcd(k, \gcd(a,b))}, \\ \frac{b}{\gcd(k,b)} &= \frac{D}{\gcd(k,D)}, \text{ where we let } D = \gcd(a,b). \end{aligned} \tag{13}$$

This means that for any prime p one has

$$\nu_p(b) - \min\{\nu_p(k), \nu_p(b)\} = \nu_p(D) - \min\{\nu_p(k), \nu_p(D)\}. \tag{14}$$

Fix a prime p. As D divides b, there are only two cases to analyze.

If $\nu_p(b) = \nu_p(D)$, then (14) clearly holds for any k.

Otherwise, if $\nu_p(D) < \nu_p(b)$, or equivalently $\nu_p(a) < \nu_p(b)$, we claim that $\nu_p(k) > \nu_p(D)$. Indeed, if $\nu_p(k) \leq \nu_p(D)$, then (14) gives $\nu_p(b) = \nu_p(D)$, which is false. Therefore $\nu_p(k) > \nu_p(D)$. When used in (14), this yields

$$\nu_p(b) - \min\{\nu_p(k), \nu_p(b)\} = \nu_p(D) - \nu_p(D) = 0,$$

hence $\nu_p(k) \geq \nu_p(b)$.

To sum up, for $k > k_{a,b}$, the ring $K[H_k]$ is a complete intersection if and only if k is a a multiple of $T = \prod_{p \text{ prime}, \, \nu_p(a) < \nu_p(b)} p^{\nu_p(b)}$. Therefore, in the family of algebras

$\{K[H_k]\}_{k>k_{a,b}}$ the complete intersection property occurs with principal period T. $\square$

According to [8], a binomial belonging to a toric ideal I is called *indispensable* if up to a scalar multiple it belongs to any binomial generating set of I. In the context of algebraic statistics it is of interest to determine if a binomial ideal is generated by indispensable binomials. This is not always the case even for 3-generated numerical semigroups. For instance, for $H = \langle 10, 12, 15\rangle$ one has $I_H = (x^3 - z^2, y^5 - z^4) = (x^3 - z^2, y^5 - x^3z^2)$.

Corollary 2.2. *With notation as in Theorem 2.1, for $k > k_{a,b}$ the toric ideal I_{H_k} is generated by indispensable binomials.*

Proof. If I_{H_k} is not CI, it follows from Herzog's [4] that the polynomials in (2) are the unique minimal binomial generating set for I_{H_k}. If I_{H_k} is CI, with notation as in (1), it follows that x^{c_1}, y^{c_2} and z^{c_3} are in the support of any generating set of I_{H_k}. By Theorem 1.2 and the discussion around equation (12) we conclude that the polynomials $x^{c_1} - z^{c_3}$ and f_2 as in (6) belong to any generating set of I_{H_k}. $\square$

If u, v are positive integers, we say that u *strictly divides* v, if $u|v$ and for any prime p such that $p|u$, then also $p|\frac{v}{u}$. The following are immediate consequences of Theorem 2.1.

Corollary 2.3. *With notation as in Theorem 2.1, assume that* $\gcd(a, b) = 1$ *or that* $\gcd(a, b)$ *strictly divides b. Then, for $k > k_{a,b}$ we have that $K[H_k]$ and $\operatorname{gr}_{\mathfrak{m}} K[H_k]$ are complete intersections if and only if k is a multiple of b.*

As it was the case of Theorem 1.2, we may formulate a shift-free form of Theorem 2.1, as follows.

Corollary 2.4. *Consider the semigroup H minimally generated by the positive integers $n_1 < n_2 < n_3$ satisfying inequality* (10). *If we let $A = n_3 - n_1$, $B = n_2 - n_1$ and*

$$C = \prod_{p \text{ prime},\ \nu_p(B)<\nu_p(A)} p^{\nu_p(A)},$$

then $K[H]$ and $\operatorname{gr}_{\mathfrak{m}} K[H]$ are complete intersections if and only if n_1 is divisible by C.

Proof. We view the generators of H as obtained from shifting up by n_1 the sequence $(0, n_2 - n_1, n_3 - n_1)$. We may then use Theorem 2.1 along the lines of the proof of Corollary 1.6. $\square$

The following consequence of Theorem 2.1 mainly recovers [7, Theorem 1.5] and a special case of [6, Theorem 1.4]. Our careful treatment of the CI-property allows to formulate a sharper statement regarding the principal period.

Theorem 2.5. *(Jayanthan and Srinivasan* [7]*, Herzog and Stamate* [6]*)*
Consider the positive integers $a < b$. Let $H_k = \langle k, a + k, b + k\rangle$ and T as in Theorem 2.1. Then, for $k > k_{a,b}$ and all i one has that $\beta_i(K[H_k]) = \beta_i(\operatorname{gr}_{\mathfrak{m}} K[H_k])$ and they are periodic in k with principal period T.

Proof. The defining ideals I_{H_k} and $I^*_{H_k}$ are Cohen-Macaulay of height 2 in $K[x,y,z]$. They are minimally generated by 2 or 3 polynomials, depending whether they are or they are not a complete intersection. Therefore the corresponding sequences of Betti numbers for $K[H_k]$ (and $\mathrm{gr}_{\mathfrak{m}} K[H_k]$) are either $(1,2,1)$ or $(1,3,2)$, and as k increases this repeats periodically with period T, by Theorem 2.1. $\square$

It is claimed in [7, Theorem 2.1] that when we shift a generating sequence $n_1 < n_2 < \cdots < n_r$ of length $r \geq 4$, if there are infinitely many complete intersections in the family $\{K[H_k]\}_{k\geq 0}$, then for $k \gg 0$ these may occur only when $(n_1 + k)$ is divisible by $(n_r - n_1)$, in particular the principal period of their occurence is $n_r - n_1$. We believe that this numerical condition is a sufficient one, but that it may not capture all the complete intersections. Example 2.8 below gives a shifted family with $r = 4$ generators and where the complete intersections occur more often than predicted by [7].

Let us recall the following theorem of C. Delorme ([3]) which states that a numerical semigroup whose semigroup ring is a complete intersection is obtained by glueing two numerical semigroups with the same property.

Theorem 2.6. *(Delorme,* [3]*)*

Let H be a semigroup minimally generated by the sequence $\mathbf{a} = a_1, a_2, \ldots, a_r$ with $\gcd(a_1, a_2, \ldots, a_r) = d$. The semigroup ring $K[H]$ is a complete intersection if and only if $r = 1$ or $\mathbf{a}$ can be written as a disjoint union

$$\mathbf{a} = k_1(b_{i_1}, \ldots, b_{i_s}) \sqcup k_2(b_{i_{s+1}}, \ldots, b_{i_r}),$$

where $k_1, k_2 > 1$, $\gcd(k_1, k_2) = d$, $k_1 \in \langle b_{i_{s+1}}, \ldots, b_{i_r}\rangle$, $k_2 \in \langle b_{i_1}, \ldots, b_{i_s}\rangle$, $k_1 \notin \{b_{i_{s+1}}, \ldots, b_{i_r}\}$, $k_2 \notin \{b_{i_1}, \ldots, b_{i_s}\}$, $\gcd(b_{i_1}, \ldots, b_{i_s}) = \gcd(b_{i_{s+1}}, \ldots, b_{i_r}) = 1$, and $K[\langle b_{i_1}, \ldots, b_{i_s}\rangle]$ and $K[\langle b_{i_{s+1}}, \ldots, b_{i_r}\rangle]$ are complete intersections.

Such a writing of $\mathbf{a}$ is called a *CI-decomposition* and $\mathbf{a}$ is a *CI-sequence.* Delorme's initial formulation treated the case $d = \gcd(a_1, a_2, \ldots, a_r) = 1$. We slightly modified it to include the case $d > 1$, necessary for us later.

Example 2.7. *Let $H_k = \langle k, k+4, k+10\rangle$. With notation as in Theorem 2.1 $D = \gcd(4, 10) = 2$, $k_{a,b} = 20$ and $T = 5$. Hence for $k > 20$ we have that $K[H_k]$ is a complete intersection if and only if $k = 5\ell$. Indeed, if ℓ is odd, then*

$$(5\ell, 5\ell+4, 5\ell+10) = (5\ell+4)(1) \sqcup 5(\ell, \ell+2)$$

is a CI-decomposition, since $5\ell + 4 = 3\cdot\ell + 2\cdot(\ell+2)$.

If ℓ is even, $\ell = 2\ell'$, then we have the CI-decomposition

$$(5\ell, 5\ell+4, 5\ell+10) = (10\ell', 10\ell'+4, 10\ell'+10) = (10\ell'+4)(1) \sqcup 10(\ell', \ell'+1).$$

We'll use Delorme's criterion and Example 2.7 to produce a shifted family where the complete intersections occur more than once for $(n_r - n_1)$ consecutive shifts, contrary to the assertions of [7, Theorem 2.1].

Example 2.8. *Start with the CI-sequence $(5\ell, 5\ell+4, 5\ell+10)$ and consider*

$$15\ell + 4 = 2\cdot(5\ell) + (5\ell+4) \in \left\langle \frac{15\ell}{3}, \frac{15\ell+12}{3}, \frac{15\ell+30}{3} \right\rangle.$$

Let $H_k = \langle k, k+4, k+12, k+30\rangle$. A SINGULAR ([2]) computation shows that the Betti numbers of $K[H_k]$ repeat periodically starting with $k_0 = 95$, with principal period equal to 30. For $k \geq k_0$ the CI-intersections correspond to $k = 15\ell$. If we let $\mathbf{a} = (15\ell, 15\ell+4, 15\ell+12, 15\ell+30)$, it has the following CI-decomposition:

- *ℓ is odd: $\mathbf{a} = (15\ell+4)(1)\sqcup 3(5\ell, 5\ell+4, 5\ell+10)$, with $15\ell+4 = 2\cdot(5\ell)+(5\ell+4)$ and $(5\ell, 5\ell+4, 5\ell+10)$ a CI-sequence.*
- *$\ell = 2\ell'$ is even: $\mathbf{a} = (30\ell'+4)(1)\sqcup 6(5\ell', 5\ell'+2, 5\ell'+5)$, with $30\ell'+4 = 4\cdot(5\ell')+2\cdot(5\ell'+2)$ and the CI-sequence $(5\ell', 5\ell'+2, 5\ell'+5) = (5\ell'+2)(1)\sqcup 5(\ell', \ell'+1)$.*

We remark that in [7] the authors have independently proposed a criterion to determine the form of the (large enough) shifts k such that $K[H_k]$ is a complete intersection, a result similar to Theorem 2.1. With our notation it says:

Theorem 2.9. *(*[7, Theorem 1.4]*, Jayanthan, Srinivasan)*
Assume $b > a$ and $k > \max\{ba, b(b-a)\}$. If we let $H_k = \langle k, k+a, k+b\rangle$, then $K[H_k]$ is a complete intersection if and only if there exist $\gcd(k,b) = d > 1$ and nonnegative integers α, β such that $d\cdot(k+a) = \alpha\cdot k + \beta\cdot(k+b)$.

The following example contradicts this statement.

Example 2.10. (and counterexample to [7, Theorem 1.4] and to Theorem 2.9*)*
Let $H_k = \langle k, k+12, k+32\rangle$. With notation as in Theorem 2.1, $D = \gcd(12,32) = 4$, $k_{a,b} = 128$, $T = 32$. Hence for $k > 128$ we have that $K[H_k]$ is a complete intersection if and only if $k = 32\ell$.

With notation as in Theorem 2.9 we have $a = 12$, $b = 32$. It is easy to see that the equation $d\cdot(k+12) = \alpha\cdot k + \beta\cdot(k+32)$ has solutions with $d = \gcd(k,32) > 1$ if and only if $k = 8\ell$, with $\ell > 0$. However, a SINGULAR computation and also Delorme's Theorem 2.6 confirm our numerics above that when $k \gg 0$ only for $k = 32\ell$ we encounter complete intersections.

In Section 5 in Table 1 we present more examples of shifted families and their infinite subsequences producing complete intersections.

3. Periodic changes in the defining equations

Let $0 < a < b$ be positive integers and $H_k = \langle k, k+a, k+b\rangle$. We show that for $k \gg 0$ the canonical generators f_1, f_2, f_3 of I_{H_k} change in a predictable way when we shift up k by b. More exactly, f_2 stays the same, as seen in Theorem 1.2, and in f_1 and f_3 the exponents of x and of z increase by the same amount $e = \gcd(a,b)/\gcd(k,a,b)$.

If $f = m_1 - m_2$ with m_1 and m_2 distinct monomials in a polynomial ring, we define the *ecart* of f as

$$\mathrm{ecart}(f) = |\deg(m_1) - \deg(m_2)|.$$

Clearly the homogeneous binomials correspond to binomials with ecart zero.

Using topological methods and for numerical semigroups with arbitrarily many generators, Vu shows in [11] that for large enough shifts k the nonhomogenous minimal generators of I_{H_k} have the same ecart.

Next we prove that for 3-generated semigroups the ecarts of the binomial generators stabilize starting from a shift that, as we see later in Section 5, is sensibly smaller than the one found in [11].

Proposition 3.1. *(see also Vu,* [11, Corollary 3.7]*) Let* $0 < a < b$ *be positive integers and* $H_k = \langle k, k+a, k+b \rangle$*. Then for* $k > k_{a,b}$*,* $\operatorname{ecart}(f_2) = 0$ *and*

$$\operatorname{ecart}(f_1) = \operatorname{ecart}(f_3) = \gcd(a,b)/\gcd(k,a,b).$$

Proof. By Theorem 1.2 we know that $\operatorname{ecart}(f_2) = 0$. For the rest we distinguish two cases:
Case 1: $K[H_k]$ is a complete intersection.

In this situation $f_1 = -f_3$, by the discussion around equation (12). The smallest positive integer solutions c_1 and c_3 of the equation

$$c_1 \cdot k = c_3 \cdot (k+b)$$

are $c_1 = (b+k)/\gcd(k,b)$ and $c_3 = k/\gcd(k,b)$. This gives the minimal relation $f_1 = x^{c_1} - z^{c_3}$ with $\operatorname{ecart}(f_1) = b/\gcd(k,b) = \gcd(a,b)/\gcd(k,a,b)$, where for the latter equality we used equation (13).
Case 2: $K[H_k]$ is not a complete intersection.

Since $\operatorname{ecart}(f_2) = 0$, using (3) and the discussion around it we get that $\operatorname{ecart}(f_1) = \operatorname{ecart}(f_3)$. Next we compute $\operatorname{ecart}(f_1)$.

Let c_1 be the smallest positive integer such that there exist $u, v \geq 0$ with

$$\begin{aligned} c_1 \cdot k &= u \cdot (k+a) + v \cdot (k+b), \text{ equivalently} && (15)\\ (c_1 - u - v) \cdot k &= u \cdot a + v \cdot b. && (16) \end{aligned}$$

Since $K[H_k]$ is not a complete intersection, u, v are unique and positive. Let $\Delta = \gcd(k,a,b)$. Then c_1, u, v introduced above are the same as for the semigroup $L_k = \langle k', k'+a', k'+b' \rangle$, where $k' = k/\Delta$, $a' = a/\Delta$, $b' = b/\Delta$ and $\gcd(k', a', b') = 1$. Let $e = \gcd(a', b') = \gcd(a/\Delta, b/\Delta) = \gcd(a,b)/\gcd(k,a,b)$. Equation (16) becomes

$$(c_1 - u - v) \cdot k' = u \cdot a' + v \cdot b'. \tag{17}$$

Since e and k' are coprime we have that $e | c_1 - u - v$. Let $c_1 - u - v = E \cdot e$ with $E \geq 1$.

If $E = 1$, then $\operatorname{ecart}(f_1) = \operatorname{ecart}(x^{c_1} - y^u z^v) = c_1 - u - v = e$ and we're done.

Assume $E \geq 2$ and let $D = \gcd(a,b)$.

Our hypothesis on k implies that $k > ab/D > D(\frac{b}{D}-1)(\frac{a}{D}-1) \geq \Delta(\frac{b}{D}-1)(\frac{a}{D}-1)$, hence $k' \geq (\frac{a'}{e}-1)(\frac{b'}{e}-1)$, which is the conductor of the numerical semigroup $\langle \frac{a'}{e}, \frac{b'}{e} \rangle$, see [9]. Therefore there exist nonnegative integers α, β such that

$$e \cdot k' = \alpha \cdot a' + \beta \cdot b'. \tag{18}$$

We may pick α and β satisfying (18) such that $\alpha + \beta$ is minimal. This implies that $0 \leq \alpha < \frac{b'}{e}$. Indeed, if $\alpha \geq \frac{b'}{e}$, then we may write $e \cdot k' = (\alpha - \frac{b'}{e}) \cdot a' + (\beta + \frac{a'}{e}) \cdot b'$ with $(\alpha - \frac{b'}{e}) + (\beta + \frac{a'}{e}) < \alpha + \beta$, a contradiction.

Equation (18) implies $(e + \alpha + \beta) \cdot k' = \alpha \cdot (k' + a') + \beta \cdot (k' + b')$. Again, by the minimality of c_1 in (15) we infer that

$$c_1 = eE + u + v \leq e + \alpha + \beta.$$

If equality holds above, by the uniqueness of u and v satisfying (15) we get that $u = \alpha$ and $v = \beta$, hence $E = 1$, a contradiction. Therefore

$$eE + u + v < e + \alpha + \beta. \tag{19}$$

After we multiply (18) by E and we subtract this from (17) we obtain

$$(u - E\alpha) \cdot a' + (v - E\beta) \cdot b' = 0.$$

As $\gcd(a', b') = e$, there exists an integer t such that

$$\begin{aligned} u &= E\alpha - t \cdot \frac{b'}{e}, \\ v &= E\beta + t \cdot \frac{a'}{e}. \end{aligned}$$

We substitute these into (19) and we obtain that

$$0 < (E - 1)(e + \alpha + \beta) < t \cdot \frac{b' - a'}{e} \text{ and } t > 0. \tag{20}$$

If $u \geq b'/e$, then the equation

$$(c_1 - u - v) \cdot k' = (u - \frac{b'}{e}) \cdot a' + (v + \frac{a'}{e}) \cdot b'$$

with $(c_1 - u - v) + (u - \frac{b'}{e}) + (v + \frac{a'}{e}) < c_1$ contradicts the minimality of c_1 in (15). Hence $0 < u < b'/e$, that is

$$t \cdot \frac{b'}{e} < E\alpha < (1 + t) \cdot \frac{b'}{e},$$

equivalently $t = \lfloor E \cdot \frac{\alpha}{b'/e} \rfloor$. Using $\alpha < \frac{b'}{e}$, we get that $t \leq E - 1$. Together with (20), this gives

$$e + \alpha + \beta < \frac{b' - a'}{e}. \tag{21}$$

Using (18) and (21) we get that $ek' \leq (\alpha + \beta) \cdot b' < b' \cdot (\frac{b' - a'}{e} - e)$, equivalently

$$\begin{aligned} k < b \cdot \left(\frac{b' - a'}{e^2} - 1 \right) &\leq b \left(\frac{b - a}{e^2 \Delta} - 1 \right) = b \left(\frac{b - a}{De} - 1 \right) \\ &\leq \frac{b}{D} \left(\frac{b - a}{e} - De \right) \leq \frac{b}{D}(b - a - D), \end{aligned}$$

which contradicts our choice of k. Therefore $E = 1$ and $\operatorname{ecart}(f_1) = e$. $\square$

We can now prove the main result of this section describing the changes that occur periodically in the minimal generating system of I_{H_k} for large k.

Proposition 3.2. *Consider the integers* $0 < a < b$. *For any* k *denote* $H_k = \langle k, k + a, k + b \rangle$ *and* $I_{H_k} \subset K[x, y, z]$ *the defining ideal of* $K[H_k]$. *Denote by* $f_{1,k}, f_{2,k}, f_{3,k}$ *the canonical generators of* I_{H_k} *obtained as in* (2).

If $k > k_{a,b}$, *then* $f_{2,k} = f_{2,k+b}$ *and the polynomials* $f_{1,k+b}$ *and* $f_{3,k+b}$ *are obtained from* $f_{1,k}$ *and* $f_{3,k}$, *respectively, by increasing the exponents of* x *and of* z *by the same amount* $e = \gcd(a, b)/\gcd(k, a, b)$.

Proof. By Theorem 1.2 we have $f_{2,k} = f_{2,k+b}$. We only prove the statement about $f_{1,k}$ and $f_{1,k+b}$; the one about $f_{3,k}$ and $f_{3,k+b}$ may be checked similarly.

Note that by Theorem 2.1 for k in our range either both ideals I_{H_k} and $I_{H_{k+b}}$ are complete intersections or none of them is.

If I_{H_k} is a complete intersection, the smallest positive integer solutions c_1 and c_3 of the equation

$$c_1 \cdot k = c_3 \cdot (k+b).$$

are $c_1 = (b+k)/\gcd(k,b)$ and $c_3 = k/\gcd(k,b)$, which gives that $f_{1,k} = x^{c_1} - z^{c_3}$. Similarly, $f_{1,k+b} = x^{c_1'} - z^{c_3'}$ with $c_1' = (b+k+b)/\gcd(k+b,b) = c_1 + \frac{b}{\gcd(k,b)}$ and $c_3' = (k+b)/\gcd(k+b,b) = c_3 + \frac{b}{\gcd(k,b)}$. By (13), $b/\gcd(k,b) = \gcd(a,b)/\gcd(k,a,b)$ and this finishes the proof of this case.

If I_{H_k} is not a complete intersection, then for $f_{1,k}$ and $f_{1,k+b}$ we look at the smallest positive integer solutions c_1 and c_1' for the equations

$$\begin{aligned} c_1 \cdot k &= u \cdot (k+a) + v \cdot (k+b), \text{ and} \\ c_1' \cdot (k+b) &= u' \cdot (k+a+b) + v' \cdot (k+2b), \text{ respectively.} \end{aligned}$$

Equivalently,

$$\begin{aligned} (c_1 - u - v) \cdot k &= u \cdot a + v \cdot b, \text{ and} \\ (c_1' - u' - v') \cdot (k+b) &= u' \cdot a + v' \cdot b. \end{aligned}$$

We subtract these two equations and after using the formula for the ecart given in Proposition 3.1 we obtain

$$e \cdot b = (u' - u) \cdot a + (v' - v) \cdot b, \tag{22}$$

therefore $\frac{b}{\gcd(a,b)} | u - u'$. On the other hand, using Theorem 1.2 combined with (3) we have that $0 < u, u' < \frac{b}{\gcd(a,b)}$, hence $u = u'$. From (22) we get $v' = v + e$, and by using Proposition 3.1 we also get that $c_1' = c_1 + e$. Thus $f_{1,k} = x^{c_1} - y^u z^v$ and $f_{1,k+b} = x^{c_1+e} - y^u z^{v+e}$, which finishes the proof. □

Example 3.3. *For $H_k = \langle k, k+4, k+10 \rangle$ we have $k_{4,10} = 20$. We computed with SINGULAR several defining ideals of $K[H_k]$ in order to put in evidence the changes that occur periodically with k. We denote $e = \gcd(a,b)/\gcd(k,a,b)$.*

Let us consider first some complete intersections. We have

- *for $k = 35$: $H_k = \langle 35, 39, 45 \rangle$, $e = 2$, $I_{H_{35}} = (x^9 - z^7, y^5 - x^3z^2)$.*
- *for $k = 45$: $H_k = \langle 45, 49, 55 \rangle$, $e = 2$, $I_{H_{45}} = (x^{11} - z^9, y^5 - x^3z^2)$.*

And examples of non-complete intersections:

- *for $k = 36$: $H_k = \langle 36, 40, 46 \rangle$, $e = 1$, $I_{H_{36}} = (x^7 - y^4z^2, y^5 - x^3z^2, z^4 - x^4y)$.*
- *for $k = 46$: $H_k = \langle 46, 50, 56 \rangle$, $e = 1$, $I_{H_{46}} = (x^8 - y^4z^3, y^5 - x^3z^2, z^5 - x^5y)$.*

Note that the CIs occur with periodicity $T = 5$ (see Example 2.7) and the common ecart e of the nonhomogeneous generators changes with period 2. Therefore we can not expect that the equations of I_{H_k} change by the rule of Proposition 3.2 with a principal period less than $b = 10$. Indeed, regarding the numerical examples above, we computed

- *for $k = 40$: $H_k = \langle 40, 44, 49 \rangle$, $e = 1$, $I_{H_{40}} = (x^5 - z^4, y^5 - x^3z^2)$.*

- *for* $k=41$*:* $H_k=\langle 41,45,50\rangle$, $e=2$, $I_{H_{41}}=(x^{12}-y^3z^7, y^5-x^3z^2, z^9-x^9y^2)$.

One possible use of Proposition 3.2 is to reduce the computation of I_{H_k} for a large value of k to the computation of I_{H_ℓ} such that $\ell\equiv k \bmod b$ and $\ell>k_0$, where k_0 is a periodicity threshold that we can estimate from a and b, e.g. by the formula in (4); see also Section 5.

Remark 3.4. *Exploring the ideas in* [11]*, given* $a_1<\cdots<a_r$ *one could show that a statement similar to Proposition 3.2 holds asymptotically in the shifted family* $\{H_k:=\langle a_1+k,\dots,a_r+k\rangle\}_{k\geq 0}$ *with arbitrary* r*. More precisely, if we let* $d=\gcd(a_r-a_1,\dots,a_r-a_{r-1})$ *for* $k>k_V$ *the inhomogeneous equations of* $K[H_{k+(a_r-a_1)}]$ *are obtained from the inhomogeneous equations of* $K[H_k]$ *by adding the same amount* $e=d/\gcd(k,d)$ *to both the exponents of* x_1 *and of* x_r*.*

4. Regularity for tangent cones

We keep the usual notation for $H_k=\langle k,k+a,k+b\rangle$ where $0<a<b$ and k are integers. In this section we study the asymptotic behaviour of the Castelnuovo-Mumford regularity of $\operatorname{gr}_{\mathfrak{m}} K[H_k]$ viewed as an $S=K[x,y,z]$-module. Recall that

$$\operatorname{reg}\operatorname{gr}_{\mathfrak{m}} K[H_k]=\max\{j-i:\beta_{ij}^S(\operatorname{gr}_{\mathfrak{m}} K[H_k])>0\}.$$

For k large enough we can access directly the maps and the shifts in the minimal graded S-free resolution of $\operatorname{gr}_{\mathfrak{m}} K[H_k]\cong S/I^*_{H_k}$.

Let $k>k_{a,b}$ and $e=\frac{\gcd(a,b)}{\gcd(k,a,b)}$.

Case 1: $K[H_k]$ is not a complete intersection.

In this case the canonical generators in (2) are obtained from the equations

$$\begin{aligned}
c_1\cdot k &= r_{12}\cdot(a+k)+r_{13}\cdot(b+k),\\
c_2\cdot(a+k) &= r_{21}\cdot k+r_{23}\cdot(b+k),\\
c_3\cdot(b+k) &= r_{31}\cdot k+r_{32}\cdot(a+k)
\end{aligned}$$

By Corollary 1.4 and the discussion before it, $I^*_{H_k}$ is minimally generated by the initial forms of canonical generators of I_{H_k}. It is easy to see from the equations above that $c_1>r_{12}+r_{13}$ and $c_3<r_{31}+r_{32}$, hence

$$I^*_{H_k}=(y^{r_{12}}z^{r_{13}}, y^{c_2}-x^{r_{21}}z^{r_{23}}, z^{c_3}).$$

It is a Cohen-Macaulay ideal of codimension 2 and the ideal of maximal minors of the matrix $\begin{pmatrix} z^{r_{23}} & 0 & -y^{r_{12}}\\ -y^{r_{32}} & z^{r_{13}} & x^{r_{21}}\end{pmatrix}$. By the Hilbert-Burch Theorem (see [1, Section 1.5]) the minimal free resolution of $S/I^*_{H_k}$ is

$$0\leftarrow S \xleftarrow{(y^{r_{12}}z^{r_{13}},\, y^{c_2}-x^{r_{21}}z^{r_{23}},\, z^{c_3})} \begin{matrix} S(-r_{12}-r_{13})\\ \oplus\\ S(-c_2)\\ \oplus\\ S(-c_3)\end{matrix} \xleftarrow{\begin{pmatrix} z^{r_{23}} & -y^{r_{32}}\\ 0 & z^{r_{13}}\\ -y^{r_{12}} & x^{r_{21}}\end{pmatrix}} \begin{matrix} S(-r_{12}-c_3)\\ \oplus\\ S(-r_{21}-c_3)\end{matrix} \leftarrow 0.$$

Note that for the last map to be homogeneous it is important that $c_2 = r_{21} + r_{23}$, which is true in our case by Theorem 1.2. The regularity of a Cohen-Macaulay K-algebra is read off the shifts in the last step of the minimal resolution, hence

$$\operatorname{reg} S/I^*_{H_k} = c_3 - 2 + \max\{r_{12}, r_{21}\}.$$

Observe that for k in our range one has, by Theorem 1.2, $r_{21} = (b-a)/\gcd(a,b)$ and that, according to Proposition 3.2, r_{12} is a constant that depends on $k \bmod b$.

Consequently

$$\operatorname{reg} S/I^*_{H_{k+b}} = \operatorname{reg} S/I^*_{H_k} + e. \tag{23}$$

Case 2: $K[H_k]$ is a complete intersection.

In this situation, by the first part of the proof of Proposition 3.2 we get $I_{H_k} = (x^{c_1} - z^{c_3}, y^{c_2} - x^{r_{21}}z^{r_{23}})$, where $c_3 = k/\gcd(b,k)$. With the same argument as in the previous case one gets that the initial forms of the generators also generate $I^*_{H_k}$,

$$I^*_{H_k} = (z^{c_3}, y^{c_2} - x^{r_{21}}z^{r_{23}}).$$

The minimal graded free resolution of $S/I^*_{H_k}$ is

$$0 \leftarrow S \xleftarrow{(z^{c_3},\, y^{c_2}-x^{r_{21}}z^{r_{23}})} \begin{matrix} S(-c_3) \\ \oplus \\ S(-c_2) \end{matrix} \xleftarrow{\begin{pmatrix} y^{c_2} - x^{r_{21}}z^{r_{23}} \\ -z^{c_3} \end{pmatrix}} S(-c_2 - c_3) \leftarrow 0,$$

hence

$$\operatorname{reg} S/I^*_{H_k} = c_2 + c_3 - 2 = \frac{b}{\gcd(a,b)} + \frac{k}{\gcd(b,k)} - 2.$$

Using (13) we conclude that equation (23) holds in this case, as well.

We summarize our findings in the following statement.

Theorem 4.1. *With notation as above, for $k > k_{a,b}$*

$$\operatorname{reg} S/I^*_{H_{k+b}} = \operatorname{reg} S/I^*_{H_k} + e.$$

In particular $\operatorname{reg} \operatorname{gr}_{\mathfrak{m}} K[H_k]$ *is a quasilinear function for $k \gg 0$ and*

$$\lim_{k\to\infty} \operatorname{reg} \operatorname{gr}_{\mathfrak{m}} K[H_k] = \infty.$$

5. Estimates for well behavior

Let $0 < a < b$ and $H_k = \langle k, a+k, b+k \rangle$. Our estimate for the threshold

$$k_{a,b} = \max\left\{ b\left(\frac{b-a}{D} - 1\right), b\frac{a}{D} \right\}$$

in Theorems 1.2, 2.1, 1.3 or 2.5 is not optimal. In practice, the phenomena described by those results may start happening earlier, from smaller shifts. However, as the examples in Table 1 show, our estimate is not too far from those and it is sometimes exact.

In [11] it is presented a bound k_V such that for all shifts $k > k_V$ the Betti numbers of I_{H_k} are periodic in k. In [6, Theorem 1.4] it is shown that starting with the same shift k_V one has $\beta_i(I_{H_k}) = \beta_i(I^*_{H_k})$ for all i, hence the periodicity in k of $\beta_i(I^*_{H_k})$. We next show that our threshold $k_{a,b}$ is better (i.e. lower) than k_V.

In this setup we recall how the constant k_V was introduced in [11]. One needs the ideal J of homogeneous polynomials in I_{H_k} for some (hence for all) $k > 0$. It is an easy exercise to see that J is generated by homogeneous binomials and from here to derive that $J = (y^{\frac{b}{D}} - x^{\frac{b-a}{D}} z^{\frac{a}{D}})$ and $\operatorname{reg} J = b/D$, where $D = \gcd(a, b)$.

If we denote by $c = \left(\frac{b}{D} - 1\right)\left(\frac{b-a}{D} - 1\right)$ the conductor of the semigroup $\langle \frac{b}{D}, \frac{b-a}{D} \rangle$, and $B = b + (b-a) + 3 + D$, we let

$$k_V = \max\left\{b(3 + \operatorname{reg} J), b(b-a)\left(\frac{Dc+b}{b-a} + B\right)\right\}.$$

Let us denote $k_V^{(1)}$ and $k_V^{(2)}$, respectively, the two quantities in the formula above defining k_V.

Proposition 5.1. *With notation as above $k_V > (b-a)k_{a,b}$.*

Proof. A conservative estimation is $B \geq 7$ and

$$b(b-a)\left(\frac{Dc+b}{b-a} + B\right) = b\left(Dc + b + B(b-a)\right) \geq b(b+7) > b\left(3 + \frac{b}{D}\right) = k_V^{(1)}.$$

On one side

$$\begin{aligned} k_V^{(2)} &= b\left(\left(\frac{b}{D} - 1\right)(b-a-D) + b + B(b-a)\right) \\ &= b\left((b-a-D)\left(\frac{b}{D} - 1 + B\right) + BD + b\right) \\ &> b(b-a-D) \cdot \frac{b}{D}. \end{aligned}$$

On the other side, we may write

$$k_V^{(2)} = b\left((b-a)\left(\frac{b}{D} - 1 + B\right) - b + D + b\right) > b(b-a)\frac{a}{D}.$$

Hence $k_V = k_V^{(2)} > (b-a)k_{a,b}$. $\square$

Based on numerical experiments, it is suggested in [11, pp. 68] that the periodicity for the Betti numbers of S/I_{H_k} might occur already when $k > k_{V^{(1)}}$. Our results confirm this bound for 3-generated semigroups, since $k_{a,b} < k_{V^{(1)}}$.

It is also clear that $k_{a,b}$ improves the threshold $k_{JS} = \max\{ab, (b-a)b\}$ introduced in [7, Theorem 1.4].

For several families of semigroups $H_k = \langle k, k+a, k+b \rangle$ we used SINGULAR ([2]) to compute

$$\begin{aligned} k_{CM} &= \min\{j \geq 0 : \operatorname{gr}_{\mathfrak{m}} K[H_k] \text{ is Cohen-Macaulay for all } k > j\}, \\ k_{S_1} &= \min\{j \geq 0 : \beta_i^S(K[H_k]) \text{ is periodic in } k \text{ for all } k > j \text{ and all } i\}, \\ k_{S_2} &= \min\{j \geq 0 : \beta_i^S(\operatorname{gr}_{\mathfrak{m}} K[H_k]) \text{ is periodic in } k \text{ for all } k > j \text{ and all } i\}. \end{aligned}$$

As a general fact, $k_{CM} \leq k_{S_2} \leq k_{S_1}$.

We recorded these values in Table 1 together with the estimates k_V, k_{JS}, $k_{V^{(1)}}$ and our bound $k_{a,b}$. We also listed the asymptotic complete intersections as given by Theorem 2.1.

TABLE 1. Thresholds

$(0,a,b)$	k_{CM}	k_{S_1}	k_{S_2}	CI	$k_{a,b}$	k_V	k_{JS}	$k_{V^{(1)}}$
$(0,4,10)$	0	16	16	$(5\ell, 5\ell+4, 5\ell+10)$	20	1520	60	80
$(0,4,6)$	0	8	8	$(3\ell, 3\ell+4, 3\ell+6)$	12	192	24	36
$(0,6,15)$	0	24	24	$(5\ell, 5\ell+6, 5\ell+15)$	30	4635	135	120
$(0,12,32)$	76	108	84	$(32\ell, 32\ell+12, 32\ell+32)$	128	42368	640	352
$(0,30,57)$	372	540	540	$(19\ell, 19\ell+30, 19\ell+57)$	570	166383	1710	1254
$(0,1,9)$	46	55	46	$(9\ell, 9\ell+1, 9\ell+9)$	63	2097	72	108
$(0,7,8)$	0	49	49	$(8\ell, 8\ell+7, 8\ell+8)$	56	168	56	88
$(0,18,20)$	0	162	162	$(20\ell, 20\ell+18, 20\ell+20)$	180	1480	360	260
$(0,2,6)$	0	4	4	$(3\ell, 3\ell+2, 3\ell+6)$	6	420	24	36
$(0,2,11)$	68	79	68	$(11\ell, 11\ell+2, 11\ell+11)$	88	3377	99	154

If we let

$$k_3 = \min\{j \geq 0 : H_k \text{ is minimally 3-generated for all } k > j\},$$

then $0 < k_3 \leq b \leq k_{a,b}$. We always considered for $K[H]$ the presentation $K[x,y,z]/I$ coming from the K-algebra map ϕ described in the Introduction. Thus it may happen that $k_{S_1} \leq k_3$. Indeed, as one can see from the table, if $(a,b) = (2,6)$ one has $k_{S_1} = 4 < k_3 = 6 = k_{a,b}$, which shows that our new bound $k_{a,b}$ is exact if we restrict the study of the periodicity to minimally 3-generated semigroups.

The data in the table indicates that k_{S_1} differs from $k_{a,b}$ by either a or $b-a$, depending on which term dominates in the max-formula defining $k_{a,b}$. More numerical experiments encourage us to formulate the following conjecture.

Conjecture 5.2. *With notation as above*

(i)

$$k_{S_1} = \begin{cases} \left(\frac{b}{D} - 1\right) \cdot a & \text{if } b \leq 2a + D, \\ \left(\frac{b}{D} - 1\right) \cdot (b-a) - b & \text{if } b > 2a + D. \end{cases}$$

(ii)

$$k_{S_2} = \begin{cases} k_{S_1} & \text{if } b \leq 2a + D, \\ k_{S_1} - 2a & \text{if } b = 2a + 2D, \\ k_{S_1} - b & \text{if } b > 2a + 2D. \end{cases}$$

Our methods allowed to improve k_V for 3-generated semigroups only. We hope this work provides motivation to look for better estimates of the various thresholds of interest, for arbitrary families of shifted semigroups. This goes hand in hand with a better understanding of the reasons that trigger the periodic behaviour of the Betti numbers, of the regularity and the others aspects discussed in this paper.

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