Strongly generic monomial ideals

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1 Introduction

The purpose of this lecture is to present a class of monomial ideals for whom a minimal graded free resolution can be easily described.

In Section 2 we recall what planar graphs are, and we give a criterion for planarity. The Buchberger graph ($Buch(I)$) of a monomial ideal I is defined.

In Section 3 we study first the strongly generic ideals I in three indeterminates. We prove in Proposition 3.8 that $Buch(I)$ is a planar and connected graph. When I is also artinian, then $Buch(I)$ is made up of the edges of a triangulation of a triangle. $Buch(I)$ can be embedded on the staircase surface of the ideal and using Construction 3.9 we show that this embedding produces a minimal multigraded free resolution of I (see Theorem 3.11). But in 3 variables, $Buch(I)$ is the 1-skeleton of Δ_I, the Scarf complex of I. Extending the Construction 3.9 to polinomial rings in n variables and labeling arbitrary simplicial complexes, we prove a theorem of Bayer, Peeva and Sturmfels which says that the Scarf complex of a strongly generic ideal is the support for a minimal free resolution of the ideal (Theorem 3.22).

Our presentation follows [4] and the original article [1] of Bayer, Peeva, and Sturmfels who introduced the notion of strongly generic ideals.

2 The Buchberger graph and planar maps

For the beginning, we present the following:

Definition 2.1. *The **Buchberger graph** $Buch(I)$ of a monomial ideal*

$$I \ = < m_1,\ \ldots,\ m_r > \subset S := \ \mathbb{K}[X_1,\ \ldots,\ X_n]$$

has vertices $1,\ 2,\ \ldots,\ r$ *and* $\{i,\ j\}$ *is an edge if and only if there is no monomial generator* m_k *with* $m_k|\ LCM(m_i,\ m_j)$ *and* $deg_{x_u} m_k < deg_{x_u} LCM(m_i,\ m_j)$ *for any variable* $x_u|\ LCM(m_i,\ m_j)$.

In [4] it is shown that $Buch(I)$ is useful when computing Gröbner bases. In the following we shall see how $Buch(I)$ can also be used to compute free resolutions (for some classes of ideals I).

Example 2.2. If $n = 2$ and $I \subset \mathbb{K}[X,\ Y]$ a monomial ideal, in $Buch(I)$ there is an edge only between "adjacent" generators. See Figure 2.2 bellow:

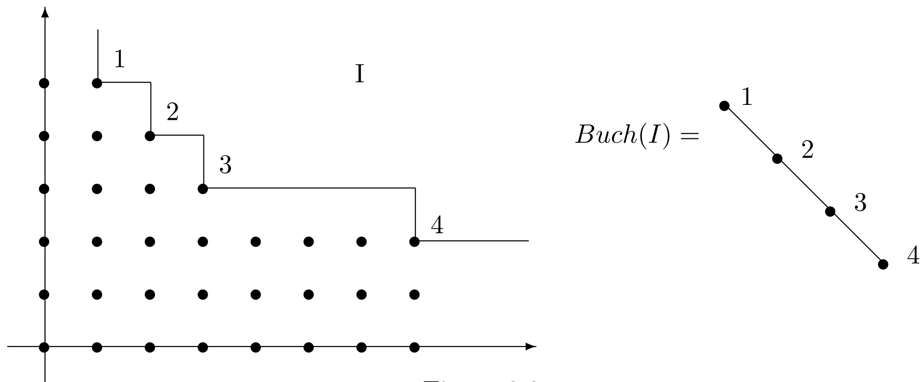
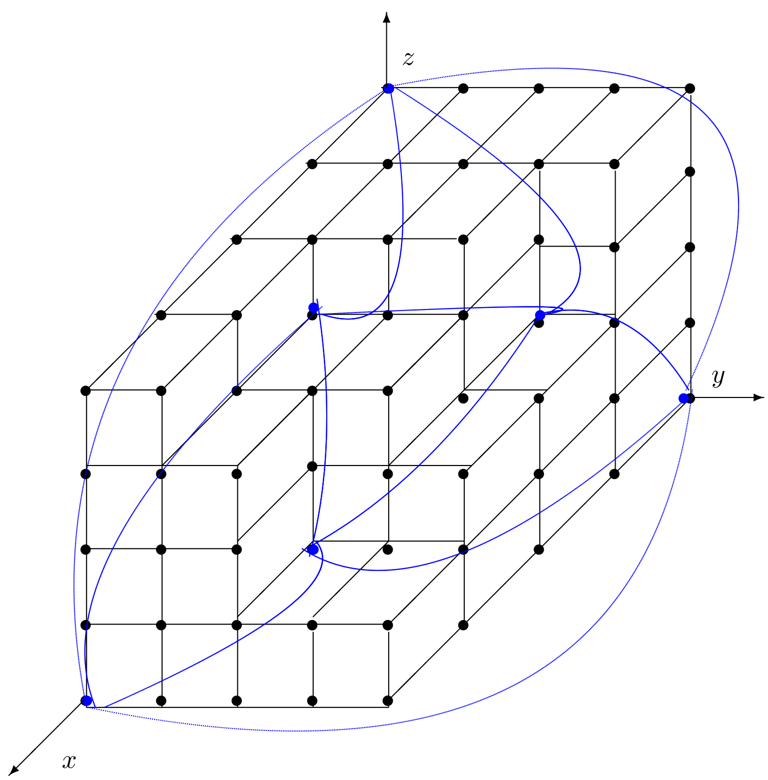


Figure 2.2

Example 2.3. For the (strongly generic) ideal

$$I = < X^4,\ Y^4,\ Z^4,\ XY^3Z^2,\ X^2YZ^3,\ X^3Y^2Z >,$$

$Buch(I)$ is depicted in Figure 2.3. We notice that it lies naturally on the shaded surface and it is thus a planar graph.



$Buch(I) =$

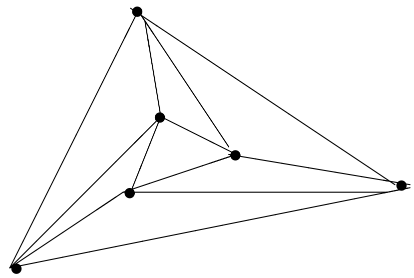


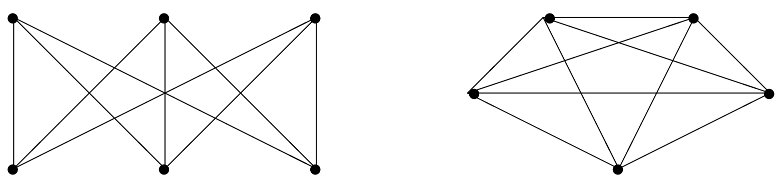
Figure 2.3

Definition 2.4. *A graph G is called **planar** if there is an embedding of the graph in a plane π, i.e. there is an injective map φ from the vertices of G to the points in π and to each edge $\{i,\ j\}$ we attach a non self-intersecting path φ_{ij} in π from $\varphi(i)$ to $\varphi(j)$ such that no two such paths intersect, but at their ends if the edges have a vertex in common.*

In the literature it is known the following result of Kuratowski:

Theorem 2.5. *[3](Kuratowski) A graph G is planar if and only if G does not contain any subgraph isomorphic with $K_{3,3}$ or with K_5.*

Remark 2.6. We recall here that $K_{3,3}$ is the complete bipartite graph on 3 vertices and K_5 is the complete graph on 5 vertices. They are drawn in Figure 2.6:



$K_{3,3}$ K_5

Figure 2.6

Remark 2.7. There exist an efficient algorithm (linear in the number of vertices) that determines whether a graph is planar or not. See [3] for more details.

Remark 2.8. Even if we know that a certain graph is planar, we need to find its embedding in the plane. This embedding is not unique.

Definition 2.9. *A **planar map** is a graph G together with an embedding into a surface homeomorphic to the plane $\mathbb{R}^2$.*

Definition 2.10. *For $I \subseteq \mathbb{K}[X,\ Y,\ Z]$ a monomial ideal, its **staircase surface** (denoted $\mathcal{S}(I)$) is the topological boundary of the set of vectors $(a,\ b,\ c) \in \mathbb{R}^3_+$ for which there exist a monomial $X^{a_0}Y^{b_0}Z^{c_0} \in I$ with $a_0 \leq a,\ b_0 \leq b$ and $c_0 \leq c$.*

Remark 2.11. $\mathcal{S}(I)$ is a bounded set (equivalently, it is compact) if and only if I is an artinian ideal (i.e. S/I is an artinian S - module).

Remark 2.12. *If* $(a,\ b,\ c) \in \mathcal{S}(I)$ *and* $X^aY^bZ^c \in I$*, then* $(a+1,\ b+1,\ c+1) \notin \mathcal{S}(I)$*.*

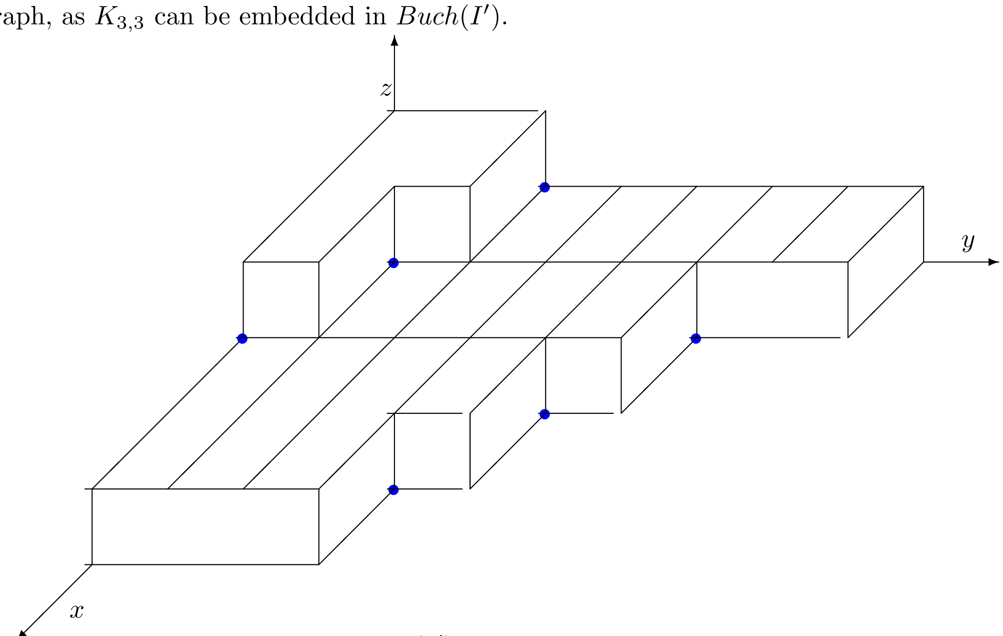
Proposition 2.13. *The orthogonal projection of* $\mathcal{S}(I)$ *on a plane with normal vector* $(1,\ 1,\ 1)$ *is a homeomorphism on the image.*

Example 2.14. In Example 2.3, $Buch(I)$ is embedded on $\mathcal{S}(I)$.

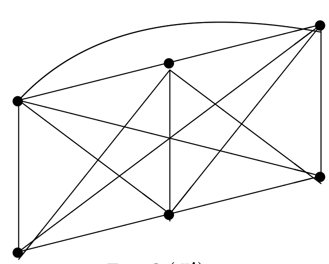
Example 2.15. Consider the ideal

$$I' = < X^2Z,\ XYZ,\ Y^2Z,\ X^3Y^5,\ X^4Y^4,\ X^5Y^3 > .$$

This is not an artinian ideal, hence $\mathcal{S}(I')$ is not bounded. In Figure 2.15 we can see the staircase surface of I' and its Buchberger graph. This is not a planar graph, as $K_{3,3}$ can be embedded in $Buch(I')$.



$\mathcal{S}(I')$



$Buch(I')$

Figure 2.15

3 Strongly generic monomial ideals

Definition 3.1. *A monomial ideal $I \subset S = \mathbb{K}[X_1, \ldots, X_n]$ is called* ***strongly generic*** *if there are no two minimal generators for I with the same nonzero exponent for the same variable.*

Example 3.2.
1. $I = < X^2, YZ, XZ > \subset \mathbb{K}[X, Y, Z]$ is not strongly generic since in the second and in the third monomial generator, the variable z has the same exponents.
2. $J = < X^2, YZ, XZ^2 > \subset \mathbb{K}[X, Y, Z]$ is strongly generic.
3. In $S = \mathbb{K}[X, Y]$ any monomial ideal I is strongly generic.

Remark 3.3. If we see monomials in S as lattice points in $\mathbb{R}^n$, then I is strongly generic if and only if for any $i \in \{1, \ldots, n\}$ and any $a \in \mathbb{N}$, on the hyperplane given by the equation $X_i - a = 0$ there is at most one minimal generator.

Remark 3.4. The notion of strongly generic ideal has been introduced by Bayer, Peeva and Sturmfels in [1] but under the name "generic ideals". Later Miller, Sturmfels and Yanagawa [5] have enlarged the class of generic ideals by changing the definition. This ideals previously called "generic" were to be called "strongly generic". The tree ideals (see [2]) are generic ideals without being strongly generic.

Proposition 3.5. *If $I = < m_1, \ldots, m_r >$ is strongly generic, then $\{i, j\} \in Buch(I)$ if and only if there exists $k \in \{1, \ldots, r\}$ with $m_k | LCM(m_i, m_j)$.*

Proof. Just use the definition for $Buch(I)$ and the strongly genericity of I. □

Remark 3.6. If $I \subseteq S$ is strongly generic and we know that the monomial $m \in S$ is of the form $m = LCM(u, v)$ for some u and v minimal generators of I, then u and v can be immediately found. Indeed, looking at the exponent in m for any variable we can identify uniquely the monomial it comes from.

Remark 3.7. The word "generic" in the definition is probably used in connection with the fact that if we identify the r generators with a lattice point in $\mathbb{R}^{rn}$, then this point corresponds to a point outside the hyperplanes with equations $a_{ij} - a_{kj} = 0, \ \forall i \neq j, \ 1 \leq i, \ k \leq n, \ 1 \leq j \leq r$.

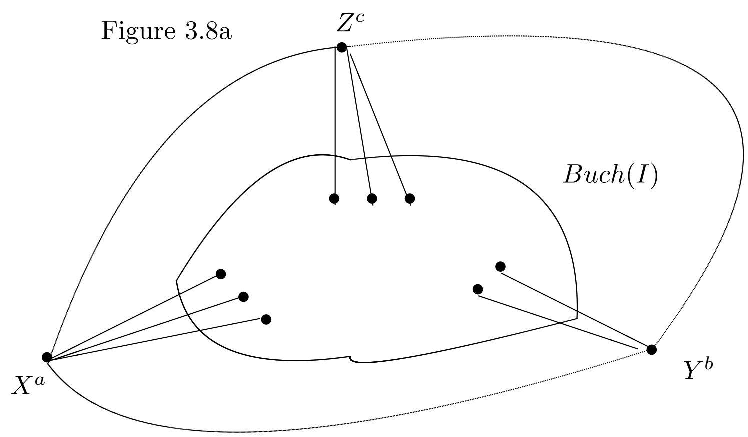
The following result is a particular case of [4, Theorem 6.13] where it is stated for generic ideals in $n \in \mathbb{N}^*$ indeterminates.

Proposition 3.8. *If I is a strongly generic ideal in $S = \mathbb{K}[X, Y, Z]$, then $Buch(I)$ is a connected planar graph. Furthermore, if I is artinian, then $Buch(I)$ is made up of the edges of a triangulation of a triangle.*

Proof. For the first part of the proposition we notice that it is enough to consider the artinian case, too. Indeed, if I is not artinian, then some (or maybe all) of the pure powers of the variables X, Y and Z are missing from I. Hence, adding

$X^a,\ Y^b$ and Z^c (or only the missing ones) to I, for $a,\ b,\ c \in \mathbb{N}$ big enough we get $I' = I + < X^a,\ Y^b,\ Z^c >$ an artinian ideal. Notice that $Buch(I) \subset Buch(I')$ since by adding these generators we do not delete any edge already in $Buch(I)$, and also no other edge between vertices in $Buch(I)$ can occur. Hence $Buch(I')$ looks like in Figure 3.8a.

Figure 3.8a



Therefore, if $Buch(I')$ were the triangulation of a triangle, $Buch(I)$ would stay planar and connected.

Let I be an artinian ideal, $I =< m_1,\ \ldots,\ m_r >$. In order to prove the planarity of $Buch(I)$ we embed it on the staircase surface $\mathcal{S}(I)$ which we know it is bounded and via projection it can be embedded in the plane (see Proposition 2.13).

Notice that if $\{i,\ j\} \in Buch(I)$, then m_i and m_j are on $\mathcal{S}(I)$, but also $LCM(m_i,\ m_j)$ is on the staircase surface (here by saying that a monomial lies on $\mathcal{S}(I)$ we mean that the point in $\mathbb{R}^n$ having as coordinates the exponent of the monomial, lies on $\mathcal{S}(I)$). Indeed, if $LCM(m_i,\ m_j) \notin \mathcal{S}(I)$ since $LCM(m_i,\ m_j) \in I$ this means that there exist another generator m_k with $m_k \mid LCM(m_i,\ m_j)$, contradiction with $\{i,\ j\} \in Buch(I)$.

For any edge $\{i,\ j\} \in Buch(I)$ we attach a path on $\mathcal{S}(I)$ such that no two such paths intersect unless they correspond to adjacent edges: first we build a path from m_i to $LCM(m_i,\ m_j)$, then another path from m_j to $LCM(m_i,\ m_j)$. They unite and give a path from m_i to m_j through $LCM(m_i,\ m_j)$. On each of the coordinate planes xOy, yOz and xOz these paths will be piecewise linear, as the staircases drawn for ideals in two variables (see Figure 3.8b).

I is strongly generic, hence any coordinate of $LCM(m_i,\ m_j)$ comes from exactly one of the monomials. Let us draw the path from m_i to $LCM(m_i,\ m_j)$: we advance on $\mathcal{S}(I)$ by increasing as much as needed the x-coordinates (keeping the others fixed), then by increasing y-coordinate (keeping the z-part fixed) and finally by increasing the z-coordinate. We end up in $LCM(m_i,\ m_j)$. It is an easy exercise to prove that these paths are indeed on $\mathcal{S}(I)$. By doing this we have obtained a path where each point with integer coordinates has at least one coordinate equal with the corresponding exponent of m_i or m_j. Hence, if $\{i,\ j\}$ and $\{k,\ l\}$ are non-adjacent edges in $Buch(I)$, the corresponding paths have no common point.

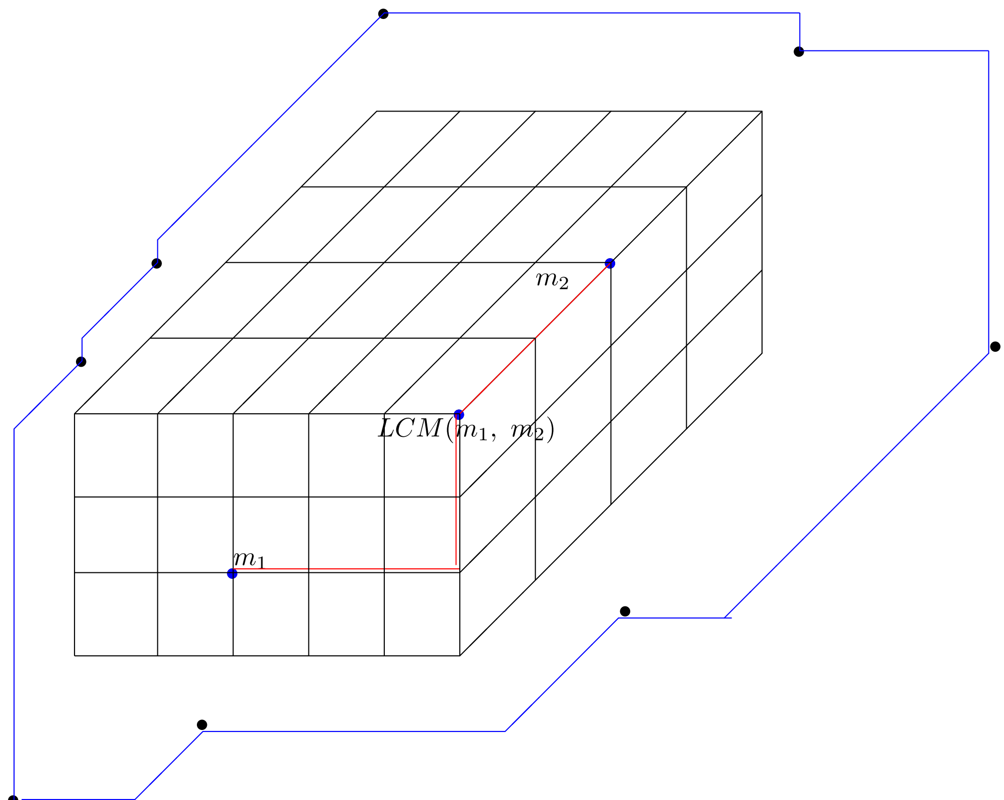


Figure 3.8b

When two edges $\{i,\ j\} \in Buch(I)$ and $\{i,\ k\} \in Buch(I),\ j \neq k$, then two paths on $\mathcal{S}(I)$ from m_i to $LCM(m_i,\ m_j)$ respectively to $LCM(m_i,\ m_k)$ may have a small part in common around m_i, which surely does not also contain the ends (since $LCM(m_i,\ m_j) \neq LCM(m_i,\ m_k)$). But we can "pull aside" each overlapping part of the paths and still stay on $\mathcal{S}(I)$. Thus we obtain paths that intersect only at the common end, and we have proved the planarity of $Buch(I)$.

We show now that $Buch(I)$ is connected. We show more: in $Buch(I)$ from any minimal monomial generator m_i there are 3 independent paths to $X^a,\ Y^b$ and Z^c respectively. We use the previous embedding of $Buch(I)$ on $\mathcal{S}(I)$. Starting from m, parallel to Ox we meet the LCM corresponding to some edge $e \in Buch(I)$. The other end of e belongs to another monomial m_j whose $y-$ and $z-$coordinates are at most those of m_i. Iterating this procedure one gets a sequence of edges in $Buch(I)$ whose vertices have strictly increasing $x-$coordinates and weakly decreasing $y-$ and $z-$coordinates. The last vertex in this sequence of edges is X^a. By changing the axis we start with we obtain all the three desired paths. Due to the monotony, they intersect only at m_i.

It only remains to prove that the embedding of $Buch(I)$ on $\mathcal{S}(I)$ is the triangulation of a triangle. We know it is a planar graph, so we need to check that each of the regions that appear is a triangle, equivalently there is no unshortable cycle of length more or equal than 4. Given an edge $e = \{i,\ j\} \in Buch(I)$, how can we find the third vertex of the face(s) that contain e? We look for k such that $LCM(m_i,\ m_j,\ m_k) \in \mathcal{S}(I)$. There are at most two choices for such k. $\square$

Let us speak about minimal free resolutions of monomial ideals in three indeterminates. Take $I =< m_1, \dots,\ m_r > \subseteq S := \mathbb{K}[X,\ Y,\ Z]$ a monomial ideal. By *Hilbert's Syzygy Theorem* we know that S/I has a minimal free $\mathbb{Z}^3$-graded resolution of length less or equal than 3:

$$0 \longrightarrow S^{\beta_2} \longrightarrow S^{\beta_1} \longrightarrow S^{r=\beta_0} \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

Since it is multigraded, the differentials are homogeneous maps and the generators need to be shifted. Hence the graded Betti numbers β_i above decompose as multigraded Betti numbers: $\beta_i = \sum\limits_{\underline{a}\in\mathbb{Z}^n} \beta_{i,\underline{a}}$:

$$0 \longrightarrow \bigoplus_{\underline{a}\in\mathbb{Z}^n} S(-\underline{a})^{\beta_{2,\underline{a}}} \longrightarrow \bigoplus_{\underline{a}\in\mathbb{Z}^n} S(-\underline{a})^{\beta_{1,\underline{a}}} \longrightarrow \bigoplus_{i=1}^{r} S(-a_i) \longrightarrow S \longrightarrow S/I \longrightarrow 0$$

where $m_i = \mathbf{X}^{a_i}$, and $a_i \in \mathbb{Z}^3$.

Construction 3.9. We shall construct such a complex starting with a planar map of a planar graph G. Suppose it has r vertices, e edges and f regions. We label the i^{th} vertex with m_i, the $\{i,\ j\}$ edge with $m_{ij} = LCM(m_i,\ m_j)$ and the region R with m_R the LCM of the labels of all its vertices. Our complex will look like:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & S^f & \xrightarrow{\partial_F} & S^e & \xrightarrow{\partial_E} & S^r & \xrightarrow{\partial_V} & S & \longrightarrow S/I \longrightarrow & 0 \\ & & \| & & \| & & \| & & & & \\ & & \bigoplus S(-a_R) & & \bigoplus S(-a_{ij}) & & \bigoplus S(-a_i) & & & & \end{array}$$

where $a_i,\ a_{ij}$ and a_R are the exponents of $m_i,\ m_{ij}$ and m_k respectively. Denote by $e_i,\ e_{ij}$ and e_k the corresponding generators. The differentials are:

$$\partial_V(e_i) = m_i$$

$$\partial_E(e_{ij}) = \tfrac{m_{ij}}{m_j} \cdot e_j - \tfrac{m_{ij}}{m_i} \cdot e_i$$

$$\partial_F(e_R) = \sum_{\substack{\{i,j\} \\ edge\ in\ R}} \pm \tfrac{m_R}{m_{ij}} \cdot e_{ij}$$

Exercise 3.10. Check that this is a complex, i.e. $\partial \circ \partial = 0$.

Theorem 3.11. *Given a strongly generic ideal $I \subseteq S = \mathbb{K}[X,\ Y,\ Z]$, the planar map of $Buch(I)$ given by the embedding on $\mathcal{S}(I)$ produces a minimal free $\mathbb{Z}^3$-graded resolution of S/I (via Construction 3.9).*

Proof.

Step 1. As with Proposition 3.8, we reduce to the artinian case. If I is not artinian, by adding some already non-existing pure powers of the variables we obtain $I' = I + < X^a,\ Y^b,\ Z^c >$ which is an artinian ideal. Suppose the planar map G' of I' gives a minimal free multigraded resolution of S/I'. If we consider the chain subcomplex in degree less or equal than $(a-1,\ b-1,\ c-1)$ we notice that it resolves S/I. Indeed, the labels of edges that contain $X^a,\ Y^b$ or Z^c have at least one nonzero component equal to $a,\ b$ or c respectively, hence these edges/summands do not appear in the chain subcomplex in degree less or equal than $(a-1,\ b-1,\ c-1)$. If we go back to Proposition 3.8 and see how $Buch(I)$ is included in $Buch(I')$, we see that this subcomplex is the same with the one obtained for $Buch(I)$ via Construction 3.9.

Step 2. Assume I is an artinian ideal. A multigraded minimal free resolution of S/I is given by the degrees of minimal generators (the shifts) in each homological degree, their multiplicities (i.e. the multigraded Betti numbers) and the monomial matrices that give the differentials. According to [4] the multigraded Betti numbers can be computed via $K^{\underline{b}}(I)$, the upper Koszul simplicial complex, for each multidegree $\underline{b}$.

□

We stop here our proof in order to review the definition of $K^{\underline{b}}(I)$ and some easy to prove properties:

If I is a monomial ideal and $\underline{b} \in \mathbb{N}^n$, then **the upper Koszul simplicial complex** is defined as:

$$K^{\underline{b}}(I) = \left\{ \text{square-free vectors } \tau \subset \{0,\ 1\}^n \,|X^{\underline{b}-\tau} \in I \right\}.$$

Proposition 3.12. *With the above notations, the following hold:*

i) $K^{\underline{b}}(I) = \emptyset \Leftrightarrow X^{\underline{b}} \notin I.$

ii) $K^{\underline{b}}(I) \neq \emptyset \Leftrightarrow X^{\underline{b}} \in I.$

iii) $K^{\underline{b}}(I) = \{\emptyset\} \Leftrightarrow X^{\underline{b}}$ *is a minimal generator of* $I.$

iv) $K^{\underline{b}}(I) = \{0,1\}^n \Leftrightarrow X^{\underline{b}-\mathbf{1}} \in I,$ *where* $\mathbf{1} = (1,1,\dots,1) \in \mathbb{N}^n$

Corollary 3.13. $\underline{b} \in \mathcal{S}(I)$ *if and only if* $\emptyset \neq K^{\underline{b}}(I)$ *and* $K^{\underline{b}}(I) \neq \{0,1\}^n$*, the* $(n-1)$*-simplex.*

These properties show that for most of the $\underline{b} \in \mathbb{N}^n$, $K^{\underline{b}}(I)$ is either $\emptyset$ or the $(n-1)-$simplex; only when $\underline{b}$ is "close" and "above" the staircase surface $\mathcal{S}(I)$, $K^{\underline{b}}(I)$ becomes interesting. A result that explains its utility is the following:

Theorem 3.14. *Given $\underline{b} \in \mathbb{N}^n$ and a monomial ideal I, then*

$$\beta_{i,\ \underline{b}}(I) = \beta_{i+1,\ \underline{b}}(S/I) = dim_{\mathbb{K}}\ \widetilde{H}_{i-1}(K^{\underline{b}}(I),\ \mathbb{K})$$

Proof. See the proof given in [4, Theorem 1.34] . □

Proof. (continuation of the proof of Theorem 3.11)

Using the above theorem we plan to find the nonzero Betti numbers. Obviously, $\beta_{1,\ \underline{b}}(S/I) = \beta_{0,\ \underline{b}}(I)$ equals 1 if $X^{\underline{b}}$ is a minimal generator for I, and it equals 0 otherwise.

$\beta_{2,\ \underline{b}}(S/I) = \beta_{1,\ \underline{b}}(I) = dim_{\mathbb{K}}\ \widetilde{H}_0(K^{\underline{b}}(I),\ \mathbb{K}) =$ (the number of connected components of $K^{\underline{b}}(I)) - 1$.

Therefore, to have a minimal 1^{st} syzygy in degree $\underline{b}$, $K^{\underline{b}}(I)$ needs to be disconnected and hence $\underline{b} \in \mathcal{S}(I)$.

Up to isomorphism, this is the list of all simplicial complexes Δ on $n = 3$ vertices:

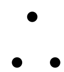
1) $\Delta = \emptyset$

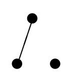
2) $\Delta = \{\emptyset\}$

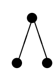
3) $\Delta =$ •

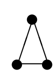
4) $\Delta =$ •   •

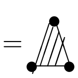
5) $\Delta =$

6) $\Delta =$ 

7) $\Delta =$ 

8) $\Delta =$ 

9) $\Delta =$ 

10) $\Delta =$ 

$K^{\underline{b}}$ is disconnected only in the cases no. 4), 7) when it has 2 connected components and in the case no. 6) when it has 3 connected components. Let us analyze each of these situations.

i) If $K^{\underline{b}}(I)$ is as in case 6) above, "around" $\underline{b}$, $\mathcal{S}(I)$ looks like in Figure 3.11a where a black dot means a point in I, "$*$" means a point not in I and "□" means a minimal generator.

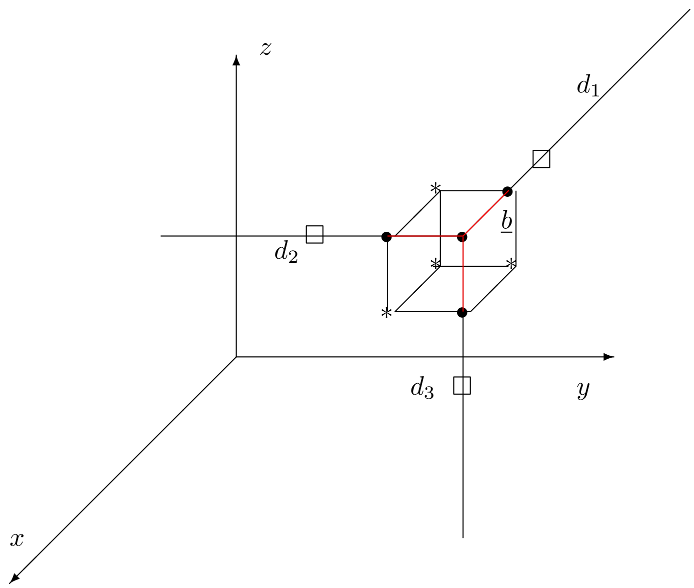


Figure 3.11a

We know that $X^{\underline{b}}$, $X^{\underline{b}-(0,0,1)}$, $X^{\underline{b}-(0,1,0)}$, $X^{\underline{b}-(1,0,0)} \in I$ and all the other vertices of the unit cube drawn in Figure 3.11a are not in I; hence on each of the three line segments d_1, d_2, d_3 starting from $\underline{b}$ parallel to the coordinate axes there has to be a minimal generator for I. But two such generators have a nonzero common coordinate, hence this configuration of $K^{\underline{b}}(I)$ can not occur if I is strongly generic.

ii) If $K^{\underline{b}}(I)$ is as in case 7) above, "around" $\underline{b}$, $\mathcal{S}(I)$ looks like in Figure 3.11b, with the same notations as before. Here, there is a minimal generator m' in the "rectangle" spanned by d_2 and d_3 and there is another minimal generator m'' on d_1. Hence $X^{\underline{b}} = LCM(m',\ m'')$. Moreover, since $X^{\underline{b}-(1,1,1)} \notin I$, there is no other minimal monomial generator of I (except m' and m'') which divides $X^{\underline{b}}$. In other words, between m' and m'' there is an edge in $Buch(I)$.

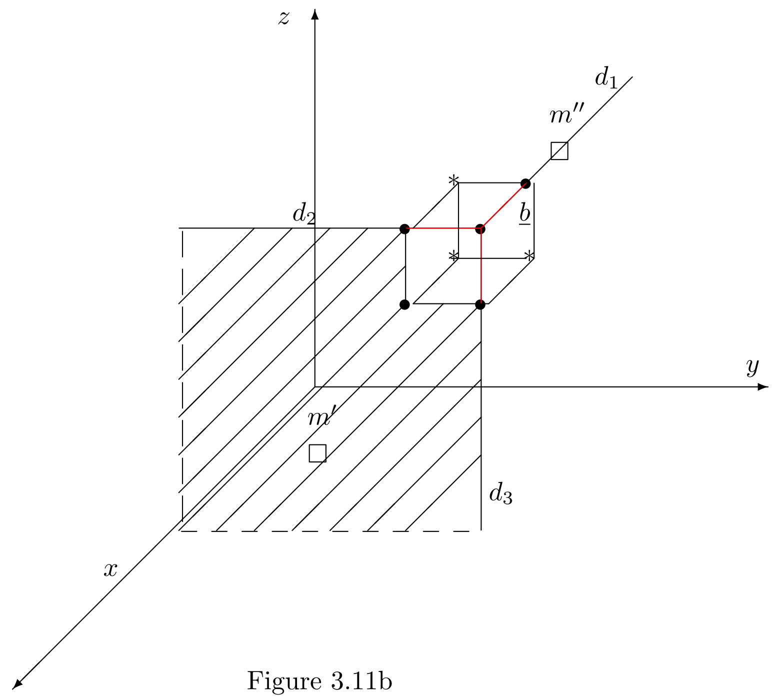


Figure 3.11b

iii) If $K^{\underline{b}}(I)$ is as in case 4) above, "around" $\underline{b}$ the staircase surface looks like in Figure 3.11c.

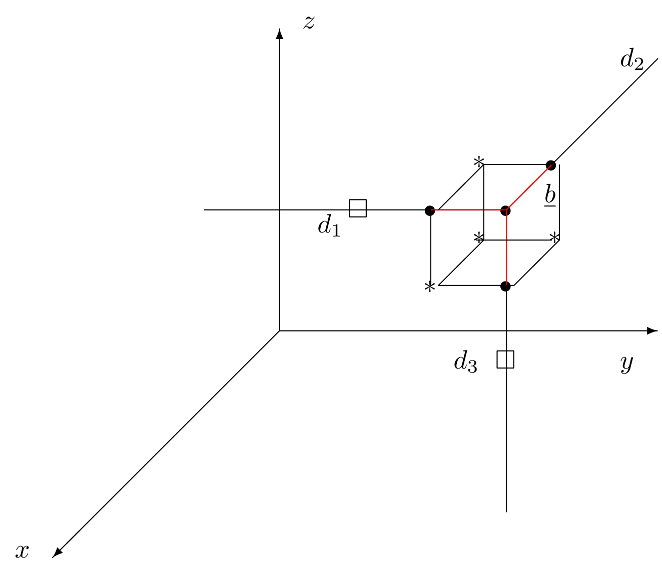


Figure 3.11c

We leave to the reader the analysis of this case. One will obtain the same result as for *ii*).

In conclusion, there is a minimal 1^{st} syzygy in degree $\underline{b}$ if and only if $\mathbf{X}^{\underline{b}} = LCM(m',\ m'')$ where m' and m'' are two minimal monomial generators connected by an edge in $Buch(I)$. Using Theorem 3.14

$$\beta_{1,\ \underline{b}}(I) = \begin{cases} 1, & if\ \mathbf{X}^{\underline{b}} = LCM(m_i,\ m_j) \quad and \quad \{i,\ j\} \in Buch(I) \\ 0, & \qquad otherwise \end{cases}$$

We pass to the 2^{nd} syzygies of I:

$\beta_{3,\ \underline{b}}(S/I) = \beta_{2,\ \underline{b}}(I) = dim_{\mathbb{K}}\ \widetilde{H}_1(K^{\underline{b}}(I),\ \mathbb{K})$. Of all possible simplicial complexes on 3 vertices listed before, only in case 9) we have nonzero reduced 1^{st} homology, and for such $\underline{b}$, $\beta_{2,\ \underline{b}}(I) = 1$. In this case, "around" $\underline{b}$ the staircase surface $\mathcal{S}(I)$ looks like in Figure 3.11c.

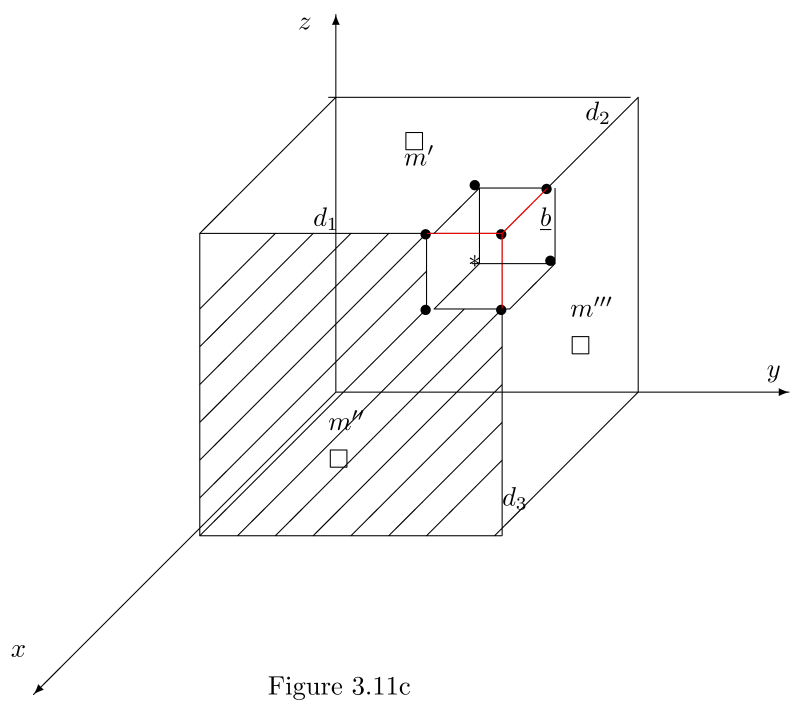


Figure 3.11c

Of all the vertices of the unit cube pictured, only $\mathbf{X}^{\underline{b}-\mathbf{1}} \notin I$. Hence, as previously shown, there is a minimal monomial generators m', m'' and m''' on each of the rectangular faces spanned by d_1 and d_2, d_2 and d_3, d_1 and d_3 respectively.

One sees easily that between every two of m', m'' and m''' there is an edge in $Buch(I)$ and $X^{\underline{b}} = LCM(m',\ m'',\ m''')$. Conversely, if $\mathbf{X}^{\underline{b}}$ can be expressed like this, we have a minimal second syzygy for I in degree $\underline{b}$.

In conclusion, looking at the way $Buch(I)$ is canonically embedded into $\mathcal{S}(I)$, we "read" the 0-syzygies of I as the "inner corners" of $\mathcal{S}(I)$, the 2^{nd}-syzygies of I as the "outer conners" of $\mathcal{S}(I)$ and the 1^{st}-syzygies of I as corners that are $LCMs$ of pairs of generators that form an edge in $Buch(I)$. On the canonical planar map of $Buch(I)$, if we label the vertices with the respective $0-$syzygies, the 1^{st}-syzygies represent the $LCMs$ of the ends of the edges in $Buch(I)$, and the 2^{nd}-syzygies are the $LCMs$ of vertices that form the triangles in the planar representation.

The chain complex constructed from this data after the recipe given in Construction 3.9 is a minimal $\mathbb{Z}^3$-graded free resolution of S/I. □

We wish to extend the Construction 3.9 from planar graphs to arbitrary simplicial complexes, doing the labeling in a similar manner. This construction has been introduced by D. Bayer, I. Peeva and B. Sturmfels in [1]:

Construction 3.15. Let Δ be a simplicial complex whose vertices are labeled by the minimal generators of a monomial ideal $I =< m_1, \ldots,\ m_r > \subseteq S := \mathbb{K}[X_1, \ldots,\ X_n]$. We label each face of Δ by the LCM of its vertices. Let $\mathcal{F}_\Delta$ be the $\mathbb{N}^n$-graded chain complex of Δ over S: it is obtained from the simplicial chain of Δ by homogenizing the differential:

$$\mathcal{F}(\Delta) := \bigoplus_{\substack{J\subset\Delta \\ face}} S(-a_J) \text{ and } d(e_J) = \sum_{i\in I} sign(i,\ J)\cdot \frac{\mathbf{X}^{a_J}}{\mathbf{X}^{a_{J\setminus\{i\}}}}\cdot e_{J\setminus\{i\}}$$

where $m_i = \mathbf{X}^{a_i},\ \forall i = 1,\ \ldots,\ r,\ \mathbf{X}^{a_J} := LCM\left\{\mathbf{X}^{a_i} \mid i\in I\right\},\ e_J$ is the generator of $S(-a_J)$ and $sign(i,\ J) = (-1)^{t-1}$ if i is the t^{th} element of the set $J \subset \{1,\ 2,\ \ldots,\ r\}$ written in increasing order.

If the complex $\mathcal{F}_\Delta$ is exact, we call it the resolution defined by the labeled simplicial complex. If this is the case, $\mathcal{F}_\Delta$ resolves S/I. Such a resolution is characteristic free.

Example 3.16. Take $n = 3,\ S = \mathbb{K}[X,\ Y,\ Z],\ \Delta =< \{1,\ 2\},\ \{1,\ 3\} >$ a simplicial complex, and we label its vertices $1,\ 2,\ 3$ with the monomials $m_1 = X^2Y,\ m_2 = XZ$ and $m_3 = YZ^3$ respectively. Set $\mathcal{F}_\Delta = \bigoplus_{\substack{J\subset\Delta \\ face}} S(-a_J)$

$$\mathcal{F}_\Delta:\quad 0 \longrightarrow \begin{array}{c} S(-a_{12}) \\ \bigoplus \\ S(-a_{13}) \end{array} \xrightarrow{\partial_1} \begin{array}{c} S(-a_1) \\ \bigoplus \\ S(-a_2) \\ \bigoplus \\ S(-a_3) \end{array} \xrightarrow{\partial_0} S \longrightarrow 0,$$

that is

$$\mathcal{F}_\Delta : \quad 0 \longrightarrow \begin{array}{c} S(-(2,1,1)) \\ \bigoplus \\ S(-(2,1,3)) \end{array} \xrightarrow{\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}} \begin{array}{c} S(-(2,1,0)) \\ \bigoplus \\ S(-(1,0,1)) \\ \bigoplus \\ S(-(0,1,3)) \end{array} \xrightarrow{(1,1,1)} S \rightarrow 0$$

$$\partial(e_{12}) = -Ze_1 + XYe_2 \ , \ \partial(e_{13}) = -Z^3e_1 + X^2e_3$$

$$\partial(e_1) = X^2Y \ , \ \partial(e_2) = XZ \ , \ \partial(e_3) = YZ^3$$

Exercise 3.17. Check that the chain complex in Example 3.16 is not a resolution of $\mathbb{K}[X,\ Y,\ Z]/ < X^2Y,\ XZ,\ YZ^3 >$.

Hint: Use the connection between Scarf complexes and minimal resolutions from [2].

Example 3.18. If $I =< m_1, \ldots,\ m_r >$, take Δ to be the $(r-1)$-simplex and we label the vertices with the generators of I. In this case, $\mathcal{F}_\Delta$ is called **the Taylor complex** which is in fact a resolution (the so-called **Taylor resolution**). See [2] for a proof.

Lema 3.19. *The complex $\mathcal{F}_\Delta$ is exact if and only if for every monomial m, the simplicial complex $\Delta[m] := \{J \in \Delta |\ m_J$ divides $m\}$ is empty or acyclic over $\mathbb{K}$.*

Proof. Since $\mathcal{F}_\Delta$ is $\mathbb{N}^n$-graded, it suffices to check exactness in each multidegree. The component of $\mathcal{F}_\Delta$ in multidegree m is a complex of finite dimensional $\mathbb{K}$-vector spaces, which can be identified with the chain complex of $\Delta[m]$ over $\mathbb{K}$. □

Definition 3.20. *For any monomial ideal $I =< m_1, \ldots,\ m_r > \subseteq S$ we define a simplicial complex*

$$\Delta_I = \{J \subseteq \{1,\ \ldots,\ r\} \,|\, m_J \neq m_{J'},\ \forall J' \subset \{1,\ \ldots,\ r\},\ J \neq J'\}$$

*and call it the **Scarf complex** of I.*

Lema 3.21. *If all nonzero Betti numbers of S/I are concentrated in the multidegrees a_J of the faces J of Δ_I, then $\mathcal{F}_{\Delta_I}$ is the minimal free resolution of S/I.*

Proof. If the minimal free resolution of S/I is strictly larger than $\mathcal{F}_{\Delta_I}$, then the Taylor resolution has at least two basis elements in some multidegree a_J for $J \in \Delta_I$. This contradicts the definition of Δ_I. □

Theorem 3.22. *(Bayer, Peeva, Sturmfels). Let I be a strongly generic monomial ideal. Then the complex $\mathcal{F}_{\Delta_I}$ defined by the Scarf complex Δ_I is a minimal free resolution of S/I over S.*

Proof. If $J \in \Delta_I$ and $j \in J$, then $m_{J\setminus\{j\}}$ properly divides m_J. Thus, all the maps in $\mathcal{F}_{\Delta_I}$ are minimal. It remains to show that $\mathcal{F}_{\Delta_I}$ is exact. We shall use the previous lemma. Consider any multidegree a_J with $J \notin \Delta_I$. But $\beta_{j,\, a_J}$ equals the $\mathbb{K}$-dimension of the homology of the Koszul complex at $\mathbb{K}_j := \Lambda^j(S/I)^n$ in degree a_J. The component of $\mathbb{K}_j$ in degree a_J is contained in the S/I module $\frac{m_J}{supp(m_J)}\mathbb{K}_j$ where $supp(m_J)$ is the maximal square-free monomial dividing m_J. To prove that this component is zero, it suffices to show that $\frac{m_J}{supp(m_J)}$ is zero in S/I. Choosing J minimal with respect to inclusion, we may assume $m_J = m_{J\cup\{\ell\}}$ for some $\ell \in \{1, \ldots, r\} \setminus J$. The monomials m_ℓ and m_I have different exponents in any fixed variable because I is generic. So m_ℓ divides $\frac{m_J}{supp(m_J)}$. This proves that all nonzero Betti numbers of S/I are concentrated in the multidegrees that label the faces of Δ_I, and we can now use Lemma 3.21 to get the conclusion. □

More results on Scarf complexes and their utility will be presented in [2].

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