The planar resolution algorithm

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1 Introduction

Let $I = < m_1,\ \ldots,\ m_r > \subset S := \mathbb{K}[x_1,\ \ldots,\ x_n]$ be an arbitrary monomial ideal, where $\mathbb{K}$ is a field. As we saw in [5], by computing the Scarf complex for I and writing down the attached chain complex, one obtains a minimal free resolution of S/I. When the ideal I is not strongly generic, we shall *"deform"* the exponents in a controlled way in order to obtain a strongly generic ideal I_ε. Then, from the minimal free resolution of S/I_ε we shall derive a free resolution of S/I, which however need not be minimal. Still, its length is at most n, the number of indeterminates, and thus it is usually much smaller and *"thinner"* than the Taylor resolution. This construction appeared in [1]. Our presentation will follow mainly [1] and [3].

In Section 2 we restrict to the case of $n = 3$ indeterminates and present an algorithm of Miller [2] to produce a minimal free resolution from the free resolution obtained as above from the deformed ideal I_ε. This minimal resolution is again cellular, corresponding to a planar map.

2 Deformation of exponents

Let $I = < m_1,\ \ldots,\ m_r > \subset S := \mathbb{K}[x_1,\ \ldots,\ x_n]$ be an arbitrary monomial ideal.

Construction 2.1. Let $\{a_i = (a_{i1}, \ldots, a_{in}) \mid 1 \le i \le r\}$ be the exponent vectors of the minimal generators of I. Choose vectors $\varepsilon_i = (\varepsilon_{i1}, \ldots, \varepsilon_{in}) \in \mathbb{R}^n_+$ for $1 \le i \le r$ such that for any $i \ne j$ and $s \ne t$

$$\text{if } a_{is} \ne 0 \text{ and } a_{js} \ne 0, \text{ then } a_{is} + \varepsilon_{is} \ne a_{js} + \varepsilon_{js}$$
$$\text{if } a_{is} = 0, \text{ then } \varepsilon_{is} = 0$$
$$\text{and if } a_{is} + \varepsilon_{is} < a_{it} + \varepsilon_{it}, \text{ then } a_{is} \le a_{it}$$

One can easily check that these conditions are satisfied for all positive and small enough vectors ε_i. Each ε_i defines a monomial $x^{\varepsilon_i} = x_1^{\varepsilon_{i1}} \cdot \ldots \cdot x_n^{\varepsilon_{in}}$ with real exponents. Denote $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_r)$. We formally introduce the (strongly generic) monomial ideal (in a polinomial ring with real exponents)

$$I_\varepsilon := < m_1 x^{\varepsilon_1}, \ldots,\ m_r x^{\varepsilon_r} > .$$

We call I_ε with the above properties a *strongly generic transformation of I* and I a *specialization* of I_ε.

Let Δ_{I_ε} be the Scarf complex of I_ε. We now label the vertex of Δ_{I_ε} corresponding to $m_i x^{\varepsilon_i}$ with the original monomial m_i. Let $\mathcal{F}_\varepsilon$ be the complex of free S-modules defined by this labeling of Δ_{I_ε} as in [5, Construction 2.15].

Example 2.2. One method to deform I in a strongly-generic way is to pick an integer $N > r$ and set $I_\varepsilon := < m_i^N \cdot \prod_{x_j | m_i} x_j^i \mid 1 \leq i \leq r >$.

This amounts to choosing $\varepsilon_{ij} = \frac{i}{N}$ in Construction 2.1, since the ideals I_ε and $< m_i \cdot \prod_{x_j | m_i} x_j^{\frac{i}{N}} \mid 1 \leq i \leq r >$ have the same Scarf complex.

Remark 2.3. Even if we allow for ε_i to be vectors with positive real values, no algebra over polynomial rings with real exponents is really used. Notice that the Scarf complex of a strongly generic monomial ideal depends only on the coordinatewise order of the exponents of the generators. If the exponents are real numbers, then we can replace them by integers while preserving their order coordinatewise. We will obtain a monomial ideal with integer exponents and the same Scarf complex. Hence, we can formally *"deform"* the exponents with small values as $\frac{1}{N_1}$, N_1 nonnegative integer, respecting our intuition of changing our ideal I with another "very close" one.

Theorem 2.4. *The complex $\mathcal{F}_\varepsilon$ is a free resolution of S/I over S.*

Proof. Fix a monomial m in S. Let J be the largest subset of $\{1, 2, \ldots, r\}$ such that $m_{\mathcal{F}}$ divides m. (Check that there exists a unique such J!). The following conditions are equivalent for a subset $A \subset \{1, 2, \ldots, r\}$:

$$m_A \text{ divides } m \ \Leftrightarrow\ A \subseteq J \ \Leftrightarrow\ m_A \text{ divides } m_J \ \Leftrightarrow\ m_A(\varepsilon) \text{ divides } m_J(\varepsilon)$$

Here $m_A(\varepsilon) := \ LCM(m_i x^{\varepsilon_i} \ :\ i \in A)$. The last equivalence follows from our choice of the ε_{ij}. By [5, Theorem 2.22.] and [5, Lemma 2.19.] applied to $I_\varepsilon[m_J(\varepsilon)]$, the set of all faces of Δ_{I_ε} which satisfy the four equivalent conditions above is an acyclic simplicial complex. □

Corollary 2.5. *The Betti numbers of I are less or equal than those of any deformation I_ε, that is, less or equal than the face numbers of the Scarf complex Δ_{I_ε}.*

Remark 2.6. The Betti numbers of I_ε depend on the choice of the generic deformation. There are only finitely many complexes Δ_{I_ε} which can be obtained by Construction 2.1 and each of them corresponds to an ε lying in an open convex polyhedral cone in $\mathbb{R}^{rn}$

Example 2.7. Let $I = < x^2,\ xy^2z,\ \mathbf{y^2z^2},\ yz^2w,\ w^2 > \subset\ S := \mathbb{K}[x,\ y,\ z,\ w]$. A strongly generic deformation is $I_\varepsilon = < x^2,\ xy^2z,\ \mathbf{y^3z^3},\ yz^2w,\ w^2 >$, and the Scarf complex $\Delta_{I_\varepsilon} = < \{1,\ 2,\ 4,\ 5\},\ \{2,\ 3,\ 4\} >$. Hence, the Betti numbers of S/I_ε are 1, 5, 7, 4, 1. Since the Betti numbers of S/I are 1, 5, 7, 4, 1, we notice that $\mathcal{F}_\varepsilon$ differs from a minimal resolution by a single summmand $0 \longrightarrow S \longrightarrow S \longrightarrow 0$ in homological degrees 2 and 3.

3 The planar resolution algorithm

Definition 3.1. *A graph G with at least 3 vertices is called* ***3-connected*** *if deleting any pair of vertices along with all edges incident to them leaves a connected graph. Given a set W of vertices in G, define the* ***suspension of*** *G* ***over*** *W by adding a new vertex to G and connecting it by edges to all vertices in W. The graph G is called* ***almost 3-connected*** *if it has a set W of three distinguished vertices such that the suspension of G over W is 3-connected.*

In the previous lecture ([5]) we showed that if $I \subset \mathbb{K}[x,\ y,\ z]$ is a strongly generic monomial ideal, then it has a minimal resolution via the planar map associated to $Buch(I)$. If $I \subset \mathbb{K}[x,\ y,\ z]$ is not strongly generic, we can deform it to a strongly generic ideal I_ε and this ideal has a minimal resolution supported by $Buch(I_\varepsilon)$. By Theorem 2.4, this planar map gives a (not necessarily minimal) free resolution for I. In order to recover a minimal free resolution for I from this one, we need to delete edges (and the corresponding regions) in $Buch(I_\varepsilon)$ and glue the regions initially separated by these edges, as the next theorem says.

Theorem 3.2. *Every monomial ideal $I \subseteq \mathbb{K}[x,\ y,\ z]$ has a minimal free resolution by some planar map. If I is artinian, then the graph G underlying such planar map is almost 3-connected.*

We do not present here the proof of this theorem, but we refer the reader to the original article [2] of E. Miller or to [3, Theorem 3.17] for a detailed explanation. As with the proof of [5, Theorem 2.11], notice that it is enough to analyze the case when I is artinian.

Miller gave an algorithm for obtaining a planar map that (minimally) resolves I. Given a deformation I_ε of $I = < m_1,\ \ldots,\ m_r >$ with $m_i = x^{a_i} y^{b_i} z^{c_i}$, write the i^{th} deformed generator as $m_{\varepsilon,i} = x^{a_{\varepsilon,i}} y^{b_{\varepsilon,i}} z^{c_{\varepsilon,i}}$.

The algorithm requires a strongly generic deformation satisfying the condition

$$(*) \text{ if } a_i = a_j \text{ and } c_i < c_j, \text{ then } a_{\varepsilon,i} < a_{\varepsilon,j}$$

and the other two conditions obtained by cyclic permutations of $(a,\ b,\ c)$. Notice that if $a_i = a_j$, then $c_i < c_j$ if and only if $b_i > b_j$. The geometric interpretation is as follows: if two generators lie at the same distance in front of the yz-plane, then the lower one (i.e. the one with the smallest z-coordinate) lies farther to the right as seen from far out on the x-axis (i.e. it has bigger y-coordinate). Condition $(*)$ says that among generators that start at the same distance from the yz-plane, the deformation pulls increasingly farther from the yz-plane as the generators move up and to the left.

The idea is to apply the deformation ε and show that specializing $I_\varepsilon \to I$ step by step makes the extra-edges in the Scarf triangulation disappear one at the time. More precisely, the algorithm specializes I_ε to I by making strict inequalities between coordinates into equalities, **only one** at the time. Before each specialization step, the (already partially specialized) ideal has a cellular minimal resolution by induction; after each specialization step, the same planar map still supports a cellular free resolution, although it may not be minimal. However, in the nonminimal case, minimality is achieved by removing exactly one edge.

The proof of Theorem 3.2 is constructive. The following algorithm ([3, Algorithm 3.18]) provides us with the desired planar map.

Algorithm 3.3. *Input: an artinian ideal* $I \subset \mathbb{K}[x,\ y,\ z]$

Output: a planar map G *supporting a cellular minimal free resolution of* I

- `initialize` $I_\varepsilon :=$ *the strongly generic deformation of* I *in* $(*)$

  $G := \ Buch(I_\varepsilon)$

- `while` $I_\varepsilon \neq I$ `do`

  - `choose` $u \in \{a,\ b,\ c\}$ *and an index* i *such that* $u_{\varepsilon,i}$ *is minimal among the deformed* u*-coordinates satisfying* $u_{\varepsilon,i} \neq u_i$. *Assume (without loss of generality) that* $u = a$, *by cyclic symmetry of* $(a,\ b,\ c)$.
  - `find` *the region of* G *whose label* $x^\alpha y^\beta z^\gamma$ *has* $\alpha = a_{\varepsilon,i}$ *and* γ *minimal.*
  - `redefine` I_ε *and* G *by setting* $a_{\varepsilon,i} := a_i$ *and leaving all other generators unmodified.*
  - `if` $a_j = a_i$ `then` *delete from* G *the edge labeled* $x^{a_i} y^\beta z^\gamma$ `else` *leave* G *unchanged.*

- `output` G.

Example 3.4. To see how this algorithm works, let

$$I = < x,\ y,\ z >^2 = < x^2,\ y^2,\ z^2,\ xy,\ xz,\ yz > .$$

Then $I_\varepsilon = < x^2,\ xy^{1.1},\ x^{1.1}z,\ y^2,\ yz^{1.1},\ x^2 >$ is a strongly generic deformation satisfying the condition $(*)$ and $Buch(I_\varepsilon)$ is the triangle with its edge midpoints connected (see Figure 1). We can run the algorithm and if we choose $u = x$, then the edge linking the edge midpoints of the edges that meet in x^2, is removed (see Figure 1). Running the `while-do` loop again will not produce further changes to the planar map. Had we chosen another u, another edge of the "inner triangle" would have been removed.

$Buch(I_\varepsilon)$ $\qquad$ G

Figure 1

Luckily there is an implementation of this algorithm written in C by J. Morton [4] and which can be downloaded from his web page, currently

`http://math.berkeley.edu/~mortonj/research.html`

One can find there the source code (the file *pgres.c*) and some other examples. If using Linux, this file can be compiled with

```
gcc pgres.c -o pgres
```

The exponents of the monomials generating I will be read from an input file (e.g. myideal) and the program generates a LaTeX file containing all the transformations of the planar map required to pass from $Buch(I_\varepsilon)$ to the final G.

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