ON INTERSECTIONS OF COMPLETE INTERSECTION IDEALS

MIRCEA CIMPOEAŞ AND DUMITRU I. STAMATE

Abstract. We prove that for certain families of toric complete intersection ideals, the arbitrary intersections of elements in the same family are again complete intersections.

Introduction

Let K be any field and $S = K[x_1, \dots x_r]$ be the polynomial ring in the variables $x_1, \dots, x_r$. An ideal $I \subset S$ is called a *complete intersection* (CI for short) if it is minimally generated by height I elements. This is a strong condition which is rarely preserved by taking intersections of such ideals.

In this note we show that for several infinite families of CI toric ideals, arbitrary intersections in the same family produce again CI ideals.

For an affine semigroup $H \subset \mathbb{N}^d$ the semigroup ring $K[H]$ is the K-subalgebra in $K[t] := K[t_1, \dots, t_d]$ generated by the monomials $t^h = t_1^{h_1} \dots t_d^{h_d}$ for all $h = (h_1, \dots, h_d) \in H$.

Consider the list of nonnegative integers $\mathbf{a} = a_1 < a_2 < \dots < a_r$. We denote $I(\mathbf{a})$ the kernel of the K-algebra map $\phi : S \to K[\langle \mathbf{a} \rangle]$ letting $\phi(x_i) = t^{a_i}$, where $\langle \mathbf{a} \rangle$ denotes the semigroup generated by $a_1, \dots, a_r$. If they generate $\langle \mathbf{a} \rangle$ minimally, we call $I(\mathbf{a})$ the toric ideal of $\langle \mathbf{a} \rangle$.

For any integer k we let $\mathbf{a} + k = a_1 + k, \dots, a_r + k$. The properties of the family of ideals $\{I(\mathbf{a} + k)\}_{k \geq 0}$ have been studied in [10], [15], [9] or [14]. Jayanthan and Srinivasan proved in [10] that for all $k \gg 0$, $I(\mathbf{a} + k)$ is CI if and only if $I(\mathbf{a} + k + (a_r - a_1))$ is CI. This was a particular case of a conjecture of Herzog and Srinivasan, proved in full generality by Vu in [15]: for all $k \gg 0$, the Betti numbers of the ideals $I(\mathbf{a} + k)$ and $I(\mathbf{a} + k + (a_r - a_1))$ are the same. See also Theorem 2.1 for related matters.

The main result of this note is Theorem 1.13, where we show that for a fixed sequence $\mathbf{a}$, the intersection of arbitrarily many CI ideals $I(\mathbf{a}+k)$ with large enough shifts k is still a CI ideal.

One observation fruitfully used in [10] and [15] is that the ideal $J(\mathbf{a}+k)$ generated by all homogeneous polynomials in $I(\mathbf{a}+k)$ plays an important role in describing the equations in $I(\mathbf{a}+k)$ whenever $k \gg 0$. Throughout this note the word homogeneous refers to the standard grading on S obtained by letting $\deg x_i = 1$ for $i = 1, \dots, r$.

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For a nonzero polynomial f in S, its initial form denoted f^* is the (nonzero) homogeneous component of least degree. For any ideal $I \subset S$ we set $I^* = (f^* : f \in I, f \neq 0)$ which is called the ideal of initial forms of I. It appears naturally as the defining ideal of the associated graded ring of S/I with respect to the maximal graded ideal $\mathfrak{m} = (x_1, \ldots, x_r)$, i.e. $S/I^* \cong \mathrm{gr}_{\mathfrak{m}}(S/I)$. One says that the polynomials $f_1, \ldots, f_s$ are a standard basis for I if $I^* = (f_1^*, \ldots, f_s^*)$.

We denote by $\bar{I}(\mathbf{a}+k) \subset S[x_0]$ the homogenization of $I(\mathbf{a}+k)$ with respect to a new variable x_0.

Most of our results deal with the situation when for the given $\mathbf{a}$ there are infinitely many shifts k such that $I(\mathbf{a}+k)$ is CI. This always takes place when $r \leq 3$, see [14, Theorem 3.1]. However, if $r > 3$ it is not always the case, see [10, Example 3.2]. A key observation in Theorem 1.3 is that if the ideal $I(\mathbf{a}+k)$ is CI for some $k \gg 0$, then it is minimally generated by a Gröbner basis with respect to revlex. If $I(\mathbf{a}+k)$ is CI for infinitely many k, then $J(\mathbf{a})$ is also CI, and in Theorem 1.13 we prove using Gröbner basis techniques that an intersection of CI ideals of the form $I(\mathbf{a}+k)$, $\bar{I}(\mathbf{a}+k)$, respectively $I(\mathbf{a}+k)^*$ is again a CI, assuming all these k are large enough.

In Example 2.4 we show that this closure property is not preserved when we intersect similarly defined Gorenstein ideals.

Infinite intersections of (not necessarily CI) ideals coming from the same shifted family are much tamer: they always produce $J(\mathbf{a})$, see Proposition 1.12.

Encouraged by numerical experiments with SINGULAR ([2]), in Section 2 we conjecture a periodic behaviour of the Betti numbers of intersections of toric ideals: for any $\mathcal{A} \subset \mathbb{N}$ with $\min \mathcal{A} \gg 0$

$$\beta_i(\cap_{k\in\mathcal{A}} I(\mathbf{a}+k)) = \beta_i(\cap_{k\in\mathcal{A}} I(\mathbf{a}+k+(a_r-a_1))) \text{ for all } i.$$

Similar statements are formulated regarding intersections of homogenizations or of ideals of initial forms, see Conjecture 2.2. We verify these in a few cases.

1. Intersections of toric complete intersections

The following result of Delorme characterizes the semigroups of $\mathbb{N}$ whose toric ideal is CI. It turns out that this is an arithmetic property of the semigroup, it does not depend on the field K. We therefore call a semigroup H, or a sequence of positive integers $\mathbf{a}$, a *complete intersection* if $K[H]$, respectively $K[\langle \mathbf{a} \rangle]$, has this property.

Theorem 1.1. *(Delorme,* [3, Proposition 10]*)*

Let H be a semigroup minimally generated by the sequence of positive integers $\mathbf{a} = a_1, a_2, \ldots, a_r$, and $d = \gcd(a_1, \ldots, a_r)$. The semigroup ring $K[H]$ is a complete intersection if and only if $r = 1$ or $\mathbf{a}$ can be written as a disjoint union

$$\mathbf{a} = k_1(b_{i_1}, \ldots, b_{i_s}) \sqcup k_2(b_{i_{s+1}}, \ldots, b_{i_r}), \tag{1}$$

where $k_1, k_2 > 1$ *with* $\gcd(k_1, k_2) = d$,

$$\begin{aligned} k_1/d &\in \langle b_{i_{s+1}}, \ldots, b_{i_r} \rangle \setminus \{b_{i_{s+1}} \ldots, b_{i_r}\}, \\ k_2/d &\in \langle b_{i_1}, \ldots, b_{i_s} \rangle \setminus \{b_{i_1}, \ldots, b_{i_s}\}, \\ \gcd(b_{i_1}, \ldots, b_{i_s}) &= \gcd(b_{i_{s+1}}, \ldots, b_{i_r}) = 1, \end{aligned}$$

and $K[\langle b_{i_1}, \ldots, b_{i_s} \rangle]$ *and* $K[\langle b_{i_{s+1}}, \ldots, b_{i_r} \rangle]$ *are complete intersections.*

A decomposition as in (1) is called a CI-split. From that we obtain one of the defining equations of $K[H]$ as follows. We may write

$$\begin{aligned} k_2/d &= \lambda_{i_1} b_{i_1} + \cdots + \lambda_{i_s} b_{i_s}, \\ k_1/d &= \lambda_{i_{s+1}} b_{i_{s+1}} + \cdots + \lambda_{i_r} b_{i_r}, \end{aligned}$$

with $\lambda_{i_1}, \ldots, \lambda_{i_r}$ nonnegative integers.

After multiplying these equations with k_1 and k_2 respectively, we get

$$\begin{aligned} k_1 k_2/d &= \lambda_{i_1} a_{i_1} + \cdots + \lambda_{i_s} a_{i_s} = \lambda_{i_{s+1}} a_{i_{s+1}} + \cdots + \lambda_{i_r} a_{i_r}, \text{ hence} \\ f &= x_{i_1}^{\lambda_{i_1}} \ldots x_{i_s}^{\lambda_{i_s}} - x_{i_{s+1}}^{\lambda_{i_{s+1}}} \ldots x_{i_r}^{\lambda_{i_r}} \in I(\mathbf{a}). \end{aligned} \tag{2}$$

We will work with the shifted family of $\mathbf{a} = a_1 < \cdots < a_r$. Notice that if $k > a_r - 2a_1$, then $a_1 + k, \ldots, a_r + k$ generate the semigroup $\langle \mathbf{a} + k \rangle$ minimally.

As the next result shows, more information is available about the CI ideals $I(\mathbf{a}+k)$ when k is large enough. We summarize some of the findings of Jayanthan and Srinivasan in [10] that we will employ.

Lemma 1.2. *([10]) Let* $\mathbf{a} = a_1 < \cdots < a_r$ *with* $r \geq 3$ *and* $k \geq (a_r - a_1)^2 - a_1$ *such that the sequence* $\mathbf{a} + k$ *is CI. Then*

(1) any CI-split for $\mathbf{a} + k$ *as in* (1) *satisfies*
  (i) $s = 1$ *or* $s = r - 1$.
  (ii) If $s = 1$, *then* $i_1 \notin \{1, r\}$.
  (iii) If $s = r - 1$, *then* $i_r \notin \{1, r\}$.

(2) $\mathbf{a} + k + \ell(a_r - a_1)$ *is CI for all* $\ell \geq 0$.

Proof. Part (1) reproduces Lemmas 1.2 and 1.3 in [10]. Part (2) may be obtained by carefully going through the proof of Theorem 2.1 in [10]. □

We use Gröbner bases techniques to derive new information about intersections of such CI ideals. We refer to [4, Chapter 15], [7] and [5] for the necessary background.

For any polynomial f in S we let $\mathrm{in}_<(f)$ be its initial term with respect to the graded reverse lexicographic order, or revlex for short. Also, $NF(f|I)$ denotes the normal form of f with respect to the reduced revlex Gröbner basis of I. The support $\mathrm{supp}(f)$ is the set of monomials in f. When f is a monomial, by abuse of notation we let $\mathrm{supp}(f) = \{i : x_i | f\}$.

Here is a first result, inspired by the work in [10].

Theorem 1.3. *Consider the sequence* $\mathbf{a} = a_1 < \cdots < a_r$ *and let* $k \geq (a_r - a_1)^2 - a_1$ *such that* $\mathbf{a} + k$ *is CI. Then the reduced Gröbner basis of the ideal* $I(\mathbf{a} + k)$ *computed with respect to revlex consists of binomials* $f_1, \ldots, f_{r-1}$ *such that* $f_1, \ldots, f_{r-2}$ *are* *homogeneous, their leading terms are pure powers of distinct variables $x_2, \dots, x_{r-1}$, and $f_{r-1} = x_1^u - x_r^v$ where $u > v > 0$.*

In particular, $I(\mathbf{a}+k)$ is minimally generated by its reduced Gröbner basis with respect to revlex.

Proof. If $r < 3$, then $I(\mathbf{a}+k)$ is a principal ideal and the statement is clear. Assume $r \geq 3$. By Lemma 1.2, we may assume the CI-split

$$\mathbf{a} + k = (a_{i_1} + k)(1) \sqcup k_2(b_1, \dots, b_{r-1}),$$

where $b_j = (a_{i_{j+1}} + k)/k_2$, for $1 \leq j \leq r-1$, $i_2 = 1$, $i_r = r$, $\gcd(b_1, \dots, b_{r-1}) = 1$ and $\langle b_1, \dots, b_{r-1} \rangle$ is CI.

As in (2), we obtain a first generator for $I(\mathbf{a}+k)$:

$$f_1 = x_{\alpha_1}^{k_2} - m_1,$$

where $\alpha_1 := i_1 \notin \{1, r\}$ and m_1 is a monomial in some of the remaining variables $T_1 = \{x_{i_2}, \dots, x_{i_r}\}$. Since $r \geq 3$, using Claim 1 in [10, Theorem 2.1] we obtain $k_2 = \deg m_1$, hence

$$k_2(a_{i_1} + k) = \sum_{\substack{1 \leq j \leq r \\ j \neq i_1}} \lambda_j (a_j + k), \text{ with } \sum_{j \neq i_1} \lambda_j = k_2 > 1.$$

Clearly, at least two λ_j's are nonzero. Denoting $w_1 = \min \operatorname{supp}(m_1)$ and $w_2 = \max \operatorname{supp}(m_1)$, we get

$$\left(\sum_{j \neq i_1} \lambda_j \right) (a_{w_1} + k) < k_2(a_{i_1} + k) < \left(\sum_{j \neq i_1} \lambda_j \right) (a_{w_2} + k),$$

$$\min \operatorname{supp}(m_1) < \alpha_1 < \max \operatorname{supp}(m_1) \text{ and } \operatorname{in}_<(f_1) = x_{\alpha_1}^{k_2}.$$

We note that $a_1 + k \geq (a_r - a_1)^2 > (a_r - a_1)^2/k_2$, hence $b_1 > (b_{r-1} - b_1)^2$ and we may apply the arguments above to the CI-sequence $b_1, \dots, b_{r-1}$. This produces a binomial generator

$$f_2 = x_{\alpha_2}^{k_2'} - m_2,$$

with $\alpha_2 \neq \alpha_1$ and m_2 a monomial in some of the variables $T_2 = T_1 \setminus \{x_{\alpha_2}\}$. If $|T_2| > 1$ then f_2 is homogeneous,

$$\min \operatorname{supp}(m_2) < \alpha_2 < \max \operatorname{supp}(m_2) \text{ and } \operatorname{in}_<(f_2) = x_{\alpha_2}^{k_2'}.$$

We continue finding homogeneous relations $f_3, \dots, f_{r-2}$ until the last step when only the variables x_1 and x_r are involved:

$$f_{r-1} = x_1^{(a_r+k)/d} - x_r^{(a_1+k)/d}, \tag{3}$$

where we let $d = \gcd(a_1 + k, a_r + k)$. Since $\gcd(\operatorname{in}_<(f_i), \operatorname{in}_<(f_j)) = 1$ for $1 \leq i < j \leq r-1$, by [5, Proposition 2.15] and the Buchberger criterion ([5, Theorem 2.14]) we conclude that $f_1, \dots, f_{r-1}$ form a Gröbner basis and a minimal generating set of $I(\mathbf{a}+k)$. By the way the polynomials f_i were constructed, they are the reduced Gröbner basis, as well. □

Corollary 1.4. *If $k \geq (a_r - a_1)^2 - a_1$ and $I(\mathbf{a}+k)$ is CI, then $\bar{I}(\mathbf{a}+k)$ and $I^*(\mathbf{a}+k)$ are CI, too, and they are minimally generated by their respective reduced Gröbner basis with respect to revlex.*

Proof. By [5, Proposition 3.15], $\bar{I}(\mathbf{a}+k)$ is generated by the homogenizations of the polynomials in the revlex Gröbner basis of $I(\mathbf{a}+k)$. With notation as in Theorem 1.3, this only changes f_{r-1} into $\bar{f}_{r-1} = x_1^{(a_r+k)/d} - x_0^{(a_r-a_1)/d} x_r^{(a_1+k)/d}$. The same argument as in the proof of Theorem 1.3 can be used to show that $\mathcal{G} = \{f_1, \ldots, f_{r-2}, \bar{f}_{r-1}\}$ is the reduced Gröbner basis of $\bar{I}(\mathbf{a}+k)$ with respect to revlex. The ideal $\bar{I}(\mathbf{a}+k)$ is the toric ideal associated to the semigroup $\langle (0, a_r+k), (a_1+k, a_r-a_1), (a_2+k, a_r-a_2), \ldots, (a_r+k, 0)\rangle \subseteq \mathbb{N}^2$, hence height $\bar{I}(\mathbf{a}+k) = r-1$ and $\bar{I}(\mathbf{a}+k)$ is CI.

At the same time, $\mathcal{G}$ is a Gröbner basis with respect to the block term order obtained by using the lexicographic order on the variable x_0, revlex on the rest, and which extends $x_0 > x_1 > \cdots > x_r$. Therefore, by [4, §15.10.3], we get $I^*(\mathbf{a}+k) = (f_1, \ldots, f_{r-2}, f^*_{r-1})$, and the rest follows from Buchberger's criterion. □

Definition 1.5. *For a sequence of nonnegative integers $\mathbf{a} = a_1, \ldots, a_r$ we let*

$$J(\mathbf{a}) = (f \in I(\mathbf{a}) : f \text{ is homogeneous}) \subseteq S.$$

It is easy to see that $J(\mathbf{a}) = J(\mathbf{a}+k)$ for all $k \geq 0$. Also, $J(\mathbf{a})$ is the toric ideal of the semigroup $\langle (a_1, 1), \ldots, (a_r, 1)\rangle \subset \mathbb{Z}^2$, hence $J(\mathbf{a})$ is a prime ideal in S of height $r-2$.

Corollary 1.6. *With notation as above, if $\mathbf{a}+k$ is CI for some $k \geq (a_r - a_1)^2 - a_1$, then $J(\mathbf{a})$ is CI. Moreover, $J(\mathbf{a})$ is minimally generated by its reduced Gröbner basis with respect to revlex.*

Proof. If $r < 3$ then $J(\mathbf{a}) = 0$ and the statement is clear. Assume $r \geq 3$. By Lemma 1.2 we may add to $\mathbf{a}+k$ any positive multiple of $(a_r - a_1)$ and still get a CI sequence. To simplify notation, we may assume that the given k is arbitrarily large. Using the notation from Theorem 1.3 we claim that

$$J(\mathbf{a}) = (f_1, \ldots, f_{r-2}).$$

Set $U = (f_1, \ldots, f_{r-2})$. Arguing as in the proof of Theorem 1.3 we obtain that the given generators of U are the reduced Gröbner basis with respect to revlex.

Clearly $U \subseteq J(\mathbf{a})$ since f_i is homogeneous for $1 \leq i \leq r-2$. If $U \neq J(\mathbf{a})$ we may pick a polynomial of minimal degree $f \in J(\mathbf{a}) \setminus U$, such that f is part of a minimal homogeneous generating system for $J(\mathbf{a})$. We may write

$$f = \sum_{i=1}^{r-2} q_i f_i + g,$$

where $q_i \in S$, $\mathrm{in}_<(q_i f_i) \leq \mathrm{in}_<(f)$ for $1 \leq i \leq r-2$ and no term of g is in $\mathrm{in}_<(U)$, see [5, Theorem 2.11]. Note that the degrees of f and of $f_1, \ldots f_{r-2}$ or g do not depend on k. Also, $\deg g \leq \deg f$. Since $f \notin U$ we get $g \neq 0$. Moreover, $\mathrm{in}_<(g)$ is not divisible by $\mathrm{in}_<(f_i)$, for $1 \leq i \leq r-2$. Yet $f_1, \ldots, f_{r-1}$ is a Gröbner basis for $I(\mathbf{a}+k)$ and $f \in I(\mathbf{a}+k)$, hence with notation as in (3) we get $x_1^{(a_r+k)/d} = \mathrm{in}_<(f_{r-1}) \mid \mathrm{in}_<(g)$, which for degree reasons is a contradiction to the fact that $k \gg 0$. Hence $J(\mathbf{a}) = U$ is a CI ideal. $\square$

Remark 1.7. Vu [15] proves that for any $\mathbf{a}$ there exists an $N > 0$ such that for any $k > N$, in the Betti table of $I(\mathbf{a} + k)$ the upper rows are the same as in the Betti table in $J(\mathbf{a})$ and only the lower rows change with k. This is another way to prove that if $\mathbf{a} + k$ is CI for some $k \gg 0$, then $J(\mathbf{a})$ is CI, too. Similarly, for $k \gg 0$ by [15, Theorem 5.7] and [9, Theorem 1.4], the ideals $\bar{I}(\mathbf{a}+k)$ and $I^*(\mathbf{a}+k)$ have the same Betti table as $I(\mathbf{a} + k)$, hence these are all CI having also the same height.

By inspecting the formula for N introduced in [15, Eq. 1.1] it is easy to see that $N > (a_r - a_1)^2 - a_1$.

Theorem 1.3 and its corollaries may be formulated without referring to the shift k.

Corollary 1.8. *If $\mathbf{a} = a_1 < \cdots < a_r$ is a CI sequence such that $a_1 \geq (a_r - a_1)^2$, then $\bar{I}(\mathbf{a}), I(\mathbf{a})^*$ and $J(\mathbf{a})$ are also CI. Moreover, the ideals $I(\mathbf{a}), \bar{I}(\mathbf{a}), I(\mathbf{a})^*$ and $J(\mathbf{a})$ are minimally generated by their reduced Gröbner basis with respect to revlex.*

We work with intersections of toric ideals coming from the same shifted family. The following observation is straightforward.

Lemma 1.9. *Let $k_1 \neq k_2$ and $f = \boldsymbol{x}^{\boldsymbol{\alpha}} - \boldsymbol{x}^{\boldsymbol{\beta}}$ in $I(\mathbf{a} + k_1) \cap I(\mathbf{a} + k_2)$. Then f is homogeneous.*

Proof. We denote $\langle \cdot, \cdot \rangle$ the standard scalar product on $\mathbb{R}^r$ and $|\boldsymbol{v}| = \sum_{i=1}^r v_i$ for any $\boldsymbol{v} = (v_1, \ldots, v_r) \in \mathbb{R}^r$. Since $f \in I(\mathbf{a} + k_1) \cap I(\mathbf{a} + k_2)$ we obtain that $\langle \boldsymbol{\alpha}, \mathbf{a} + k_1 \rangle = \langle \boldsymbol{\beta}, \mathbf{a} + k_1 \rangle$ and $\langle \boldsymbol{\alpha}, \mathbf{a} + k_2 \rangle = \langle \boldsymbol{\beta}, \mathbf{a} + k_2 \rangle$. By subtracting these equations we get that $k_1(|\boldsymbol{\alpha}| - |\boldsymbol{\beta}|) = k_2(|\boldsymbol{\alpha}| - |\boldsymbol{\beta}|)$, hence $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ and f is homogeneous. $\square$

Remark 1.10. While the toric ideals $I(\mathbf{a} + k_1)$ and $I(\mathbf{a} + k_2)$ are generated by binomials, this is no longer true for their intersection.

Indeed, if for $k_1 \neq k_2$ the ideal $I(\mathbf{a} + k_1) \cap I(\mathbf{a} + k_2)$ is generated by binomials, by Lemma 1.9 these are homogeneous, hence $J(\mathbf{a}) \subseteq I(\mathbf{a} + k_1) \cap I(\mathbf{a} + k_2)$. The reverse inclusion always holds, hence $J(\mathbf{a}) = I(\mathbf{a} + k_1) \cap I(\mathbf{a} + k_2)$. For $1 \leq i \leq 2$, pick $f_i = m_{i,1} - m_{i,2}$ in $I(\mathbf{a} + k_i)$ with $m_{i,1}, m_{i,2}$ monomials and $\deg m_{i,1} > \deg m_{i,2}$. Then $f_1 f_2 \in J(\mathbf{a})$, hence its homogeneous component of maximal degree, namely $m_{1,1} m_{2,1}$, is also in $J(\mathbf{a})$, which is false since toric ideals do not contain monomials.

Definition 1.11. *Let $\mathbf{a} = a_1 < \cdots < a_r$ be a sequence of nonnegative integers and $\mathcal{A} \subset \mathbb{N}$. We introduce*

$$\begin{aligned} \mathcal{I}_{\mathcal{A}}(\mathbf{a}) &= \bigcap_{k \in \mathcal{A}} I(\mathbf{a} + k), \\ \mathcal{J}_{\mathcal{A}}(\mathbf{a}) &= \bigcap_{k \in \mathcal{A}} I^*(\mathbf{a} + k), \\ \mathcal{H}_{\mathcal{A}}(\mathbf{a}) &= \bigcap_{k \in \mathcal{A}} \bar{I}(\mathbf{a} + k). \end{aligned}$$

The next result shows that when we intersect infinitely many toric ideals (or the ideals of their initial forms) in the same shifted family, the result does not the depend on the family $\mathcal{A}$ of shifts.

Proposition 1.12. *Assume $\mathcal{A}$ is an infinite set of nonnegative integers. Then*

$$\begin{aligned}\mathcal{I}_{\mathcal{A}}(\mathbf{a}) &= \mathcal{J}_{\mathcal{A}}(\mathbf{a}) = J(\mathbf{a}),\\ \mathcal{H}_{\mathcal{A}}(\mathbf{a}) &= J(\mathbf{a})S[x_0].\end{aligned}$$

Proof. Since $J(\mathbf{a})$ is generated by homogeneous polynomials and $J(\mathbf{a}) = J(\mathbf{a}+k)$, we have the inclusions $J(\mathbf{a}) \subseteq \mathcal{I}_{\mathcal{A}}(\mathbf{a})$, $J(\mathbf{a}) \subseteq \mathcal{J}_{\mathcal{A}}(\mathbf{a})$ and $J(\mathbf{a})S[x_0] \subseteq \mathcal{H}_{\mathcal{A}}(\mathbf{a})$. We settle the reverse inclusions one by one.

Let $f \in \mathcal{I}_{\mathcal{A}}(\mathbf{a})$. If $f = 0$, we are done. If $f \neq 0$, let $m = \deg f$. We may write $f = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} = \sum_{i=0}^{m} (\sum_{|\boldsymbol{\alpha}|=i} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}})$, with $c_{\boldsymbol{\alpha}} \in K$. Pick $k \in \mathcal{A}$ such that $k > \max\{\langle \boldsymbol{\alpha}, \mathbf{a}\rangle : c_{\boldsymbol{\alpha}} \neq 0\}$. Since $f \in I(\mathbf{a}+k)$ we get

$$0 = \sum_{i=0}^{m} \left(\sum_{|\boldsymbol{\alpha}|=i} c_{\boldsymbol{\alpha}} t^{\langle \boldsymbol{\alpha}, \mathbf{a}+k\rangle} \right) = \sum_{i=0}^{m} \left(\sum_{|\boldsymbol{\alpha}|=i} c_{\boldsymbol{\alpha}} t^{\langle \boldsymbol{\alpha}, \mathbf{a}\rangle} \right) t^{ki}.$$

Letting $f_i = \sum_{|\boldsymbol{\alpha}|=i} c_{\boldsymbol{\alpha}} t^{\langle \boldsymbol{\alpha}, \mathbf{a}\rangle}$ we notice that when $i \neq j$ the polynomials $f_i t^{ki}$ and $f_j t^{kj}$ have no common monomials. Hence $f_i = 0$ and the i^{th} graded component of f, $\sum_{|\boldsymbol{\alpha}|=i} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in J(\mathbf{a})$ for all i. This gives $f \in J(\mathbf{a})$ and $\mathcal{I}_{\mathcal{A}}(\mathbf{a}) = J(\mathbf{a})$.

Let $f \in \mathcal{J}_{\mathcal{A}}(\mathbf{a})$. Since an intersection of homogeneous ideals is again homogeneous, we may reduce to the case $f = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$ is homogeneous of degree d. Pick $k \in \mathcal{A}$ such that $k > \max\{\langle \boldsymbol{\alpha}, \mathbf{a}\rangle : c_{\boldsymbol{\alpha}} \neq 0\}$. Then $f \in I(\mathbf{a}+k)^*$, hence there exists $g(\boldsymbol{x}) = \sum_{\boldsymbol{\beta}} d_{\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}}$ in S such that $(f+g)^* = f$ and $f+g \in I(\mathbf{a}+k)$. Therefore $|\boldsymbol{\beta}| > d$ whenever $d_{\boldsymbol{\beta}} \neq 0$ and

$$\begin{aligned}\left(\sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} t^{\langle \boldsymbol{\alpha}, \mathbf{a}+k\rangle} \right) &+ \sum_{\boldsymbol{\beta}} d_{\boldsymbol{\beta}} t^{\langle \boldsymbol{\beta}, \mathbf{a}+k\rangle} = 0,\\ \left(\sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} t^{\langle \boldsymbol{\alpha}, \mathbf{a}\rangle} \right) t^{kd} &+ \sum_{\boldsymbol{\beta}} d_{\boldsymbol{\beta}} t^{\langle \boldsymbol{\beta}, \mathbf{a}\rangle} \cdot t^{k|\boldsymbol{\beta}|} = 0.\end{aligned}$$

By our choice of k we get that all monomials in the first summand of the previous equation have degree smaller then $(k+1)d$, thus $\sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} t^{\langle \boldsymbol{\alpha}, \mathbf{a}\rangle} = 0$ and since f is homogeneous $f \in J(\mathbf{a})$, too. Thus $\mathcal{J}_{\mathcal{A}}(\mathbf{a}) = J(\mathbf{a})$.

Let $f \in \mathcal{H}_{\mathcal{A}}(\mathbf{a})$. Arguing as above we reduce to the case when f is homogeneous of degree d in $S[x_0]$. We may write $f(x_0, \boldsymbol{x}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} x_0^{d-|\boldsymbol{\alpha}|}$. We dehomogenize by substituting $x_0 = 1$ and we get $f(1, \boldsymbol{x}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \in I(\mathbf{a}+k)$ for all k in $\mathcal{A}$. Hence $f(1, \boldsymbol{x}) \in \mathcal{I}_{\mathcal{A}}(\mathbf{a}) = J(\mathbf{a})$. The homogeneous components of $f(1, \boldsymbol{x})$ are in $J(\mathbf{a})$, too. Hence $f \in J(\mathbf{a})S[x_0]$. This shows the desired equality $\mathcal{H}_{\mathcal{A}}(\mathbf{a}) = J(\mathbf{a})S[x_0]$. □

We shall now state the main result of this note.

Theorem 1.13. *Let $\mathcal{A} \subseteq \mathbb{N}$ such that $\min \mathcal{A} \geq (a_r - a_1)^2 - a_1$. If $I(\mathbf{a}+k)$ is CI for all $k \in \mathcal{A}$, then $\mathcal{I}_{\mathcal{A}}(\mathbf{a})$, $\mathcal{J}_{\mathcal{A}}(\mathbf{a})$ and $\mathcal{H}_{\mathcal{A}}(\mathbf{a})$ are CI, as well.*

Proof. We first consider the case when $\mathcal{A}$ is finite. If $|\mathcal{A}| = 1$, by Corollary 1.4 there is nothing more to prove. Assume $|\mathcal{A}| > 1$. By Corollary 1.6, for any $k \in \mathcal{A}$ we have

$$I(\mathbf{a}+k) = J(\mathbf{a}) + (f_{r-1,k}),$$

where $f_{r-1,k} = x_1^{(a_r+k)/d_k} - x_r^{(a_1+k)/d_k}$ and $d_k = \gcd(a_1+k, a_r+k)$. Denote

$$f_{\mathcal{A}} = \mathrm{lcm}(f_{r-1,k} : k \in \mathcal{A}).$$

We prove that

$$\mathcal{I}_{\mathcal{A}}(\mathbf{a}) = J(\mathbf{a}) + (f_{\mathcal{A}}). \tag{4}$$

The "$\supseteq$" inclusion is obvious. For the other one, let $f \in \mathcal{I}_{\mathcal{A}}(\mathbf{a})$. If $f \in J(\mathbf{a})$, we are done. Assume $f \notin J(\mathbf{a})$.

For any $k \in \mathcal{A}$ there exist $j_k \in J(\mathbf{a})$ and $g_k \in S$ such that $f = j_k + g_k f_{r-1,k}$. Without loss of generality we may assume $g_k = NF(g_k|J(\mathbf{a}))$. Indeed, if we let $g_k = j'_k + NF(g_k|J(\mathbf{a}))$ we may use the decomposition $f = (j_k + j'_k f_{r-1,k}) + NF(g_k|J(\mathbf{a})) \cdot f_{r-1,k}$.

Let $k \neq \ell$ shifts in $\mathcal{A}$. Then $f = j_k + g_k f_{r-1,k} = j_\ell + g_\ell f_{r-1,\ell}$. We claim that $j_k = j_\ell$. If we assume $F = g_k f_{r-1,k} - g_\ell f_{r-1,\ell}$ is nonzero, then $\mathrm{in}_<(F) = c \cdot m_1 \cdot m_2$ where $c \in K$, m_1 is a monomial in $\mathrm{supp}(g_k) \cup \mathrm{supp}(g_\ell)$ and m_2 is a monomial in $\mathrm{supp}(f_{r-1,k}) \cup \mathrm{supp}(f_{r-1,\ell}) = \{x_1^{\alpha_k}, x_1^{\alpha_\ell}, x_r^{\beta_k}, x_r^{\beta_\ell}\}$. Since $F \in J(\mathbf{a})$, by Corollary 1.6 there exists $1 \leq i \leq r-2$ such that $\mathrm{in}_<(f_i) = x_{q_i}^{\alpha_i} | \mathrm{in}_<(F)$. Hence $\mathrm{in}_<(f_i)|m_1$ and g_k or g_ℓ may be reduced modulo $J(\mathbf{a})$, a contradiction. Set $j = j_k$ for some (actually for all) $k \in \mathcal{A}$. Since $f = j + g_k \cdot f_{r-1,k}$, we get $f_{r-1,k}|f-j$ for all $k \in \mathcal{A}$ and $f_{\mathcal{A}}|f-j$. Therefore $f \in J(\mathbf{a}) + (f_{\mathcal{A}})$, which finishes the proof of (4).

Note that $f_{\mathcal{A}}$ is regular on the domain $S/J(\mathbf{a})$, and using Corollary 1.6 we conclude that $\mathcal{I}_{\mathcal{A}}(\mathbf{a})$ is CI.

The statement about $\mathcal{H}_{\mathcal{A}}(\mathbf{a})$ is proven along the same lines as above using Corollary 1.4 and the observation, similar to (4), that $\mathcal{H}_{\mathcal{A}}(\mathbf{a}) = J(\mathbf{a})S[x_0] + (\bar{f}_{\mathcal{A}})$.

For k in our range, by Corollary 1.4 $I(\mathbf{a}+k)$ is generated by a standard basis. Therefore $I^*(\mathbf{a}+k) = J(\mathbf{a}) + (f^*_{r-1,k}) = J(\mathbf{a}) + (x_r^{\beta_k})$, hence $\mathcal{J}_{\mathcal{A}}(\mathbf{a}) = I^*(\mathbf{a}+k_0)$ for some k_0 in $\mathcal{A}$.

If the set $\mathcal{A}$ of shifts is infinite, by Proposition 1.12 the desired intersections are $\mathcal{I}_{\mathcal{A}}(\mathbf{a}) = \mathcal{J}_{\mathcal{A}}(\mathbf{a}) = J(\mathbf{a})$ and $\mathcal{H}_{\mathcal{A}}(\mathbf{a}) = J(\mathbf{a})S[x_0]$, which are CI by Corollary 1.6. This completes the proof of the theorem. $\square$

Corollary 1.14. *Under the above conditions $\mathcal{I}_{\mathcal{A}}(\mathbf{a})^*$ is CI.*

Proof. We first assume $\mathcal{A}$ is finite. Denote by $\widetilde{\ }$ the image under the K-algebra map $\pi : S \to K[x_2, \ldots, x_r]$ letting $\pi(x_1) = 0$ and $\pi(x_i) = x_i$ for all $i > 1$. Using the notation from the proof of Theorem 1.13, equation (4) gives $\widetilde{\mathcal{I}_{\mathcal{A}+k}(\mathbf{a})} = \widetilde{J(\mathbf{a})} + (\widetilde{f_{\mathcal{A}}})$, where $f_{\mathcal{A}}$ is the lcm in S of a finite number of binomials of the form $x_1^{e_1} - x_r^{e_r}$ with $e_1 > e_r > 1$ and $\gcd(e_1, e_r) = 1$. We claim that binomials of this form are irreducible in S. Indeed, we have an isomorphim of K-algebras $K[x_1, x_r]/(x_1^{e_1} - x_r^{e_r}) \cong K[t^{e_r}, t^{e_1}]$. The latter is a domain, hence $x_1^{e_1} x_r^{e_r}$ is irreducible in $K[x_1, x_r]$ and in $K[x_1, \ldots, x_r]$. Therefore $\widetilde{f_{\mathcal{A}}} = x_r^e$ and $f^*_{\mathcal{A}} = x_r^e$ for some positive integer e.

Since $J(\mathbf{a})$ is generated by homogeneous binomials, we see that $\widetilde{\mathcal{I}_{\mathcal{A}}(\mathbf{a})}$ is generated by a set of monomials and homogeneous binomials that naturally form a standard basis $\mathcal{G}$. It is immediate to see that for any $g \in \mathcal{G}$ there exists $f \in S$ such that $\widetilde{f} = g$ and $\deg f^* = \deg g^*$. By a result of Herzog in [8] (see also [9, Lemma 1.2] for a formulation which is better suited to our situation) we conclude that the $r-1$ generators of $\mathcal{I}_{\mathcal{A}}(\mathbf{a})$ in (4) are also a standard basis, hence $\mathcal{I}_{\mathcal{A}+k}(\mathbf{a})^*$ is CI.

When $\mathcal{A}$ is infinite, by Proposition 1.12 we have $\mathcal{I}_{\mathcal{A}}(\mathbf{a}) = J(\mathbf{a})$, hence $\mathcal{I}_{\mathcal{A}}(\mathbf{a})^* = J(\mathbf{a})$. Conclusion follows by Corollary 1.6. $\square$

2. Questions and examples

Several periodic features have been noticed for the Betti numbers of the toric ideal and other ideals attached to large enough shifts of a numerical semigroup, see [15], [9], [10], [14]. We summarize the most important ones below.

Let $\mathbf{a} = a_1, \ldots, a_r$ be an increasing sequence of nonnegative integers.

Theorem 2.1. *For all $k \gg 0$ and all i one has*

(i) (Vu, [15, Theorem 1.1]*)* $\beta_i(I(\mathbf{a}+k)) = \beta_i(I(\mathbf{a}+k+(a_r-a_1)))$,

(ii) (Herzog–Stamate, [9, Theorem 1.4]*)* $\beta_i(I(\mathbf{a}+k)) = \beta_i(I(\mathbf{a}+k)^*)$,

(iii) (Vu, [15, Theorem 5.7]*)* $\beta_i(I(\mathbf{a}+k)) = \beta_i(\bar{I}(\mathbf{a}+k))$.

Numerical experiments with SINGULAR ([2]) encourage us to believe that similar periodicities occur for the Betti numbers of intersections of these ideals, as well.

Conjecture 2.2. *With notation as above, if* $\min \mathcal{A} \gg 0$ *then for all i one has*

(i) $\beta_i(\mathcal{I}_{\mathcal{A}}(\mathbf{a})) = \beta_i(\mathcal{I}_{\mathcal{A}+(a_r-a_1)}(\mathbf{a}))$,

(ii) $\beta_i(\mathcal{J}_{\mathcal{A}}(\mathbf{a})) = \beta_i(\mathcal{J}_{\mathcal{A}+(a_r-a_1)}(\mathbf{a}))$,

(iii) $\beta_i(\mathcal{I}_{\mathcal{A}}(\mathbf{a})) = \beta_i(\mathcal{I}_{\mathcal{A}}(\mathbf{a})^*) = \beta_i(\mathcal{H}_{\mathcal{A}}(\mathbf{a}))$.

Proposition 2.3. *In any of the following situations, Conjecture 2.2 holds:*

(i) $\mathcal{A}$ is infinite,

(ii) $I(\mathbf{a}+k)$ is CI for all $k \in \mathcal{A}$ and $\min \mathcal{A} \geq (a_r - a_1)^2 - a_1$.

Proof. By Proposition 1.12, this conjecture is verified when $\mathcal{A}$ is infinite since all the intersections that occur are $J(\mathbf{a})$ or its extension in $S[x_0]$.

For part *(ii)*: by Lemma 1.2 we have that $I(\mathbf{a}+k+(a_r-a_1))$ is again CI for all $k \in \mathcal{A}$. Using Theorem 1.13 we have that the intersections $\mathcal{I}_{\mathcal{A}}(\mathbf{a})$, $\mathcal{J}_{\mathcal{A}}(\mathbf{a})$, $\mathcal{H}_{\mathcal{A}}(\mathbf{a})$, and $\mathcal{I}_{\mathcal{A}+(a_r-a_1)}(\mathbf{a})$, $\mathcal{J}_{\mathcal{A}+(a_r-a_1)}(\mathbf{a})$, $\mathcal{H}_{\mathcal{A}+(a_r-a_1)}(\mathbf{a})$ are all CI of the same codimension. This settles (i) , (ii) and one of the equalities in part (iii) of Conjecture 2.2. For the remaining equation we use Corollary 1.14. $\square$

The main results in Section 1 hold for shifts $k \gg 0$. Even though $I(\mathbf{a}+k)$ may be CI for infinitely many k, when we intersect two CI ideals $I(\mathbf{a}+k_1)$ and $I(\mathbf{a}+k_2)$ for k_1 or k_2 not large enough, the result might not be again a CI.

Let $\mathbf{a} = 0, 6, 15$. As noted in [14, Table 1], for $k \geq 25$, $I(\mathbf{a}+k)$ is CI if and only if k is divisible by 5. Still, the ideal $I(\mathbf{a}+k) \subset K[x,y,z]$ is a CI for $k = 8$ and $k = 10$, and a SINGULAR ([2]) computation shows that

$$\begin{aligned} I(8,14,23) \cap I(10,16,25) &= (z^2 - x^4y, x^7 - y^4) \cap (y^5 - x^3z^2, x^5 - z^2) \\ &= (y^5 - x^3z^2, x^9y - x^5z^2 - x^4yz^2 + z^4, x^{12} - x^5y^4 - x^7z^2 + y^4z^2) \end{aligned}$$

is not a CI.

It is natural to ask if the CI property may be replaced by Gorenstein in Theorem 1.13 or in Corollary 1.6. We give a negative answer by using Example 2.4 and the series of remarks that follow it.

Example 2.4. Let $\mathbf{a} = 0, 1, 2, 3$. According to [12, Corollary 6.2] (see also [6, §2]), the ideal $I(\mathbf{a} + k) \subset S = K[x_1, x_2, x_3, x_4]$ is Gorenstein if and only if $k \equiv 2 \bmod 3$. For the rest of the paper set $I_k = I(\mathbf{a} + k)$.

If $k = 3\ell + 2$ for some $\ell > 0$, by [11] (or [6])

$$I_k = (x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_2x_4, x_1^{\ell+1}x_2 - x_4^{\ell+1}, x_1^{\ell+2} - x_4^{\ell}x_3), \tag{5}$$

and thus

$$J(\mathbf{a}) = (x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_2x_4). \tag{6}$$

Remark 2.5. For any $k = 3\ell + 2 > 2$, the ideal $I_k \cap I_{k+3}$ is not Gorenstein.

Using any algorithm for computing the intersection of two ideals (e.g. [5, Proposition 3.5]) one can check that

$$\begin{aligned} I_k \cap I_{k+3} = J(\mathbf{a}) + (\ & x_1^{2\ell+4}x_3 - x_1^{\ell+2}x_2x_4^{\ell+1} - x_1^{\ell+1}x_2x_4^{\ell+2} + x_4^{2\ell+3}, \\ & x_1^{2\ell+4}x_2 - x_1^{\ell+3}x_4^{\ell+1} - x_1^{\ell+2}x_4^{\ell+2} + x_3x_4^{2\ell+2}, \\ & x_1^{2\ell+5} - x_1^{\ell+3}x_3x_4^{\ell} - x_1^{\ell+2}x_3x_4^{\ell+1} + x_2x_4^{2\ell+2}). \end{aligned}$$

As x_1 is regular on both S/I_k and S/I_{k+3}, it is regular on $S/(I_k \cap I_{k+3})$, too. Using reduction modulo x_1, it is enough to show that the ideal $(x_1, I_k \cap I_{k+3})$ is not Gorenstein. Letting $R = S/(x_1, I_k \cap I_{k+3})$ we notice that

$$(x_1, I_k \cap I_{k+3}) = (x_1, x_3^2 - x_2x_4, x_2x_3, x_2^2, x_4^{2\ell+3}, x_2x_4^{2\ell+2}, x_3x_4^{2\ell+2}) \tag{7}$$

and that the residue classes $u = \widehat{x_4^{2\ell+2}}$ and $v = \widehat{x_2x_4^{2\ell+1}}$ are in $\mathrm{Soc}(R)$.

We claim that u and v are linearly independent over K, hence the Cohen-Macaulay type of the Artinian ring R is not one, and R is not a Gorenstein ring. Indeed, if $\mu u + \lambda v = 0$ for some $\mu, \lambda \in K$, then $w := \mu x_4^{2\ell+2} + \lambda x_2x_4^{2\ell+1} \in (x_1, I_k \cap I_{k+3})$. It is routine to check that the generators in (7) are also a Gröbner basis with respect to revlex, thus if $\mu, \lambda \neq 0$, then $\mathrm{in}_<(w)$ divides the leading term of some polynomial in the Gröbner basis in (7), which gives a contradiction.

Remark 2.6. The ideal $J(\mathbf{a})$ in (6) is not Gorenstein.

Indeed, as x_1 is regular on the domain $S/J(\mathbf{a})$ and a routine check shows

$$(x_1, J(\mathbf{a})) : (x_4) = (x_1, x_2^2, x_2x_3, x_3^2 - x_2x_4) : (x_4) = (x_1, x_2^2, x_2x_3, x_3^2 - x_2x_4),$$

we get that $\{x_1, x_4\}$ is a regular sequence on $S/J(\mathbf{a})$. The type of $S/J(\mathbf{a})$ equals $\dim_K \mathrm{Soc}(S/(x_1, x_4, J(\mathbf{a}))) = \dim_K \mathrm{Soc}(S/(x_1, x_4, x_2^2, x_2x_3, x_3^2)) = 2$, hence $S/J(\mathbf{a})$ is not a Gorenstein ring.

Remark 2.7. According to [13, Corollary 2.4] (or the proof of [9, Proposition 2.5]), the generators in (5) are also a standard basis for I_k, hence

$$I_k^* = J(\mathbf{a}) + (x_4^{\ell+1}, x_4^{\ell}x_3).$$

Clearly $I_k^* \supset I_{k+3}^*$, and using [9, Proposition 2.5] both are Gorenstein ideals because I_k and I_{k+3} are so. Thus $I_k^* \cap I_{k+3}^* = I_{k+3}^*$ is a Gorenstein ideal, and this shows that in general

$$\beta_i(I_k \cap I_{k+3}) \neq \beta_i(I_k^* \cap I_{k+3}^*).$$

Hence Conjecture 2.2(iii) can not be improved by adding the equality $\beta_i(\mathcal{I}_{\mathcal{A}}(\mathbf{a})) = \beta_i(\mathcal{J}_{\mathcal{A}}(\mathbf{a}))$, which is nevertheless true when $|\mathcal{A}| = 1$, cf. Theorem 2.1(ii).

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References

[1] W. Bruns, J. Herzog, *Cohen–Macaulay rings*, revised ed., Cambridge Studies in Advanced Mathematics, vol. **39**, Cambridge University Press, Cambridge (1998).

[2] W. Decker, G.-M. Greuel, G. Pfister, H. Schönemann, SINGULAR 3-1-6 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2012).

[3] C. Delorme, *Sous-monoïdes d'intersection complète de N*, Ann. Sci. Ecole Norm. Sup. (4) **9** (1976), no. 1, 145–154.

[4] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, vol. **150**, Springer, 1995.

[5] V. Ene, J. Herzog, *Gröbner Bases in Commutative Algebra*, Graduate Studies in Mathematics, vol. **130**, American Mathematical Society, 2012.

[6] P. Gimenez, I. Sengupta, I. Srinivasan, *Minimal graded free resolutions for monomial curves defined by arithmetic sequences*, J. Algebra **388** (2013) 294–310.

[7] G.-M. Greuel, G. Pfister, *A Singular Introduction to Commutative Algebra*, 2nd ed., Springer, 2008.

[8] J. Herzog, *When is a regular sequence super regular?*, Nagoya Math. J. **83** (1981), 183–195.

[9] J. Herzog, D. I. Stamate, *On the defining equations of the tangent cone of a numerical semigroup ring*, J. Algebra **418** (2014), 8–28.

[10] A.V. Jayanthan, H. Srinivasan, *Periodic occurence of complete intersection monomial curves*, Proc. Amer. Math. Soc. **141** no 12 (2013), 4199–4208.

[11] D.P. Patil, *Minimal sets of generators for the relation ideals of certain monomial curves*, Manuscripta Math. **80** (1993), 239–248.

[12] D.P. Patil, I. Sengupta, *Minimal set of generators for the derivation module of certain monomial curves*, Comm. Algebra **27** (1999), 5619–5631.

[13] L. Sharifan, R. Zaare-Nahandi, *Minimal free resolution of the associated graded ring of monomial curves of generalized arithmetic sequences*, J. Pure and Applied Algebra **213** (2009), 360–369.

[14] D. I. Stamate, *Asymptotic properties in the shifted family of a numerical semigroup with few generators*, to appear in Semigroup Forum, doi:10.1007/s00233-015-9724-2.

[15] T. Vu, *Periodicity of Betti numbers of monomial curves*, J. Algebra **418** (2014), 66–90.

Mircea Cimpoeaş, Simion Stoilow Institute of Mathematics of the Romanian Academy, Research unit 5, P.O.Box 1-764, Bucharest 014700, Romania
E-mail address: mircea.cimpoeas@imar.ro

Dumitru I. Stamate, Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, Bucharest, Romania, and
Simion Stoilow Institute of Mathematics of the Romanian Academy, Research group of the project PN-II-RU-PD-2012-3-0656, P.O.Box 1-764, Bucharest 014700, Romania
E-mail address: dumitru.stamate@fmi.unibuc.ro