CUMULATIVE SCIENTIFIC REPORT FOR THE PROJECT PN-II-RU-PD-2012-3-0656 DURING MAY 2013– DECEMBER 2014

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General data

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Contents

1. Brief introduction in the theme of the project

Given a graded algebra $R = \oplus_i R_i$ and M a finitely generated graded R-module, we are interested in studying the minimal free graded resolution of M over R. Its invariants (the Betti numbers $\beta_i^R(M)$) are very important in understanding the equations describing M. In many relevant cases, e.g. R is a polynomial ring algebra and $M = I$ is a graded ideal, there is some extra structure available for this module, and that reflects the properties of the module. Several techniques from combinatorics, toplogy or homological algebra become available. This is an area of much current interest, for details we refer to the monographs [4], [22], [30], [23].

When R or M are not already graded, it is a useful way to reduce to the graded case by considering the filtration induced by the powers of an ideal $\mathfrak{m} \subset R$ and one constructs the associated graded ring

$$\operatorname{gr}_{\mathfrak{m}} R = R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \cdots$$

and similarly for $\operatorname{gr}_{\mathfrak{m}} M$. Of course, several properties may disappear in this transformation, however one knows that the Betti numbers may only increase: $\beta_i^R(M) \leq \beta_i^{\operatorname{gr}_{\mathfrak{m}} R}(\operatorname{gr}_{\mathfrak{m}} M)$. We refer to [1] for more details on this topic. From the geometric point of view, if $(R, \mathfrak{m})$ is the local ring at a point (say the origin) of a variety, its associated graded ring is the ring of coordinates of the tangent cone at the point on the variety, cf. [9], [27].

A special class of graded rings $R = \oplus_i R_i$ is the one where R_0 is a field (or a semisimple ring) and it has an R-linear resolution. In this case, R is called Koszul. For an algebra that is not graded *ab initio*, we may use the idea above to pass to the associated graded ring (with respect to a suitable filtration) and we may study the Koszul property for this, as well. This idea has been successfully exploited by V. Reiner and the director of the grant in [25], and there are perspectives to extend its applications.

Already for a standard graded algebra it is rather difficult to check the Koszul property (see [12], [24], [6]). Conca, Trung si Valla ([7]), exploring an idea of Herzog et al([17]) showed that if an algebra R has a so called *Koszul filtration*, then R is Koszul. This suffiecient condition was useful in determining the Koszul property for several classes of algebras, see the survey [6] and the references within.

2. Scientific objectives

We focused on the following problems and we set the following objectives for the period covered in this report:

(1) Asymptotic properties of toric algebras.
Objective: 1 accepted paper in an ISI indexed journal.
(2) The persistance of homological properties to deformations of Gröbner type.
Objective: 1 preprint.

3. Main results

The (numerical) objectives above have been reached through the paper [18] that was recently published in Journal of Algebra and by the preprint [5] which is currently circulated among some specialists before being sumitted to a journal.

We also made important progress towards meeting the goals for the year 2015, e.g. [19] si [28].

In what follows we briefly describe the contents of the papers already written in this project and we outline the current progress on the themes that will be our main focus for the rest of the grant period.

Our paper [18] has appeared:

Jürgen Herzog, **Dumitru I. Stamate**
On the defining equations of the tangent cone of a numerical semigroup ring, J. Algebra **418** (2014), 8–28.

We describe its contents, that will be useful to describe the other papers, as well.

Let $\mathbf{a} = a_1 < \cdots < a_r$ be a sequence of nonnegative integers. We denote $\langle a_1, \ldots, a_r\rangle$ (or simply $\langle \mathbf{a}\rangle$) the subsemigroup of $\mathbb{N}$ generated by $a_1, \ldots, a_r$. In other words, $\langle \mathbf{a}\rangle$ consists of all $\mathbb{N}$-linear combinations of $a_1, \ldots, a_r$. If $H = \langle a_1, \ldots, a_r\rangle$ we call $a_1, \ldots, a_r$ a *generating system* for H. From now on, any subsemigroup $H \subset \mathbb{N}$ with $0 \in H$ will be called a *numerical semigroup*. Such a semigroup is finitely generated and it admits a unique minimal generating system whose cardinality we denote by $\mu(H)$. It is usually part of the definition of numerical semigroups that the greatest common divisor of its generators be 1. In the context of this paper it is convenient to drop this requirement.

For any integer k, let $\mathbf{a} + k$ denote the shifted sequence $a_1 + k, \ldots, a_r + k$. If H is minimally generated by $\mathbf{a} = a_1, \ldots, a_r$, we let $H_k = \langle \mathbf{a} + k\rangle$. We refer to the family of semigroups $\{H_k\}_{k\in\mathbb{N}}$ as the *shifted family* associated to H. One can see that although the a_i's generate H minimally, for some shifts k the sequence $\mathbf{a} + k$ may not be a minimal generating system for H_k. In particular, $(H_k)_\ell$ may differ from $H_{k+\ell}$. For example, when $H = \langle 3, 5, 7\rangle$ we get $H_1 = \langle 4, 6, 8\rangle = \langle 4, 6\rangle$. On the other hand, if $H = \langle \mathbf{a}\rangle$ is minimally generated by $\mathbf{a} = a_1 < \cdots < a_r$, then for all $k > a_r - 2a_1$, H_k is minimally generated by the sequence $\mathbf{a} + k$.

Let K be any field and $S = K[x_1, \ldots, x_r]$ the polynomial ring over K in the variables $x_1, \ldots, x_r$. Let $\mathbf{a} = a_1 < \cdots < a_r$ be a sequence of positive integers and $\varphi : S \to K[t]$ is the K-algebra map letting $\varphi(x_i) = t^{a_i}$ for $i = 1, \ldots, r$, where $K[t]$ is the polynomial ring over K in the variable t. If we denote $H = \langle a_1, \ldots, a_r\rangle$, then the image of φ is the semigroup ring $K[H]$, i.e. the K-subalgebra of $K[t]$ generated by $t^{a_1}, \ldots, t^{a_r}$ over K. Let $I(\mathbf{a})$ be the kernel of φ. When $\mathbf{a}$ is a minimal generating system for H, the ideal $I(\mathbf{a})$ depends only on H and we denote $I_H = I(\mathbf{a})$.

One knows from [16] that the minimal number of generators $\mu(I_H)$ of I_H is at most 3 for $r \le 3$. On the other hand, already for $r = 4$, the value of $\mu(I_H)$ may be arbitrarily large, cf. [3]. The more it is surprising that for any numerical semigroup H there exists an upper bound for $\mu(I_{H_k})$ independent of k, see [31, Theorem 1.1]. This result was conjectured by J. Herzog and H. Srinivasan and it was proved first by P. Gimenez, I. Sengupta şi H. Srinivasan în [14] for numerical semigroups generated by an arithmetic sequence. This conjecture and a stronger form of it have been recently proved by T. Vu in [31].

Although for $r = 3$ we have a $\mu(I_H) \le 3$, however the number of generators for I_H^* may be arbitrarily large. A first family of such examples was found by T. Shibuta, see [15]. For this family its *width* is not bounded, where by the width of the semigroup H, denoted $\mathrm{wd}(H)$, we understand the difference between the largest and the smallest element in the minimal generating system of H. In our Corollary 1.16 we prove that there exists un upper bound for $\mu(I_H^*)$ which is valid for all semigroups of fixed width. This follows from a recent theorem of Vu, cf. [31, Theorem 1.1] and our next theorem.

Theorem 1.4. *Let H be a numerical semigroup. There exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$, the ideal I_{H_k} is minimally generated by a standard basis, and $\beta_i(I_{H_k}) = \beta_i(I^*_{H_k})$ for all i. In particular, $\operatorname{gr}_{\mathfrak{m}} K[H_k]$ is Cohen–Macaulay for all $k \geq k_0$.*

The methods that we use for showing the existence of a uniform upper bound for $\mu(I^*_H)$ for all numerical semigroups H of fixed width do not give an explicit value. However, numerical experiments with SINGULAR [8] give us reasons to believe that $\binom{\mathrm{wd}(H)+1}{2}$ is such an upper bound, and it can not be improved since it is reached by numerical semigroups generated by certain intervals of integers. We prove that this conjectured bound is valid for any numerical semigroup with the property that $\mu(I^*_H) \leq \mu(I^*_{\widetilde{H}})$, where $\widetilde{H}$ is the semigroup generated by all integers in the interval whose ends are the smallest and the largest minimal generator of H.

To support our conjecture, we prove in Proposition 2.10 that for a numerical semigroup H generated by an arithmetic sequence we even have $\beta_i(I^*_H) \leq \beta_i(I^*_{\widetilde{H}})$, for all i. It is possible that such an inequality take place for any numerical semigroup!

In the last section of the paper we study several families of semigroups where we tested the above conjectures and we describe the ideal I^*_H for all members H in such families. The first family is based on a well-known result of J. Sally in [26], where the author describes the equations of the tangent cone of a Gorenstein local ring with the property $r = e + d - 3$. Here r denotes the embedding dimension, e denotes the multiplity, and d is the Krull dimension of the ring. We define a *Sally semigroup* to be a numerical semigroup whose semigroup ring satisfies the above identity. We prove that there exists Sally semigroups of any multiplicity $e \geq 4$. Another family we study is due to H. Bresinsky [3]. This is the first known family of 4-generated numerical semigroups such that $\mu(I_H)$ may be arbitrarily large when H runs in this family. We show that a Bresinsky semigroup is Cohen-Macaulay and we exhibit a minimal generating system of equations that is also a standard basis.

The other two families refer to 3-generated semigroups, and although their members have arbitrary width, the behaviour of $\mu(I^*_H)$ is quite different from above. For $a > 3$, the ideal I^*_H associated to the semigroup $H = \langle a, a+1, 2a+3 \rangle$ is generated by $\lfloor \frac{a-1}{3} \rfloor + 3$ monomials. For that family, the number of generators of I^*_H is a quasi-linear function on the width of H, and it goes to infinity as $\mathrm{wd}(H)$ goes to infinity, too. When $a = 3b$ we come across an example found by T. Shibuta, that was treated in [15, Example 5.5] with different methods.

On the other hand, for any $a, b > 3$ coprime, if we let $H = \langle a, b, ab - a - b \rangle$, then also $\mu(I^*_H) = 4$, although the widths of such semigroups may be arbitrarily large.

The work on the paper [5]:
Mircea Cimpoeaş, **Dumitru I. Stamate**,
On intersections of complete intersection ideals, preprint 2014.

was stimulated by numerical experiments and by the experience gained when preparing the previously described paper.

An ideal I in a (Noetherian) ring R is called a *complete intersection* (CI for short) if it can be generated by height I elements, i.e. the minimum allowed by a famous theorem of Krull. This class of Gorenstein rings has many homological properties and a geometric relevance. Therefore its extremal character has attracted the attention of many researchers.

In general the class of CI ideals is not closed to the main operations with ideals, e.g. sum, intersection, except some very trivial cases. However, in this paper we present infinite families of CI ideals that are (each) closed to taking intersections.

In order to explain the constructions we continue to use the notation introduced in the previous paragraphs. We saw that for a list $\mathbf{a}$ and a shift $k \gg 0$, the Betti numbers for $I_{\mathbf{a}+k}$ are periodic in k. In particular, if $I_{\mathbf{a}+k}$ is CI for some $k \gg 0$, then there exist an infinity of shifts j such that $I_{\mathbf{a}+j}$ and $I^*_{\mathbf{a}+j}$ are also CI.

For a finite family of indices $\mathcal{A} \subset \mathbb{N}$ we consider

$$\mathcal{I}_{\mathcal{A}} = \bigcap_{j \in \mathcal{A}} I_{\mathbf{a}+j}$$

and similarly

$$\mathcal{J}_{\mathcal{A}} = \bigcap_{j \in \mathcal{A}} I^*_{\mathbf{a}+j}.$$

These ideals are no longer binomial and many of the properties of toric ideals are lost. Using Gröbner bases techniques and the work in [20] we prove that when $I_{\mathbf{a}+j}$ is CI for all $j \in \mathcal{A}$ and $\min \mathcal{A} \gg 0$, then $\mathcal{I}_{\mathcal{A}}$ and $\mathcal{J}_{\mathcal{A}}$ are also CI.

More precise results are obtained for 3-generated semigroups using the characterization of shifts $k \gg 0$ such that $I(\mathbf{a}+k)$ is CI, cf. [29].

Numerical experiments make us believe that this stability to intersections of CI ideals in a shifted family of semigroups is only a facet of a larger periodicity phenomenon. Also, Gröbner deformations of these shifted ideals have the same periodicity to intersections. We believe these are reasons for the further study of the "asymptotic properties" and/of the Gröbner deformations of the (ideal of) affine semigroup rings, not necessarily numerical.

The mobilities we already made favored some collaborations ([28], [18]) that correspond to the objectives for 2015. We briefly sketch the progress:

In [28]:
Alexandra Seceleanu, **Dumitru I. Stamate**, *On Sally semigroup rings*, in preparation.

we study the class of Sally semigroups described above. We have experimentally noticed that for artinian Sally rings, their Betti numbers ar the same. We confirm this observation and we give a precise formula for them. We employ recent results of Elias and Rossi ([10]) about the classification of short artinian algebras, and also a theorem regarding the decomposition into connected sums of rings, cf. Anathnarayan et al ([2]). Such a formula is useful because for the Sally numerical semigroup we presented in [18] their Betti numbers are rather large, close to the maximum conjectured in [18].

Complementing [18], we present many new examples of Sally numerical semigroups.

In [19]:

J. Herzog, **D.I. Stamate**, *Strongly Koszul ungraded toric rings*, in preparation.

We extend the concept of strongly Koszul algebra (that was introduced in [17]) to toric algebras that are not necessarily graded. We prove that the associated graded ring is Koszul in the classical sense, and it has the same Poincaré series as $K[H]$. In particular, the latter is rational. For the proof one uses some topological characterizations of the Koszul property, previously obtained by us in [25] for $\mathrm{gr}_{\mathfrak{m}} K[H]$.

4. Dissemination of results and other activities

Mobilities played an important part in the planning of the activities. We considered very important to meet with experts in our area of research from prestigious universities. On one hand we could present our results, and on the other hand these visits materialized into several research project initiated within the theme of our project. For instance one joint paper was recently published in a prestigious journal, another one is in the preprint stage, while other ones are in various stages: writing or configuration of the results for presentation, for details see the previous section in this report.

I was invited to give seminar talks at all the universities that I visited. These presentations have been excellent opportunities to obtain feed-back and sugestions for subsequent developments. Next we outline the research visits we made, which in general goes hand in hand with the dissemination talks.

Research visits:

(1) University Osnabrueck, Germania, June 2013. Host: prof. Tim Römer.
(2) University Duisburg-Essen, Essen, Germania, July – August 2013. Host: prof. Jürgen Herzog.
(3) University of Nebraska, Lincoln, NE, SUA, September 2013. Host: prof. Roger Wiegand.
(4) University of Minnesota, Minneapolis, MN, SUA, October 2013. Host: prof. Victor Reiner.
(5) University of Missouri, Columbia, MO, SUA, October 2013. Host: prof. Hema Srinivasan.
(6) University of Nebraska, Lincoln, NE, SUA, January 2014. Host: prof. Roger Wiegand.
(7) University of Minnesota, Minneapolis, MN, SUA, February 2014. Host: prof. Victor Reiner.
(8) Universite de Montpellier 2, Institut de Mathematiques et de Modelisations de Montpellier, Montpellier, France, May 2014. Host: dr. Ignacio Garcia Marco.
(9) Universita di Genova, Italy, June 2014. Host: prof. Aldo Conca.
(10) University Duisburg-Essen, Essen, Germania, July – August 2014. Host: prof. Jürgen Herzog.
(11) University of Nebraska, Lincoln, NE, SUA, September 2014. Host: dr. Alexandra Seceleanu.

Invited experts:

During 2–7 September 2014, prof. Jorge Ramirez Alfonsin, Université Montpellier 2, France visited us. On this occasion he gave a series of 4 talks about the algebraic properties of matroids, in the framework of the National Algebra School.

We also discussed his very recent results concerning the Moebius function of intervals in numerical semigroups and new combinatorial approaches in the study of (numerical) semigroups.

Dissemination of results:

(1) D. Stamate, *On the CI property of the tangent cone of a toric ring*, Workshop for Young Researchers in Mathematics, Ovidius University Constanţa, 8–10 May 2013.
(2) D. Stamate, *Shifting semigroups*, short talk, Workshop "Syzygies in Berlin", Freie Universität, Berlin, Germania, 28 May 2013.
(3) D. Stamate, *Shifted semigroup rings*, Oberseminar University of Osnabrueck, Germania, 4 June 2013.
(4) D. Stamate, *On the CI property of the tangent cone of a toric ring*, AMS-RMS Joint meeting, Special Session on Commutative Algebra, Alba Iulia, 30 June 2013.
(5) D. Stamate, *On the equations of toric rings*, University Duisburg-Essen, Essen, 29 August 2013.
(6) D. Stamate, *Tools of Combinatorial Commutative Algebra 2*, National Algebra School "Algebraic methods in Combinatorics", 3 September 2013.
(7) D. Stamate, *Matroids and realisability*, National Algebra School "Algebraic methods in Combinatorics", 4 September 2013.
(8) D. Stamate, *On the defining equations of the tangent cone of a numerical semigroup ring*, Comm. Algebra Seminar talk, University of Nebraska, Lincoln, NE, SUA, 18 September 2013.
(9) D. Stamate, *On the defining equations of the tangent cone of a numerical semigroup ring*, Comm. Algebra Seminar talk, University of Minnesota, Minneapolis, MN, SUA, 14 October 2013.
(10) D. Stamate, *On the defining equations of the tangent cone of a numerical semigroup ring*, Comm. Algebra Seminar talk, University of Missouri, Columbia, MO, SUA, 22 October 2013.
(11) D. Stamate, *Asymptotic properties of numerical semigroups I, II*, Commutative Algebra seminar IMAR & Univ. Bucureşti, 19 şi 26 November 2013.
(12) D. Stamate, *About the structure of Sally rings*, Comm. Algebra Seminar talk, University of Minnesota, Minneapolis, MN, SUA, 14 February 2014.
(13) D. Stamate, *On the CI property of the tangent cone of a toric ring*, Workshop for Young Researchers in Mathematics, Ovidius University Constanţa, 22–23 May 2014.
(14) D. Stamate, *On numerical semigroup rings and their defining relations*, Seminaire Algebre et geometrie combinatoires, Université de Montpellier 2, Franţa, 27 May 2014.

(15) D. Stamate, *On the defining equations of the tangent cone of a numerical semigroup ring*, Commutative Algebra Seminar talk, University of Genova, Italia, 3 June 2014.
(16) D. Stamate, *Asymptotic properties of numerical semigroups*, National Algebra School "Algebraic and Combinatorial Applications of Toric Ideals", 3 September 2014.
(17) D. Stamate, *Flavors of Koszul rings*, Comm. Algebra Seminar talk, University of Nebraska, Lincoln, NE, SUA, 17 September 2014.

Other seminar talks:

(1) M. Cipu, *Algebraic tools for discrete tomography*, Commutative Algebra seminar IMAR & Univ. Bucureşti, 18 February 2014.
(2) M. Cipu, *Quantic Ehrhart polynomials*, Commutative Algebra seminar IMAR & Univ. Bucureşti, 1 April 2014.
(3) D. Stamate, *On the subadditivity problem for maximal shifts in free resolutions, after Herzog et al*, Commutative Algebra seminar IMAR & Univ. Bucureşti, 6 May 2014.
(4) M. Cipu, *A conjectural characterization for complete intersection numerical semigroups*, Commutative Algebra seminar IMAR & Univ. Bucureşti, 4 November 2014.

Other visits supported by this project:

(1) Conference GMZ50 honoring Gunter Ziegler, Freie Universität Berlin, Germania, organized by Christian Haase, Raman Sanyal, Nadja Wisniewski, 25 May 2013.
(2) COCOA School, Universität Osnbrück, Germania, organized by W. Bruns, L. Robbiano, A. Bigatti, 10–14 June 2013.
(3) ETMAGT-International Conference Experimental and Theoretical Methods in Algebra, Geometry and Topology, Eforie Nord, 20–24 June 2013. (Dumitru Stamate and Mihai Cipu)
(4) Recent Trends in Algebraic and Geometric Combinatorics, Madrid, 26-30 November 2013. (Mihai Cipu)
(5) Encuentros de Algebra Computacional y Aplicaciones-EACA, Barcelona, Spania, 17–22 June 2014. (Mihai Cipu)
(6) Meeting On Combinatorial Commutative Algebra, MOCCA, Levico Terme, Italia, 8–12 September 2014. (Mihai Cipu)

The discussions with prof. Tim Römer on the occasion of the visit to Osnbrück in June 2013 favored the **organization** of the 21st edition of the National Algebra School – "Algebraic methods in Combinatorics" where prof. Tim. Römer was a keynote speaker. We organized this event at IMAR, 2-6 September 2013, together with Viviana Ene, Mihai Epure, Miruna Roşca, Andrei Zarojanu. Scientific committee: Dorin Popescu, Tim Römer, Marius Vlădoiu.

In this year, on the occasion of the visit to Montpellier I invited prof. Jorge Ramirez-Alfonsin to give a series of lectures in Bucharest at the 22nd edition of the National Algebra School–"Algebraic and Combinatorial Applications of Toric Ideals". Scientific committee: Hara Charalambous, Mihai Cipu (member of the team of the grant), Jorge Ramirez Alfonsin. We organized this school at IMAR, 1-5 September 2014, toghether with Florin Ambro, Viviana Ene, Mihai Epure, Miruna Roşca, Marius Vlădoiu and Andrei Zarojanu.

At both editions there were many participants, including students and PhD students.

5. Plans for the future

In complience with the Scheduling of the activities of this project, we continue to study the persistence to Gröbner basis deformations of homological properties of the toric algebra associated to an affine semigroup, since it is a new area that attracted the interest rapidly. Specialized computer software, SINGULAR [8] in particular, will be very useful for concrete numerical experiments At the beginning of 2015, after receiving some feed-back from colleagues and other specialists we'll submit our preprint [5].

We'll finalize [19] in order to identify new situations of combinatorially described Koszul algebras.

We'll also study the intervals of a numerical semigroup from a combinatorial and topological point of view. That will be useful in verifying the Koszul property for the associated graded ring $\operatorname{gr}_{\mathfrak{m}} K[H]$ when H is a numerical semigroup, which is one of the objectives of this research project.

For the coming period we have in mind a dense set of activities. The timely financing from UEFISCDI has been instrumental in realizing the activities so far. We hope this happens in the future, too.

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