ON THE OCCURRENCE OF COMPLETE INTERSECTIONS IN SHIFTED FAMILIES OF NUMERICAL SEMIGROUPS

DUMITRU I. STAMATE

Dedicated to Professor Jürgen Herzog on his 80th anniversary

ABSTRACT. We give a necessary and sufficient condition for the existence of infinitely many complete intersections in the shifted family of a numerical semigroup.

In this note we give a necessary and sufficient condition for the existence of infinitely many complete intersections in the shifted family of a numerical semigroup. We first introduce the terminology and the necessary background before giving the result in Theorem 7 and Corollary 8.

Let K be any field and $\mathbf{a} = a_1 < \cdots < a_r$ a list of nonnegative integers, $r \geq 2$. We denote by $I(\mathbf{a})$ the kernel of the K-algebra map $\varphi : K[x_1, \ldots, x_r] \to K[t]$ letting $\varphi(x_i) = t^{a_i}$ for $i = 1, \ldots, r$. The image of φ is the semigroup ring $K[\langle \mathbf{a} \rangle]$ over K of the numerical semigroup $\langle \mathbf{a} \rangle := \sum_{i=1}^{r} \mathbb{N} a_i \subseteq \mathbb{N}$ spanned by $\mathbf{a}$. Note that we do not impose that the elements in $\mathbf{a}$ be coprime.

It has been an ongoing topic of research to describe generators for $I(\mathbf{a})$ explicitly, or its algebraic properties in terms of the integers $\mathbf{a}$. Unless $\mathbf{a}$ has some special form, it is computationally challenging to write down a minimal generating set for $I(\mathbf{a})$, see [10] or the survey [9].

When $r = 2$, $I(\mathbf{a})$ is a principal ideal. When $r = 3$, in one of his very first and also one of his most cited papers [6], Herzog proved that $I(\mathbf{a})$ is generated by at most three binomials, which he shows how to obtain in terms of $\mathbf{a}$. Moreover, Herzog in loc. cit. describes for which a_1, a_2, a_3 the ideal $I(\mathbf{a})$ is generated by two binomials, i.e. $I(\mathbf{a})$ is a complete interesection ideal. Recall that an ideal I in the polynomial ring $K[x_1, \ldots, x_r]$ is called a complete intersection (CI for short) if it can be generated by height I elements. One also says thay $K[x_1, \ldots, x_r]/I$ is a CI ring.

Extending [6], in [3] Delorme proves for arbitrary r that $I(\mathbf{a})$ is a CI ideal if and only if the sequence $\mathbf{a}$ can be obtained by a recursive procedure which nowadays is called gluing.

For any nonnegative integer k we let $\mathbf{a} + k = a_1 + k, \ldots, a_r + k$. The sequence $\{\langle \mathbf{a} + k \rangle\}_k$ is called the *shifted family* of (numerical) semigroups generated by $\mathbf{a}$. Based on numerical experiments, Herzog and Srinivasan conjectured, and Vu proved in [11] that the i^{th} Betti number of $I(\mathbf{a} + k)$ is eventually periodic in k for $k \gg 0$.

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Theorem 1. *(Vu, [11, Theorem 1.1]) For all $i \geq 0$, $\beta_i(I(\mathbf{a}+k)) = \beta_i(I(\mathbf{a}+k+(a_r-a_1)))$ for $k \gg 0$.*

In particular, for $k \gg 0$, $I(\mathbf{a}+k)$ is CI if and only if $I(\mathbf{a}+k+(a_r-a_1))$ is CI, as it was already proved by Jayanthan and Srinivasan in [7], see Theorem 3. The question remains, if complete intersections do occur for $k \gg 0$ in the shifted family $\langle \mathbf{a}+k\rangle$. Equivalently, whether $I(\mathbf{a}+k)$ is CI for infinitely many values of k. When $r=3$, the answer is positive as indicated in [7], and in [8, Theorem 3.1] we give the precise values of k (large enough) for which $I(\mathbf{a}+k)$ is CI.

When $r \geq 4$, the number of CIs may be finite or not in shifted families, as noticed in [7]. Of course, we can compute a threshold k_0 proved in [11] (usually it is very large) from where the Betti numbers of $I(\mathbf{a}+k)$ start repeating periodically, and then finding with Singular ([2]) or other specialized software minimal systems of generators for $I(\mathbf{a}+k_0), \dots, I(\mathbf{a}+k_0+(a_r-a_1-1))$. Yet, we would like to know without computing many toric ideals in the shifted family, whether CIs occur infinitely many times.

We consider the ideal

$$J(\mathbf{a}) = (f \in I(\mathbf{a}) : f \text{ homogeneous}),$$

where by homogeneous we refer to the standard grading with $\deg x_i = 1$ for all i. Note that $J(\mathbf{a}) = J(\mathbf{a}+k)$ for all $k \geq 0$.

It was remarked in [7] and in [11] that the ideal $J(\mathbf{a})$ plays a role in studying the asymptotic properties in the shifted family $\{I(\mathbf{a}+k)\}_{k\geq 0}$. Related to the CI property, in [1] we proved the following.

Proposition 2. *([1, Corollary 1.6]) Let $r \geq 3$ and $\mathbf{a} = a_1 < \cdots < a_r$. Then if $I(\mathbf{a}+k)$ is CI for some $k \geq (a_r-a_1)^2 - a_1$, the ideal $J(\mathbf{a})$ is CI and it is minimally generated by its reduced Gröbner basis with respect to the reverse lexicographic order.*

The next result is from [7], but we give it in the form from [1, Lemma 1.2 (2)].

Theorem 3. *(Jayanthan and Srinivasan, [7, Theorem 2.1]) Assume $r \geq 3$ and $k \geq (a_r-a_1)^2 - a_1$. If $I(\mathbf{a}+k)$ is CI, then $I(\mathbf{a}+k+\ell(a_r-a_1))$ is CI for all $\ell \geq 0$.*

In particular, if $I(\mathbf{a}+k)$ is CI for infinitely many shifts k, the ideal $J(\mathbf{a})$ is a CI ideal. The purpose of this note is to show in Theorem 7 and Corollary 8 that the converse of this statement also holds.

A key observation is that $J(\mathbf{a})$ is a toric ideal. Lacking a proper reference, we include a proof here.

Lemma 4. *Given the nonnegative integers $\mathbf{a} = a_1 < \cdots < a_r$, the ideal $J(\mathbf{a})$ is the defining ideal of the semigroup ring associated to the affine semigroup generated by $(a_1,1), \dots, (a_r,1)$.*

Proof. We denote $L = \Sigma_{i=1}^r \mathbb{N}(a_i,1) \subset \mathbb{N}^2$ and $K[L]$ the attached semigroup ring. The defining ideal of $K[L]$ is the kernel of the monomial map $\psi : K[x_1,\dots,x_r] \to K[s,t]$ given by $\psi(x_i) = s^{a_i}t$ for $i=1,\dots,r$. As explained in [10], $\operatorname{Ker}\psi$ is generated by the binomials $\prod_{i=1}^r x_i^{\alpha_i} - \prod_{i=1}^r x_i^{\beta_i}$ where $\sum_{i=1}^r a_i\alpha_i = \sum_{i=1}^r a_i\beta_i$ and $\sum_{i=1}^r \alpha_i = \sum_{i=1}^r \beta_i$.

Clearly, $\operatorname{Ker}\psi \subseteq J(\mathbf{a})$. For the reverse inclusion, we consider the grading on $K[x_1, \ldots, x_r]$ induced by $\mathbf{a}$ where $\deg_{\mathbf{a}}(x_i) = a_i$ for $i = 1, \ldots, r$. It is known that $I(\mathbf{a})$ is homogeneous with respect to this $\mathbf{a}$-grading.

Let $0 \neq f$ be a homogeneous polynomial in $I(\mathbf{a})$ with respect to the standard grading. We prove that $f \in \operatorname{Ker}\psi$. We decompose f into $\mathbf{a}$-graded components, and each of these will be homogeneous (in the standard grading). So, without loss of generality we may assume that $f \in I(\mathbf{a})$ is homogeneous with respect to both the standard grading and the $\mathbf{a}$-grading.

We write $f = \sum_{i=1}^{p} c_i m_i$ where $c_i \in K$ and m_i is a monomial for $i = 1, \ldots, p$ with $\deg_{\mathbf{a}}(m_i)$ the same for all i. Since $f \in I(\mathbf{a})$ and $I(\mathbf{a})$ is the kernel of the K-algebra map $\varphi : K[x_1, \ldots, x_r] \to K[s]$ letting $\varphi(x_i) = s^{a_i}$ for $i = 1, \ldots, r$, we get that $p \geq 2$ and $\sum_{i=1}^{p} c_i = 0$.

Consequently, $f = \sum_{i=1}^{p=1} c_i(m_i - m_p)$ is generated by binomials in $\operatorname{Ker}\psi$, which proves $J(\mathbf{a}) = \operatorname{Ker}(\psi)$. □

Extending the work of Delorme [3], for affine semigroups in $\mathbb{N}^d$ the CI property was characterized by Fisher, Morris and Shapiro ([4]) as follows.

Theorem 5. *(Fischer, Morris and Shapiro* [4, Theorem 3.1]*)*
Let H be an affine semigroup that is not a free abelian semigroup and assume that H is minimally generated by $\mathcal{V} = \{v_1, \ldots, v_r\} \subset \mathbb{N}^d$. The semigroup ring $K[H]$ is CI if and only if there exists a partition $\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2$ and $0 \neq v \in \langle \mathcal{V}_1 \rangle \cap \langle \mathcal{V}_2 \rangle$ such that $\mathbb{Z}\mathcal{V}_1 \cap \mathbb{Z}\mathcal{V}_2 = \mathbb{Z}v$ and both semigroup rings $K[\langle \mathcal{V}_1 \rangle]$ and $K[\langle \mathcal{V}_2 \rangle]$ are CI.

A partition of $\mathcal{V}$ as above will be called *a CI-split*.

Lemma 6. *Let $r \geq 3$ and the nonnegative integers $\mathbf{a} = a_1 < \cdots < a_r$. Assume $J(\mathbf{a})$ is a CI ideal. Fix $k \geq 0$ and set $\mathcal{V} = \{(a_1 + k, 1), \ldots, (a_r + k, 1)\}$. If $\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2$ is any CI-split for $\mathcal{V}$ and $|\mathcal{V}_1| \leq |\mathcal{V}_2|$ then*

(i) $|\mathcal{V}_1| = 1$,
(ii) $\mathcal{V}_1 \cap \{(a_1 + k, 1), (a_r + k, 1)\} = \emptyset$.

Proof. Let $\mathbb{Z}\mathcal{V}_1 \cap \mathbb{Z}\mathcal{V}_2 = \mathbb{Z}v$ and $v = (a, b) \in \mathbb{N}^2$ is as given by Theorem 5.

(i) If we assume $2 \leq |\mathcal{V}_1|$, then we may subtract two generators in $\mathcal{V}_1$ and in $\mathcal{V}_2$, respectively, and we obtain that $(e_1, 0) \in \mathbb{Z}\mathcal{V}_1$ and $(e_2, 0) \in \mathbb{Z}\mathcal{V}_2$ for some e_1, e_2 positive integers. Clearly $(e_1 \cdot e_2, 0) \in \mathbb{Z}\mathcal{V}_1 \cap \mathbb{Z}\mathcal{V}_2$, therefore $b = 0$. Since $v = (a, 0)$ is also a linear combination of vectors in $\mathcal{V}$ with nonnegative integer coefficients we get that $a = 0$ and $e_1 e_2 = 0$, a contradiction. Hence $|\mathcal{V}_1| = 1$.

(ii) Let $\mathcal{V}_1 = \{a_p + k\}$ with $1 \leq p \leq r$. Then $v = c(a_p + k, 1) = \sum_{i=1}^{r} c_i(a_i + k, 1)$ for some nonnegative integers $c, c_1, \ldots, c_r$, with $c > 0$ and $c_p = 0$. Thus $c = \sum_{i=1}^{r} c_i$. Also, since $r > 2$ we obtain

$$\left(\sum_{i=1}^{r} c_i\right)(a_1 + k) < \sum_{i=1}^{r} c_i(a_i + k) < \left(\sum_{i=1}^{r} c_i\right)(a_r + k),$$

which gives $a_1 + k < a_p + k < a_r + k$ and $\mathcal{V}_1 \cap \{(a_1 + k, 1), (a_r + k, 1)\} = \emptyset$. □

Theorem 7. *Let $r \geq 3$ and the nonnegative integers $\mathbf{a} = a_1 < \cdots < a_r$. If $J(\mathbf{a})$ is a CI ideal, then $I(\mathbf{a} + \ell(a_r - a_1) - a_1)$ is a CI ideal for all $\ell \geq a_r - a_1$.*

Proof. We prove the statement by induction on $r \geq 3$.

If $r = 3$, then it is an easy exercise to show that if we let $d = \gcd(a_2 - a_1, a_3 - a_1)$, then $J(\mathbf{a}) = (x_2^{(a_3-a_1)/d} - x_1^{(a_3-a_2)/d} x_3^{(a_2-a_1)/d})$, hence it is always a CI. It is proven in [8, Theorem 3.1] that for shifts $k > (a_3 - a_1)^2 - a_3$, $I(\mathbf{a} + k)$ is CI if and only if $a_1 + k$ is a multiple of a certain constant T that is a divisor of $\gcd(a_3 - a_1, a_2 - a_1)$. In particular, $I(\mathbf{a} + \ell(a_3 - a_1) - a_1)$ is CI for all $\ell \geq a_3 - a_1$.

Assume $r > 3$ and $k = \ell(a_r - a_1) - a_1$ with $\ell \geq a_r - a_1$. If we let $\mathcal{V} = \{(a_1 + k, 1), \dots, (a_r + k, 1)\}$, by Theorem 5 and Lemma 6, there exists a CI-split

$$\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2 \tag{1}$$

with $\mathcal{V}_1 = \{(a_p, 1)\}$ for some $1 < p < r$. Denote by $\mathbf{b}$ the sequence obtained from $\mathbf{a}$ by removing a_p. Since $J(\mathbf{b})$ is CI, by the induction hypothesis we get that $I(\mathbf{b} + k)$ is CI for our choice of k.

Notice that

$$\begin{aligned} \mathbb{Z}\mathcal{V}_2 &= \mathbb{Z}\{(a_i + k, 1) : 1 \leq i \leq r, i \neq p\} \\ &= \mathbb{Z}\{(a_1 + k, 1), (a_i - a_1, 0) : 2 \leq i \leq r, i \neq p\} \\ &= \mathbb{Z}\{(a_1 + k, 1), (d, 0)\}, \end{aligned}$$

where we let $d = \gcd(a_2 - a_1, \dots, \widehat{a_p - a_1}, a_r - a_1)$ and we marked by $\widehat{\ }$ the absence of that term from the enumeration.

Using this observation, it is routine to check that

$$\mathbb{Z}(a_p + k, 1) \cap \mathbb{Z}\mathcal{V}_2 = \mathbb{Z}(D(a_p + k), D),$$

where $D = d/\gcd(d, a_p - a_1)$. Clearly $D > 1$, otherwise $(a_p + k, 1) \in \langle \mathcal{V}_2 \rangle$, which is false.

From the CI-split (1) we also obtain that $D(a_p + k, 1) \in \langle \mathcal{V}_2 \rangle$. Therefore, there exist nonnegative integers $c, c_1, \dots, c_r$ with $c > 0, c_p = 0$ such that $\sum_{i=0}^r c_i = D$ and

$$D \cdot (a_p + k) = \sum_{1 \leq i \leq r, i \neq p} c_i (a_i + k). \tag{2}$$

As $\mathbb{Z}(\mathbf{b} + k) = \mathbb{Z}\{a_1 + k, d\} = \mathbb{Z}d$, it follows that $\mathbb{Z}(a_p + k) \cap \mathbb{Z}(\mathbf{b} + k) = \mathbb{Z}v$, where $v = \operatorname{lcm}(a_p + k, d) = (a_p + k) \cdot d/\gcd(a_p - a_1, d) = (a_p + k) \cdot D$. Equation (2) gives that $v \in \mathbb{Z}(\mathbf{b} + k)$, hence

$$\mathbf{a} + k = \{a_p + k\} \sqcup \{\mathbf{b} + k\}$$

is a CI-split for $\mathbf{a} + k$, by Theorem 5. This finishes the proof. $\square$

Corollary 8. *Let $\mathbf{a} = a_1 < \cdots < a_r$ and $r \geq 2$. The ideal $I(\mathbf{a} + k)$ is CI for infinitely many $k > 0$ if and only if $J(\mathbf{a})$ is a CI ideal.*

Example 9. (1) For $\mathbf{a} = (0, 4, 12, 30)$ a computation with Singular ([2]) shows that $J(\mathbf{a}) = (x_2^3 - x_1^2 x_3, x_3^5 - x_1^3 x_4^2)$, which is a CI ideal. Therefore, by Theorem 7 the ideal $I(k, k+4, k+12, k+30)$ is CI for $k = 30\ell$ and $\ell \geq 30$. In fact, as noted in [8, Example 3.8], when $k \geq 95$ the ideal $I(k, k+4, k+12, k+30)$ is CI if and only if k is a multiple of 15.

(2) On the other hand, when $\mathbf{a} = (0,1,2,3)$, using Singular we obtain that $J(\mathbf{a}) = (x_3^2 - x_2x_4, x_2x_3 - x_1x_4, x_2^2 - x_1x_3)$, which is not a CI ideal since height $J(\mathbf{a}) = 2$. Therefore, by Corollary 8 the ideal $I(k, k+1, k+2, k+3)$ is not CI for any $k \gg 0$. In fact, the latter is not CI for any $k \geq 0$, see [5].

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Dumitru I. Stamate, Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, Bucharest –010014, Romania

Email address: dumitru.stamate@fmi.unibuc.ro