# The Hilbert-Kunz Functions of Surfaces of Type ADE

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# Outline

# The Hilbert-Kunz function

- Let $(R, \mathfrak{m})$ be a local, Noetherian ring of (Krull-)dimension $d$,
- $k \subset R$ a field of characteristic $p$,
- and $I = (f_1, \ldots, f_n)$ be an $\mathfrak{m}$-primary ideal.
- Denote by $F \colon R \to R$, $r \mapsto r^p$ the Frobenius morphism and
- let $I^{[q]} := (f_1^q, \ldots, f_n^q) = F^e(I)$ with $q = p^e$.

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- let $I^{[q]} := (f_1^q, \ldots, f_n^q) = F^e(I)$ with $q = p^e$.

Then the map

$$e \mapsto \dim_k((R/I^{[p^e]}))$$

is called the *Hilbert-Kunz function of $R$ with respect to $I$*.

# The Hilbert-Kunz multiplicity

$$e_{HK}(R, I) := \lim_{e \to \infty} \frac{\dim_k(R/I^{[q]})}{q^d}$$

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Question: Is $e_{HK}(R)$ rational?

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Both rings have the same Hilbert-Kunz multiplicity (one). BUT: $R$ is regular, while $S$ is not.

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For the converse one needs to add ”equidimensional” to the assumptions. In this case the Hilbert-Kunz multiplicity measures the singularity of the ring (high values for bad singularities).

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They are of the form $k[X, Y, Z]/(f)$ with

$A_n$: $f = X^{n+1} + YZ, \omega = (2, n+1, n+1)$

$D_n$: $f = X^2 + Y^{n-1} + YZ^2, \omega = (n-1, 2, n-2), n \geq 4$

$E_6$: $f = X^2 + Y^3 + Z^4, \omega = (6, 4, 3)$

$E_7$: $f = X^2 + Y^3 + YZ^3, \omega = (9, 6, 4)$

$E_8$: $f = X^2 + Y^3 + Z^5, \omega = (15, 10, 6)$

By Watanabe & Yoshida these rings have Hilbert-Kunz multiplicity $2 - 1/|G|$.

# Reduction to a standard-graded case I

Let $R$ be a surface of type ADE. The map

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$$X \mapsto U^{\deg(X)},\ Y \mapsto V^{\deg(Y)},\ Z \mapsto W^{\deg(Z)}$$

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induces a map from $R$ to $S_R := k[U, V, W]/(F)$, where $F$ is the image of $f$ under this map.
The induced map $R \to S_R$ is local and flat.

# Reduction to a standard-graded case II

**Theorem**

Let $(R, \mathfrak{m}) \to S$ be a flat, local morphism and $M$ an $R$-module.
Assume $k \subset R$, $S$.
Then

$$\dim_k(M) \cdot \dim_k(S/\mathfrak{m}S) = \dim_k(M \otimes_R S).$$

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This reduces our computation to a standard-graded case.

# Some notation

- Let $R$ be a normal, standard-graded domain and $Y$ its Proj.
- Let $k \subset R$ an algebraically closed field with char($k$)$= p$,
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- $f_1, \ldots, f_n \in R$ homogeneous elements of degree $d_i$, generating an $R_+$-primary ideal
- We have the short exact sequence

$$(*) \qquad 0 \to \mathrm{Syz}_Y(f_1, \ldots, f_n) \to \bigoplus_{i=1}^{n} \mathcal{O}_Y(-d_i) \to \mathcal{O}_Y \to 0.$$

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$$0 \to F^{*e}(\mathrm{Syz}_Y(f_1, \ldots, f_n))(m) \to \bigoplus_{i=1}^{n} \mathcal{O}_Y(m - qd_i) \to \mathcal{O}_Y(m) \to 0.$$

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Note that we have $F^{*e}(\mathrm{Syz}_Y(f_1, \ldots, f_n)) = \mathrm{Syz}_Y(f_1^q, \ldots, f_n^q)$.

# The connection to Hilbert-Kunz functions

Taking global sections yields

$$\begin{aligned} 0 \quad &\rightarrow \quad H^0(\mathrm{Syz}_Y(f_1^q, \ldots, f_n^q)(m)) \rightarrow \oplus_{i=1}^n H^0(\mathcal{O}_Y(m - qd_i)) \\ &\rightarrow \quad H^0(\mathcal{O}_Y(m)) \rightarrow \end{aligned}$$

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Summing up the dimensions, we get

$$\begin{aligned} \dim(R/(I^{[q]})_m) &= h^0(\mathcal{O}_Y(m)) - \textstyle\sum_{i=1}^n h^0(\mathcal{O}_Y(m-qd_i)) \\ &\quad + h^0(\mathrm{Syz}_Y(f_1^q,\dots,f_n^q)(m)). \end{aligned}$$

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Idea: Hypersurfaces of type *ADE* are Cohen-Macaulay finite ($=$ there are up to isomorphism only finitely many indecomposable, maximal CM modules).

# Syzygies are MCM

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Since we are in a two-dimensional situation, maximal Cohen-Macaulay and reflexive are equivalent.

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Since $M$ is an $R$-module, we have $f \cdot M = 0$. We get

$$f \cdot S^n \subseteq \varphi(S^n).$$

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Let $\Gamma$ be the "map" that attaches to $M$ the tuple $(\varphi, \psi)$.

# The category of matrix factorizations I

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- A pair of matrices $(\alpha, \beta)$ is called a *morphism of matrix factorizations* $(\varphi_1, \psi_1)$ and $(\varphi_2, \psi_2)$ if the following diagram commutes:

$$\begin{tikzcd} S^n \arrow[r, "\psi_1"] \arrow[d, "\alpha"] & S^n \arrow[r, "\varphi_1"] \arrow[d, "\beta"] & S^n \arrow[d, "\alpha"] \\ S^m \arrow[r, "\psi_2"] & S^m \arrow[r, "\varphi_2"] & S^m \end{tikzcd}$$

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- We call $(\varphi_1, \psi_1)$ and $(\varphi_2, \psi_2)$ *equivalent* if $\alpha$ and $\beta$ are isomorphisms.

# The category of matrix factorizations II

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The sequence of maps and modules

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reduces modulo $f$ to an 2-periodic $R$-free of $\operatorname{coker}(\varphi)$.
Because $\operatorname{Syz}^i(M)$ is either 0 or MCM for all $M \in R\text{−mod}$ and every $i \geq \dim R$, $\operatorname{coker}(\varphi)$ is MCM over $R$.

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$$\begin{aligned} & \text{reduced, indecomposable matrix factorizations of } f \\ & \text{up to equivalence} \\ \xrightarrow{\cong} \quad & \text{non-free, indecomposable, maximal Cohen-Macaulay} \\ & R-\text{modules up to isomorphism.} \end{aligned}$$

# An observation

In 2006 Kajura, Saito and Takahashi computed the matrix factorizations of surfaces of type ADE.
In the $E_6$ case ($X^2 + Y^3 + Z^4 = 0$) there are three that give a MCM module of rank 2.

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One of these is

$$\varphi = \psi = \begin{pmatrix} -X & 0 & Y^2 & Z^3 \\ 0 & -X & Z & -Y \\ Y & Z^3 & X & 0 \\ Z & -Y^2 & 0 & X \end{pmatrix}.$$

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$$\varphi = \psi = \begin{pmatrix} -X & Y^2 & Z^3 \\ 0 & Z & -Y \\ Y & X & 0 \\ Z & 0 & X \end{pmatrix}.$$

Deleting the second column, the rows of the remaining $4 \times 3$ matrix generate

$$\mathrm{Syz}_R(-X, Y, Z)$$

# Representation I

We now show that $\operatorname{coker}(\varphi)$ and $\operatorname{Syz}_R(-X, Y, Z)$ become isomorphic as sheafs on the punctured spectrum $U := \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$.

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- Because $\varphi = \psi$ we get a 1-periodic free $R$-resolution

$$\cdots \xrightarrow{\varphi} R^4 \xrightarrow{\varphi} R^4 \xrightarrow{\varphi} R^4 \rightarrow \operatorname{coker}(\varphi) \rightarrow 0.$$

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We now show that $\operatorname{coker}(\varphi)$ and $\operatorname{Syz}_R(-X, Y, Z)$ become isomorphic as sheafs on the punctured spectrum $U := \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$.

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- This gives $\operatorname{coker}(\varphi) = \operatorname{im}(\varphi)$, which is generated by the columns of $\varphi$.
- Show that on $D(Y) \cup D(Z)$ the second column belongs to $\widetilde{N}$, where $N$ is the module generated by the columns 1,3,4.

# Representation II

- The commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_U \arrow[r, "{(-X,Y,Z)}"] \arrow[d, "\cong"] & \mathcal{O}_U^3 \arrow[r, "\varphi^{(2)}"] \arrow[d, "\cong"] & \widetilde{N} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{O}_U \arrow[r, "{(-X,Y,Z)}"] & \mathcal{O}_U^3 \arrow[r] & \mathrm{Syz}_U(-X, Y, Z)^{\vee} \arrow[r] & 0 \end{tikzcd}$$

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  induces an isomorphism $\widetilde{N} \cong \mathrm{Syz}_U(-X, Y, Z)^\vee$.
- Since $M$ and $\mathrm{Syz}_R(-X, Y, Z)$ are reflexive, the above isomorphism lifts to $R$.
- Show $\mathrm{Syz}_R(-X, Y, Z)^\vee \cong \mathrm{Syz}_R(-X, Y, Z)$ by the theory of matrix factorizations.

# A short exact Sequence

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**Theorem (Brenner, Kaid 2007,_)**

Let $R := k[x, y, z]/(x^2 - f(y, z))$ with weights $\omega = (\alpha, \beta, \gamma)$ where $f$ is homogeneous of degree $2 \cdot \alpha$. For $a = 2 \cdot l + 1$, with $l \in \mathbb{N}$ and $b$, $c \in \mathbb{N}$ we have a short exact sequence:

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$$\begin{aligned} 0 &\to \text{Syz}_R(x^a, y^b, z^c)(-\alpha) \\ &\to \text{Syz}_R(f^l, y^b, z^c)(-\alpha) \oplus \text{Syz}_R(f^{l+1}, y^b, z^c) \\ &\to \text{Syz}_R(x^a, y^b, z^c) \to 0. \end{aligned}$$

Note that this Theorem holds also in much bigger generality.

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- One gets

$$\mathrm{Syz}(X^q, Y^q, Z^q) \cong M(-n/2).$$

**Theorem (_)**

The Hilbert-Kunz function of $D_n$ is the map

$$e \mapsto \left(2 - \frac{1}{4n-8}\right)(p^e)^2 + \frac{m^2}{4n-8} - \frac{m+1}{2},$$

where $m \equiv p^e \ (2n-4)$.

**Theorem (_)**

The Hilbert-Kunz function of $E_6$ is the map

$$e \mapsto \left(2 - \frac{1}{24}\right)(p^e)^2 - \frac{23}{24}.$$

### Theorem (_)

The Hilbert-Kunz function of $E_7$ is the map

$$e \mapsto \begin{cases} \left(2 - \frac{1}{48}\right)(p^e)^2 - \frac{71}{48}, & \text{if } p^e \ (24) \in \{\pm 5, \pm 11\} \\ \left(2 - \frac{1}{48}\right)(p^e)^2 - \frac{47}{48}, & \text{else} \end{cases}$$

### Theorem (_)

The Hilbert-Kunz function of $E_8$ is the map

$$e \mapsto \begin{cases} \left(2 - \frac{1}{120}\right)(p^e)^2 - \frac{191}{120}, & \text{if } p^e \ (30) \in \{\pm 7, \pm 13\} \\ \left(2 - \frac{1}{120}\right)(p^e)^2 - \frac{119}{120}, & \text{else} \end{cases}$$