The Hilbert-Kunz Functions of Surfaces of Type ADE

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Mangalia, 05. September 2012

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Outline



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- The geometric approach of Brenner/ Trivedi
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 - Matrix factorizations
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 - Representations of MCMs as syzygies
 - How to control the Frobenius pullbacks?

The Hilbert-Kunz functions of surfaces of type ADE

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Basic definitions The geometric approach of Brenner/ Trivedi

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The Hilbert-Kunz function

- Let (R, m) be a local, Noetherian ring of (Krull-)dimension d,
- $k \subset R$ a field of characteristic p,
- and $I = (f_1, \ldots, f_n)$ be an m-primary ideal.
- Denote by $F: R \to R, r \mapsto r^p$ the Frobenius morphism and
- let $I^{[q]} := (f_1^q, ..., f_n^q) = F^e(I)$ with $q = p^e$.

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- let $I^{[q]} := (f_1^q, \dots, f_n^q) = F^e(I)$ with $q = p^e$.

Then the map

$$e \mapsto \dim_k((R/I^{[p^e]}))$$

is called the Hilbert-Kunz function of R with respect to I.

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The Hilbert-Kunz multiplicity

$$e_{HK}(R,I) := \lim_{e o \infty} rac{\dim_k(R/I^{[q]})}{q^d}$$

is a positive real number (Monsky, 1983) and is called the *Hilbert-Kunz multiplicity of R with respect to I.*

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We call $e_{HK}(R) := e_{HK}(R, \mathfrak{m})$ the Hilbert-Kunz multiplicity of R.

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Question: Is $e_{HK}(R)$ rational?

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Examples

We want to compute the Hilbert-Kunz functions with respect to the maximal ideal of the rings

R := k[[X, Y]] and S := k[[X, Y, Z]]/(ZX, ZY).

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 $\begin{aligned} R &:= k\llbracket X, Y \rrbracket \quad \text{and} \quad S &:= k\llbracket X, Y, Z \rrbracket / (ZX, ZY). \\ \dim_k(R/(X^q, Y^q)) \end{aligned}$

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$$R := k[[X, Y]] \text{ and } S := k[[X, Y, Z]]/(ZX, ZY).$$

$$\dim_k(R/(X^q, Y^q)) = \left| \{X^i Y^j | 0 \le i, j \le q-1\} \right|$$

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$$\dim_k(S/(X^q, Y^q, Z^q)) = \left| \{X^i Y^j \mid 0 \le i, j \le q - 1\} \cup \{Z^i \mid 0 \le i \le q - 1\}$$

$$= q^2 + q - 1$$

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Both rings have the same Hilbert-Kunz multiplicity (one).

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$$= q^2 + q - 1$$

Both rings have the same Hilbert-Kunz multiplicity (one). BUT: R is regular, while S is not.

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The previous examples shows that studying only Hilbert-Kunz multiplicities is not enough.

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Regular rings have Hilbert-Kunz multiplicity one. The converse does not hold by the example.

For the converse one needs to add "equidimensional" to the assumptions. In this case the Hilbert-Kunz multiplicity measures the singularity of the ring (high values for bad singularities).

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Surfaces of type ADE

Surfaces of type ADE are the rings of invariants of k[u, v] under the group actions of finite subgroups of $SL_2(k)$, where |G| is invertible in k.

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They are of the form k[X, Y, Z]/(f) with

$$\begin{array}{l} A_n: \ f = X^{n+1} + YZ, \ \omega = (2, n+1, n+1) \\ D_n: \ f = X^2 + Y^{n-1} + YZ^2, \ \omega = (n-1, 2, n-2), \ n \geq 4 \\ E_6: \ f = X^2 + Y^3 + Z^4, \ \omega = (6, 4, 3) \\ E_7: \ f = X^2 + Y^3 + YZ^3, \ \omega = (9, 6, 4) \\ E_8: \ f = X^2 + Y^3 + Z^5, \ \omega = (15, 10, 6) \end{array}$$

By Watanabe & Yoshida these rings have Hilbert-Kunz multiplicity 2 - 1/|G|.

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Reduction to a standard-graded case I

Let *R* be a surface of type ADE. The map

$$k[X,Y,Z] \longrightarrow k[U,V,W],$$

$$X \mapsto U^{\deg(X)}, \ Y \mapsto V^{\deg(Y)}, \ Z \mapsto W^{\deg(Z)}$$

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induces a map from *R* to $S_R := k[U, V, W]/(F)$, where *F* is the image of *f* under this map. The induced map $R \to S_R$ is local and flat.

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Reduction to a standard-graded case II

Theorem

Let $(R, \mathfrak{m}) \rightarrow S$ be a flat, local morphism and *M* an *R*-module. Assume $k \subset R, S$. Then

$$\dim_k(M) \cdot \dim_k(S/\mathfrak{m}S) = \dim_k(M \otimes_R S).$$

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In our situation we get

$$\dim_k(M) = \frac{\dim_k(M \otimes_R S_R)}{\deg(X) \cdot \deg(Y) \cdot \deg(Z)}.$$

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This reduces our computation to a standard-graded case.

Some notation

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- Let *R* be a normal, standard-graded domain and *Y* its Proj.
- Let $k \subset R$ an algebraically closed field with char(k)= p,
- *f*₁,..., *f*_n ∈ *R* homogeneous elements of degree *d*_i, generating an *R*₊-primary ideal

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- Let $k \subset R$ an algebraically closed field with char(k)= p,
- *f*₁,..., *f*_n ∈ *R* homogeneous elements of degree *d*_i, generating an *R*₊-primary ideal
- We have the short exact sequence

(*)
$$0 \rightarrow \operatorname{Syz}_{Y}(f_{1}, \ldots, f_{n}) \rightarrow \bigoplus_{i=1}^{n} O_{Y}(-d_{i}) \rightarrow O_{Y} \rightarrow 0.$$

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Frobenius pullbacks

 Let F : Y → Y be the absolute Frobenius morphism (induced by the ordinary Frobenius on section rings).

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- Taking the *e*-th iterated pullback of (*) and tensoring it with O_Y(m) yields

$$0 \to F^{*e}(\operatorname{Syz}_{Y}(f_{1},\ldots,f_{n}))(m) \to \bigoplus_{i=1}^{n} O_{Y}(m-qd_{i}) \to O_{Y}(m) \to 0.$$

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Note that we have $F^{*e}(Syz_Y(f_1, \ldots, f_n)) = Syz_Y(f_1^q, \ldots, f_n^q)$.

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The connection to Hilbert-Kunz functions

Taking global sections yields

$$0 \rightarrow H^{0}(\operatorname{Syz}_{Y}(f_{1}^{q}, \dots, f_{n}^{q})(m)) \rightarrow \oplus_{i=1}^{n} H^{0}(O_{Y}(m - qd_{i}))$$

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$$\rightarrow H^{0}(O_{Y}(m)) \rightarrow (R/I^{[q]})_{m} \rightarrow 0.$$

Summing up the dimensions, we get

$$\dim(R/(I^{[q]})_m) = h^0(O_Y(m)) - \sum_{i=1}^n h^0(O_Y(m-qd_i)) + h^0(\operatorname{Syz}_Y(f_1^q, \dots, f_n^q)(m)).$$

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How does this help?

We have to control $h^0(\operatorname{Syz}_Y(f_1^q, \ldots, f_n^q)(m))$.

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How does this help?

We have to control $h^0(\operatorname{Syz}_Y(f_1^q,\ldots,f_n^q)(m))$.

The easiest case would be a splitting into free sheaves.

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The idea is to look for a finite list of "easier" sheafs of O_Y -modules such that $Syz_Y(f_1^q, \ldots, f_n^q)(m)$ is one of those (up to twist).

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Idea: Hypersurfaces of type *ADE* are Cohen-Macaulay finite (= there are up to isomorphism only finitely many indecomposable, maximal CM modules).

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Syzygies are MCM

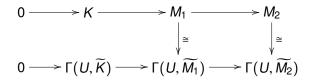
Let *K* be the kernel of the map $M_1 \rightarrow M_2$ between reflexive *R*-modules and $U := \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}.$

Basic definitions The geometric approach of Brenner/ Trivedi

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Syzygies are MCM

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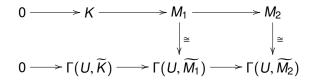


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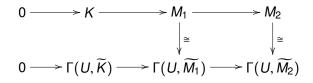
induces an isomorphism $K \cong (K^{\vee})^{\vee}$.

Basic definitions The geometric approach of Brenner/ Trivedi

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induces an isomorphism $K \cong (K^{\vee})^{\vee}$.

Since we are in a two-dimensional situation, maximal Cohen-Macaulay and reflexive are equivalent.

MCMs over hypersurfaces Matrix factorizations

MCM modules over hypersurfaces I

Let (S, \mathfrak{n}) be a regular, local, Noetherian ring, $f \in \mathfrak{n}^2$ non-zero and R := S/(f).

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For a MCM *R*-module *M* the Auslander-Buchsbaum formula $(pd_S(M)+depth_S(M) = depth_S(S))$ says that *M* as an *S*-module has a free resolution of length one:

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Since *M* is an *R*-module, we have $f \cdot M = 0$. We get

$$f \cdot S^n \subseteq \varphi(S^n).$$

MCMs over hypersurfaces Matrix factorizations

MCM modules over hypersurfaces II

Because φ is injective, we get for a given $x \in S^n$ an unique $y \in S^n$ with $fx = \varphi(y)$.

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MCMs over hypersurfaces Matrix factorizations

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Let ψ be the map $x \mapsto y$. Then ψ is linear and $\varphi \psi = f \cdot id$.

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Moreover, $\varphi \psi \varphi = f \cdot \varphi = \varphi(f \cdot id)$ gives $\psi \varphi = f \cdot id$, because φ is injective.

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Let Γ be the "map" that attaches to *M* the tuple (φ, ψ) .

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MCMs over hypersurfaces Matrix factorizations

The category of matrix factorizations I

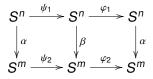
 A pair (φ, ψ) ∈ Mat_{n×n}(S) with φψ = ψφ = f · id is called matrix factorization of f.

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MCMs over hypersurfaces Matrix factorizations

The category of matrix factorizations I

- A pair (φ, ψ) ∈ Mat_{n×n}(S) with φψ = ψφ = f · id is called matrix factorization of f.
- A pair of matrices (α, β) is called a morphism of matrix factorizations (φ₁, ψ₁) and (φ₂, ψ₂) if the following diagram commutes:

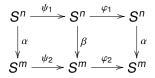


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We call (φ₁, ψ₁) and (φ₂, ψ₂) equivalent if α and β are isomorphisms.

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MCMs over hypersurfaces Matrix factorizations

The category of matrix factorizations II

Given a matrix factorization (φ, ψ) , we get a short exact sequence

$$0 \to S^n \xrightarrow{\varphi} S^n \to \operatorname{coker}(\varphi) \to 0.$$

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The sequence of maps and modules

$$\cdots \to S^n \xrightarrow{\varphi} S^n \xrightarrow{\psi} S^n \xrightarrow{\varphi} S^n \to \operatorname{coker}(\varphi) \to 0$$

reduces modulo *f* to an 2-periodic *R*-free of $coker(\varphi)$.

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reduces modulo *f* to an 2-periodic *R*-free of $coker(\varphi)$. Because $Syz^{i}(M)$ is either 0 or MCM for all $M \in R$ -mod and every $i \ge \dim R$, $coker(\varphi)$ is MCM over *R*.

MCMs over hypersurfaces Matrix factorizations

The equivalence of categories

Theorem (Eisenbud 1980)

The functors coker and Γ induce an equivalence of categories

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MCMs over hypersurfaces Matrix factorizations

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MCMs over hypersurfaces Matrix factorizations

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reduced, indecomposable matrix factorizations of *f* up to equivalence



non-free, indecomposable, maximal Cohen-Macaulay

R – modules up to isomorphism.

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Representations of MCMs as syzygies How to control the Frobenius pullbacks?

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An observation

In 2006 Kajura, Saito and Takahashi computed the matrix factorizations of surfaces of type ADE. In the E_6 case ($X^2 + Y^3 + Z^4 = 0$) there are three that give a MCM module of rank 2.

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$$\varphi = \psi = \begin{pmatrix} -X & 0 & Y^2 & Z^3 \\ 0 & -X & Z & -Y \\ Y & Z^3 & X & 0 \\ Z & -Y^2 & 0 & X \end{pmatrix}.$$

Representations of MCMs as syzygies How to control the Frobenius pullbacks?

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$$arphi = \psi = egin{pmatrix} -X & Y^2 & Z^3 \ 0 & Z & -Y \ Y & X & 0 \ Z & 0 & X \end{pmatrix}.$$

Deleting the second column, the rows of the remaining 4×3 matrix generate

$$\operatorname{Syz}_{R}(-X, Y, Z)$$

Representations of MCMs as syzygies How to control the Frobenius pullbacks?

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Representation I

We now show that $\operatorname{coker}(\varphi)$ and $\operatorname{Syz}_R(-X, Y, Z)$ become isomorphic as sheafs on the punctured spectrum $U := \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}.$

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$$\cdots \xrightarrow{\varphi} R^4 \xrightarrow{\varphi} R^4 \xrightarrow{\varphi} R^4 \longrightarrow \operatorname{coker}(\varphi) \to 0.$$

Representations of MCMs as syzygies How to control the Frobenius pullbacks?

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This gives coker(φ) = im(φ), which is generated by the columns of φ.

Representations of MCMs as syzygies How to control the Frobenius pullbacks?

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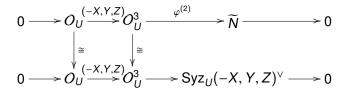
- This gives coker(φ) = im(φ), which is generated by the columns of φ.
- Show that on D(Y) ∪ D(Z) the second column belongs to N, where N is the module generated by the columns 1,3,4.

Representations of MCMs as syzygies How to control the Frobenius pullbacks?

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Representation II

The commutative diagram

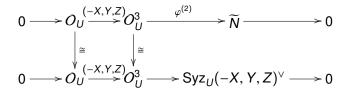


Representations of MCMs as syzygies How to control the Frobenius pullbacks?

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Representation II

• The commutative diagram



induces an isomorphism $\widetilde{N} \cong \operatorname{Syz}_U(-X, Y, Z)^{\vee}$.

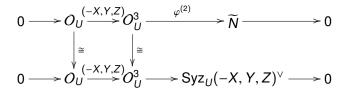
Representations of MCMs as syzygies How to control the Frobenius pullbacks?

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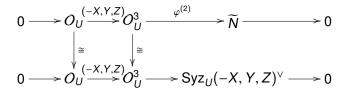
 Since M and Syz_R(-X, Y, Z) are reflexive, the above isomorphism lifts to R.

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- Since M and Syz_R(-X, Y, Z) are reflexive, the above isomorphism lifts to R.
- Show Syz_R(-X, Y, Z)[∨] ≅ Syz_R(-X, Y, Z) by the theory of matrix factorizations.

Representations of MCMs as syzygies How to control the Frobenius pullbacks?

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A short exact Sequence

We need a criterium to distinguish the different MCM of rank 2 and to compute the isomorphism class of the Frobenius pullbacks of Syz(X, Y, Z).

Representations of MCMs as syzygies How to control the Frobenius pullbacks?

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A short exact Sequence

We need a criterium to distinguish the different MCM of rank 2 and to compute the isomorphism class of the Frobenius pullbacks of Syz(X, Y, Z). This can be done by the ordinary Hilbert series.

Theorem (Brenner, Kaid 2007,_)

Let $R := k[x, y, z]/(x^2 - f(y, z))$ with weights $\omega = (\alpha, \beta, \gamma)$ where f is homogeneous of degree $2 \cdot \alpha$. For $a = 2 \cdot l + 1$, with $l \in \mathbb{N}$ and b, $c \in \mathbb{N}$ we have a short exact sequence:

Representations of MCMs as syzygies How to control the Frobenius pullbacks?

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$$\begin{array}{rcl} 0 & \rightarrow & \operatorname{Syz}_R(x^a, y^b, z^c)(-\alpha) \\ & \rightarrow & \operatorname{Syz}_R(f^l, y^b, z^c)(-\alpha) \oplus \operatorname{Syz}_R(f^{l+1}, y^b, z^c) \\ & \rightarrow & \operatorname{Syz}_R(x^a, y^b, z^c) \to 0. \end{array}$$

Note that this Theorem holds also in much bigger generality.

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To control the Frobenius-pullbacks of Syz(X, Y, Z) we have to do the following:

Daniel Brinkmann Universität Osnabrück The Hilbert-Kunz Functions of Surfaces of Type ADE

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To control the Frobenius-pullbacks of Syz(X, Y, Z) we have to do the following:

• Represent all MCM of rank 2 as syzygy modules

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- Compute their Hilbert series (and recognize that they are different)

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- This will be of the form t^n times the Hilbert series of a rank 2 MCM $M = Syz(f_1, f_2, f_3)$.

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- This will be of the form t^n times the Hilbert series of a rank 2 MCM $M = Syz(f_1, f_2, f_3)$.
- One gets

$$\operatorname{Syz}(X^q, Y^q, Z^q) \cong M(-n/2).$$

Theorem (_)

The Hilbert-Kunz function of D_n is the map

$$e\mapsto \left(2-rac{1}{4n-8}
ight)(p^e)^2+rac{m^2}{4n-8}-rac{m+1}{2},$$

where
$$m \equiv p^e (2n - 4)$$
.

Theorem (_)

The Hilbert-Kunz function of E_6 is the map

$$e\mapsto \left(2-\frac{1}{24}\right)(p^e)^2-\frac{23}{24}.$$

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Theorem (_)

The Hilbert-Kunz function of E_7 is the map

$$e \mapsto \begin{cases} \left(2 - \frac{1}{48}\right)(p^e)^2 - \frac{71}{48}, & \text{if } p^e \ (24) \in \{\pm 5, \pm 11\}\\ \left(2 - \frac{1}{48}\right)(p^e)^2 - \frac{47}{48}, & \text{else} \end{cases}$$

Theorem (_)

The Hilbert-Kunz function of E₈ is the map

$$e \mapsto \begin{cases} \left(2 - \frac{1}{120}\right) (p^e)^2 - \frac{191}{120}, & \text{if } p^e \ (30) \in \{\pm 7, \pm 13\} \\ \left(2 - \frac{1}{120}\right) (p^e)^2 - \frac{119}{120}, & \text{else} \end{cases}$$

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