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Toric Ideals, an Introduction

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Toric ideals

Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_m\} \subseteq \mathbb{Z}^n$. We consider $\mathbb{N}A := \{l_1\mathbf{a}_1 + \cdots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}_0\}$. Let K be a field.

Definition

The toric ideal I_A associated to A is the kernel of the K-algebra homomorphism

$$\phi : \; K[x_1, \cdots, x_m] \to K[t_1, \cdots, t_n, t_1^{-1} \cdots t_n^{-1}]$$

given by

$$\phi(x_i) = t^{\mathbf{a}_i} = t_1^{a_{i,1}} t_2^{a_{i,2}} \cdots t_n^{a_{i,n}} \;,$$

where $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \cdots, a_{i,n})$.

Example

Let $A = \{2, 1, 1\} \subset \mathbb{N}$.

$$\phi: \ K[x_1, x_2, x_3] \to K[t]: \ x_1 \mapsto t^2, x_2 \mapsto t, x_3 \mapsto t$$

Some elements in the kernel of ϕ (i.e. I_A) are:

- $x_1 - x_2^2 \in I_A$,
- $x_1 - x_3^2 \in I_A$,
- $x_2 - x_3 \in I_A$,
- $x_2^5 - x_1^2 x_3 \in I_A$, etc

It is not hard to prove that there are at least two minimal generating sets, (exercise).

$$I_A = (x_1 - x_2^2, x_2 - x_3) = (x_1 - x_3^2, x_2 - x_3)$$

Example

Let $A = \{(2,1,0),(1,2,0),(2,0,1),(1,0,2),(0,2,1),(0,1,2)\}$.

$$A \leftrightarrow \begin{bmatrix} 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix}$$

$$\phi : \ K[x_1, x_2, x_3, x_4, x_5, x_6] \to K[t_1, t_2, t_3]$$

$x_1 \mapsto t_1^2 t_2$, $x_2 \mapsto t_1 t_2^2$, $x_3 \mapsto t_1^2 t_3$, $x_4 \mapsto t_1 t_3^2$, $x_5 \mapsto t_2^2 t_3$, $x_6 \mapsto t_2 t_3^2$.
One can prove that
$I_A = (x_1x_6 - x_2x_4, x_1x_6 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6,$
$x_1x_4^2 - x_3^2x_6, x_4^2x_5 - x_3x_6^2, x_1x_4x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2)$.

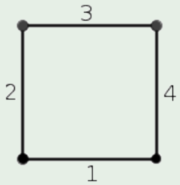
Note that the sum of the first and sixth column equals the sum of the second and fourth column, (dependence relation among the columns).

Example

$$A \leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$I_A = (x_1x_3 - x_2x_4)$$

The columns of the matrix of A correspond to the edges of the following graph:

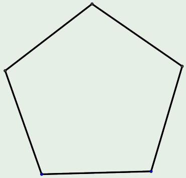


I_A is called the toric ideal of the graph.

Example

$$A \leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The corresponding graph is



The toric ideal is 0: $I_A = 0$.

Properties of toric ideals

- toric ideals are prime ideals (easy proof)
- toric ideals are generated by binomials (needs more work–initial terms of polynomials).

Does the converse hold?
How do we compute toric ideals?

Grading and Toric ideals

$A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subseteq \mathbb{Z}^n$. We grade the polynomial ring $K[x_1, \ldots, x_m]$ by the semigroup $\mathbb{N}A$:

$$\deg_A(x_i) = \mathbf{a}_i \ , \quad i = 1, \ldots, m \ .$$

For $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$ and $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \cdots x_m^{u_m}$ we let

$$\deg_A(\mathbf{x}^{\mathbf{u}}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A.$$

Theorem

The toric ideal I_A *is generated by all the binomials* $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ *such that* $\deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}})$.

If $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A$ we define $\deg_A(\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}) := \deg_A(\mathbf{x}^{\mathbf{u}})$.

Example

Example

Let $A = \{2, 1, 1\}$. In $\phi : \ K[x_1, x_2, x_3]$ we set $\deg_A(x_1) = 2$, $\deg_A(x_2) = \deg_A(x_1) = 1$.

$$I_A = (x_1 - x_2^2, x_2 - x_3)$$

$$\deg_A(x_1 - x_2^2) = 2, \ \ \deg_A(x_2 - x_3) = 1$$

Example

Example

$$A \leftrightarrow \begin{bmatrix} 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix}$$

$I_A = (x_1x_6 - x_2x_4, x_1x_6 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6,$
$x_1x_4^2 - x_3^2x_6, x_4^2x_5 - x_3x_6^2, x_1x_4x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2).$

$$\deg_A(x_1x_6 - x_2x_4) = (2, 2, 2)\ .$$

Generators of Toric Ideals

Remark

If $\mathbf{u} \in \mathbb{Z}^m$ *then* $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ *where* $\mathbf{u}^+, \mathbf{u}^- \in \mathbb{N}^m$*.*

Recall that $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$.
We let $\pi : \mathbb{Z}^m \longrightarrow \mathbb{Z}A$ where $(u_1, \ldots, u_m) \mapsto u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m$.
If $\mathbf{w}, \mathbf{v} \in \mathbb{N}^m$ and $\deg_A(\mathbf{w}) = \deg_A(\mathbf{w})$ then $u = w - v \in \ker \pi$.

Theorem

$$I_A = \langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \ : \ \mathbf{u} \in \ker \pi \rangle$$

Example

Example

Let $A = \{2, 1, 1\}$. We have seen that

$$I_A = (x_1 - x_2^2, x_2 - x_3)$$

Let $\pi : \mathbb{Z}^3 \longrightarrow \mathbb{Z}A$ where $\mathbf{u} = (u_1, u_2, u_3) \mapsto 2u_1 + u_2 + u_3$.
$\mathbf{u} = (1, -2, 0) \in \ker \pi$ and $\mathbf{u} = (1, 0, 0) - (0, 2, 0)$.
Note that another way to get $\mathbf{u}$ is as the difference
$\mathbf{u} = (3, 1, 2) - (2, 3, 2)$.

The corresponding binomials are

$$x_1 - x_2^2 \text{ and } x_1^3 x_2 x_3^2 - x_1^2 x_2^3 x_3^2 \in I_A \ .$$

Important binomials of I_A

- primitive binomials
- circuit binomials
- binomials that belong to a reduced Grobner basis of I_A
- binomials that belong to a minimal generating set of I_A (Markov binomials)
- indispensable binomials that belong to all generating sets of I_A

Graver basis

Definition

An irreducible binomial $x^{\mathbf{u}^+} - x^{\mathbf{u}^-} \in I_A$ *is called* primitive *if there exists no other binomial* $x^{\mathbf{v}^+} - x^{\mathbf{v}^-} \in I_A$ *such that* $x^{\mathbf{v}^+}$ *divides* $x^{\mathbf{u}^+}$ *and* $x^{\mathbf{v}^-}$ *divides* $x^{\mathbf{u}^-}$.

The set of all primitive binomials of a toric ideal I_A *is called the Graver basis of* I_A.

Example

We have seen that $I = (x_1 - x_2^2, x_2 - x_3)$ is the toric ideal corresponding to $A = \{2, 1, 1\}$.

All elements of the minimal generating set of I are primitive.

$x_2^5 - x_1^2 x_3 \in I$ is not primitive since $x_2^2 - x_1 \in I$.

Circuits

$$supp(x^{\mathbf{u}}) := \{i \mid x_i \ \text{ divides } x^{\mathbf{u}}\}$$

and

$$supp(x^{\mathbf{u}} - x^{\mathbf{v}}) := supp(x^{\mathbf{u}}) \cup supp(x^{\mathbf{v}}) \ .$$

Definition

An irreducible binomial $B \in I_A$ is called a circuit if there is no binomial $B' \in I_A$ such that $supp(B') \subsetneq supp(B)$.

No two circuits can have the same support, (why?).

Example

Example

$A = \{1, 2, 4\}$. Then $I_A = (x_1^2 - x_2, x_1^4 - x_3, x_2^2 - x_3)$.
All 3 elements of this generating set of I_A are circuits:

$supp(x_1^2 - x_2) = \{1, 2\}$, $supp(x_1^4 - x_3) = \{1, 3\}$,
$supp(x_2^2 - x_3) = \{2, 3\}$.

All 3 elements of this generating set of I_A are primitive.

The element $x_3 - x_1^2 x_2$ is primitive but is not a circuit.

Circuits and Primitive binomials

What is the relation between Circuits and Primitive binomials?

Theorem

(B. Sturmfels) Circuits are primitive.

Theorem

(B. Sturmfels) The set of circuits of I_A generate up to radical I_A.

Review on Monomial Orders

By T^n we denote the set of monomials $x^{\mathbf{a}}$ in $k[x_1, \ldots, x_n]$, where $x^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $\mathbf{a} = (a_1, a_2, \cdots, a_n)$.

Definition

By a monomial order on T^n we mean a total order on T^n such that

- $1 < x^{\mathbf{a}}$ *for all* $x^{\mathbf{a}} \in T^n$ *with* $x^{\mathbf{a}} \neq 1$
- *If* $x^{\mathbf{a}} < x^{\mathbf{b}}$ *then* $x^{\mathbf{a}} x^{\mathbf{c}} < x^{\mathbf{b}} x^{\mathbf{c}}$ *for all* $x^{\mathbf{c}} \in T^n$.

If $n \geq 2$ then there are infinitely many monomial orders on T^n.

Review on leading monomials: initial terms

Let $<$ be a monomial order on $k[x_1, \ldots, x_n]$. Let f be a nonzero polynomial in $k[x_1, \ldots, x_n]$. We may write

$$f = a_1 x^{\mathbf{u}_1} + a_2 x^{\mathbf{u}_2} + \ldots a_r x^{\mathbf{u}_r},$$

where $a_i \neq 0$ and $x^{\mathbf{u}_1} > x^{\mathbf{u}_2} > \cdots > x^{\mathbf{u}_r}$. If $a_1 = 1$ we call f monic.

Definition

For $f \neq 0$ in $k[x_1, \ldots, x_n]$, we define the leading monomial of f to be $in_<(f) = x^{\mathbf{u}_1}$.

Proof that I_A is generated by binomials

$A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$, $\phi :\ K[x_1, \cdots, x_m] \to K[t_1, \cdots, t_n, t_1^{-1} \cdots t_n^{-1}]$, $f(x_1, \ldots, x_m) \mapsto f(t^{\mathbf{a}_1}, \ldots, t^{\mathbf{a}_m})$. We will show that

$$I_A = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} :\ \deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}}) \rangle \ .$$

Sketch of Proof

One direction (RHS $\subset$ LHS) is clear.

For the other containment fix a monomial order $<$. If LHS $\not\subset$ RHS, find $f \in I_A$, f monic, $f \notin$ RHS. Choose f so that $\mathrm{in}_<(f) = \mathbf{x}^{\mathbf{u}}$ is minimal among all elements with this property.

Since f is in $I_A = \ker \phi$, f must have at least one other term $x^{\mathbf{v}}$, whose image will cancel the image of $\mathbf{x}^{\mathbf{u}}$ (so that $\deg_A(x^{\mathbf{u}}) = \deg_A(x^{\mathbf{v}})$).

But then $f - (x^{\mathbf{u}} - x^{\mathbf{v}}) \in I_A$ and this gives a contradiction, (why?)

Review: Grobner bases

Definition

A set of non-zero polynomials $G = \{g_1, \ldots, g_t\}$ contained in an ideal I, is called Grobner basis for I if and only if for all nonzero $f \in I$ there exists $i \in \{1, \ldots, t\}$ such that $in_<(g_i)$ divides $in_<(f)$.

For any nonzero ideal I and for any term order there exists a Grobner basis for I. There are infinitely many Grobner bases for I.

Reduced Grobner bases

Definition

A Grobner basis $G = \{g_1, \ldots, g_t\}$ is called a reduced Grobner basis for I if

- *The polynomials g_i are monic for $i \in \{1, \ldots, t\}$ and*
- *no monomial of g_i is divisible by any $in_<(g_j)$ for any $j \neq i$.*

Theorem

(Buchberger) Let $<$ be a monomial order on $k[x_1, \ldots, x_n]$ and I a nonzero ideal. Then I has a unique reduced Grobner basis with respect to the monomial order $<$.

Reduced Grobner bases of toric ideals

Theorem

Let I be a toric ideal and $<$ a monomial order on the underlying ring. The reduced Grobner basis of I consists of binomials.

(Buchberger's algorithm applied to a binomial generating set of I)

How to compute a binomial generating set of I_A.

Usually one starts with the generators of an ideal and then finds a Grobner basis of it. For the task in question we work in the opposite direction!
Suppose that $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subseteq \mathbb{N}^n$. Consider the ideal J in $K[t_1, \ldots, t_n, x_1, \ldots, x_m]$:

$$J = \langle x_1 - t^{\mathbf{a}_1}, \ldots, x_m - t^{\mathbf{a}_m} \rangle$$

Lemma

Let $\phi : \; K[x_1, \cdots, x_m] \to K[t_1, \cdots, t_n]$ *given by* $x_i \mapsto \mathbf{t}^{\mathbf{a}_i} = t_1^{a_{i,1}} t_2^{a_{i,2}} \cdots t_n^{a_{i,n}}$ *for* $i = 1, \ldots m$. *Then*

$$I_A = \ker \phi = J \cap K[x_1, \ldots, x_m]$$

Sketch of proof

How to compute a binomial generating set of I_A.

$$\ker \phi = \langle x_1 - t^{\mathbf{a}_1}, \ldots, x_m - t^{\mathbf{a}_m} \rangle \cap K[x_1, \ldots, x_m]$$

(one direction: inclusion RHS to LHS) Let f be an element on the RHS of the equality. Then

$$f(x_1, \ldots, x_m) = \sum_{i=1}^{m} (x_i - \mathbf{t}^{\mathbf{a}_i}) h_i(t_1, \ldots, t_n, x_1, \ldots, x_m) \ .$$

We see that $f(\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_m}) = 0$ and $f \in \ker \phi = I_A$

How to compute a binomial generating set of I_A.

$$\ker \phi = \langle x_1 - t^{\mathbf{a}_1}, \ldots, x_m - t^{\mathbf{a}_m} \rangle \cap K[x_1, \ldots, x_m]$$

(Other direction: LHS inside the RHS)
Let $g(x_1, \ldots, x_m) = \sum_{\mathbf{u}} c_{\mathbf{u}} {x_1}^{u_1} \cdots {x_m}^{u_m} \in \ker \phi$. Then $g(\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_m}) = \sum_{\mathbf{u}} c_{\mathbf{u}} \mathbf{t}^{u_1 \mathbf{a}_1} \cdots \mathbf{t}^{u_m \mathbf{a}_m} = 0$. Thus in $K[t_1, \ldots, t_n, x_1, \ldots, x_n]$

$$g(x_1, \ldots, x_m) = g(x_1, \ldots, x_m) - g(\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_m}) =$$

$$\sum_{\mathbf{u}} c_{\mathbf{u}} ({x_1}^{u_1} \cdots {x_m}^{u_m} - \mathbf{t}^{u_1 \mathbf{a}_1} \cdots \mathbf{t}^{u_m \mathbf{a}_m}) \in J \ .$$

How to compute a binomial generating set of I_A

Theorem

Consider an elimination monomial order in $K[t_1, \ldots, t_n, x_1, \ldots, x_m]$, with $t_1, \ldots, t_n$ all greater than $x_1, \ldots, x_m$. Let G be the reduced Grobner basis of J in $K[t_1, \ldots, t_n, x_1, \ldots, x_m]$. Then

$$G \cap K[x_1, \ldots, x_m] \textit{ is a Grobner basis of } J \cap K[x_1, \ldots, x_m] = I_A$$

The above gives the algorithm for finding a reduced Grobner basis of I_A when $A \subset \mathbb{N}^n$. This is a generating set of I_A.

Exercise

Find an algorithm to compute I_A when $A \subset \mathbb{Z}^n$.

Initial ideals

Definition

The Universal Grobner basis of an ideal I is the union of all reduced Grobner bases.

Is this set finite?

Let $<$ be a monomial order on $k[x_1, \ldots, x_n]$ and I a nonzero ideal. The initial ideal of ideal of I is the monomial ideal generated by the leading monomials of any polynomial of I.

Theorem

Let I be a nonzero ideal of $k[x_1, \ldots, x_n]$. Then I has finitely many distinct initial ideals.

Universal Grobner basis of a toric ideal

Theorem

(Weispfenning, N. Schwartz) The universal Grobner basis of any ideal I is a finite set.

Let I_A be a toric ideal.

Theorem

The universal Grobner basis of I_A is a finite set of binomials.

The universal Grobner basis is a finite subset of I_A and is a Grobner basis for I_A with respect to all term orders simultaneously.

Circuits, Universal Grobner bases and Primitive polynomials

Let UGB_A denote the Universal Grobner basis of A.

Theorem

(Sturmfels) For any toric ideal I_A the following containments hold:

$$Circuits_A \subset UGB_A \subset Graver_A$$

- B. Sturmfels, *Grobner bases and Convex Polytopes, University Lecture Series, Vol 8, AMS (1995)*
- *B. Sturmfels, Equations defining toric varieties, Algebraic Geometry - Santa Cruz 1995, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997, pp. 437-449.*