DISCRETE INVARIANTS IN COMMUTATIVE ALGEBRA AND IN ALGEBRAIC GEOMETRY
20th National School on Algebra
Mangalia, Romania
2 - 8 September 2012

Toric Ideals and Minimal systems of generators

Hara Charalambous

Department of Mathematics University of Thessaloniki, Greece
Circuits, Universal Grobner bases and Primitive polynomials

Let $A = \{a_1, \ldots, a_m\} \subset \mathbb{Z}^n$ and $I_A$ the corresponding toric ideal in $K[x_1, \ldots, x_m]$.

- $\text{deg}_A(x_1^{u_1} \cdots x_m^{u_m}) := u_1a_1 + \cdots + u_ma_m \in \mathbb{N}A$
- $I_A = \langle x^u - x^v : \text{deg}_A(x^u) = \text{deg}_A(x^v) \rangle$.

Theorem

(Sturmfels) For any toric ideal $I_A$ the following containments hold:

$\text{Circuits}_A \subset \text{UGB}_A \subset \text{Graver}_A$
Definition

An irreducible binomial $B \in I_A$ is called a circuit if there is no binomial $B' \in I_A$ such that $\text{supp}(B') \subsetneq \text{supp}(B)$.

where if

$$B = x^u - x^v \text{ then } \text{supp}(B) = \text{supp}(x^u) \cup \text{supp}(x^v)$$

and

$$\text{supp}(x^u) = \{i \mid x_i \text{ divides } x^u\}$$

Definition

An irreducible binomial $x^{u^+} - x^{u^-} \in I_A$ is called primitive if there exists no other binomial $x^{v^+} - x^{v^-} \in I_A$ such that $x^{v^+}$ divides $x^{u^+}$ and $x^{v^-}$ divides $x^{u^-}$. 
Example

Let $A = \{1, 2, 4\}$.

Then $I_A = \langle x_1^2 - x_2, x_1^4 - x_3, x_2^2 - x_3 \rangle$.

- $x_1^2 - x_2, x_1^4 - x_3, x_2^2 - x_3$ are circuits.
- $x_1^2 x_2 - x_3$ is primitive, but not a circuit.
- $x_1^2 x_2 - x_3$ is primitive, but not in the Universal Grobner basis of $I_A$.

Proof: Let $x_1^2 x_2 - x_3$ be in the reduced Grobner basis $G$.

Fact: $x_1^2 - x_2 \in I_A$. Thus there is $g \in G$ such that $\text{in}_{<}(g)$ divides $\text{in}_{<}(x_1^2 - x_2)$.

Case 1: $\text{in}_{<}(x_1^2 - x_2) = x_1^2$. Then $\text{in}_{<}(g)$ divides $x_1^2 x_2$, a contradiction.

Case 2: $\text{in}_{<}(x_1^2 - x_2) = x_2$. Again we derive a contradiction.

What is a minimal generating set of $I_A$? Is it unique?

Question

When is a primitive binomial not in the Universal Grobner basis? When is $\text{UGB}_A = \text{Graver}_A$?
Pointed semigroups

Definition

The affine semigroup $\mathbb{N}A := \{l_1a_1 + \cdots + l_ma_m \mid l_i \in \mathbb{N}\}$ is pointed if

$$\{x : x \in \mathbb{N}A \text{ and } -x \in \mathbb{N}A\} = \{0\}.$$

Example

- $A = \{1, -1\}$. $\mathbb{N}A$ is not pointed.
- $A = \{1, 2, 3\}$. $\mathbb{N}A$ is pointed.
- $A \subset \mathbb{N}^n$ then $\mathbb{N}A$ is pointed.
- The semigroups of toric ideals of graphs are pointed.
Example where $\mathbb{N}A$ is pointed

Let $A = \{(2, 1, 0), (1, 2, 0), (2, 0, 1), (1, 0, 2), (0, 2, 1), (0, 1, 2)\}$.

- $I_A = \langle x_1 x_6 - x_2 x_4, x_1 x_6 - x_3 x_5, x_2^2 x_3 - x_1^2 x_5, x_2 x_3^2 - x_1^2 x_4, x_1 x_5^2 - x_2 x_6, x_1 x_4^2 - x_3 x_6, x_4^2 x_5 - x_3 x_6, x_1 x_4 x_5 - x_2 x_3 x_6, x_4 x_5^2 - x_2 x_6^2 \rangle$.
- $I_A = \langle x_1 x_6 - x_2 x_4, x_2 x_4 - x_3 x_5, x_2^2 x_3 - x_1^2 x_5, x_2 x_3^2 - x_1^2 x_4, x_1 x_5^2 - x_2 x_6, x_1 x_4^2 - x_3 x_6, x_4^2 x_5 - x_3 x_6, x_1 x_4 x_5 - x_2 x_3 x_6, x_4 x_5^2 - x_2 x_6^2 \rangle$.
- $I_A = \langle x_3 x_5 - x_2 x_4, x_1 x_6 - x_3 x_5, x_2^2 x_3 - x_1^2 x_5, x_2 x_3^2 - x_1^2 x_4, x_1 x_5^2 - x_2 x_6, x_1 x_4^2 - x_3 x_6, x_4^2 x_5 - x_3 x_6, x_1 x_4 x_5 - x_2 x_3 x_6, x_4 x_5^2 - x_2 x_6^2 \rangle$.

Are there other generating sets of $I_A$? What do these minimal generating sets of $I_A$ have in common?
The $A$-degrees of the binomials are:
$(2, 2, 2), (2, 2, 2), (2, 2, 5), (1, 4, 4), (4, 1, 4), (2, 5, 2), (4, 4, 1), (5, 2, 2), (3, 3, 3)$. 
Example where \( \mathbb{N}A \) is not pointed

**Example**

Let \( A = \{1, -1\} \). In \( k[x_1, x_2] \), \( \deg_A(x_1) = 1 \), \( \deg_A(x_2) = -1 \).

- \( I_A = (x_1x_2 - 1) \).
- **Claim**: \( I_A = (x_1^2x_2^2 - 1, x_1^3x_2^3 - 1) \).

  Proof \( x_1x_2 - 1 = x_1x_2(x_1^2x_2^2 - 1) - (x_1^3x_2^3 - 1) \).

- \( I_A = (x_1^6x_2^6 - 1, x_1^{10}x_2^{10} - 1, x_1^{15}x_2^{15} - 1) \).

- Is it true that for every \( n \) there is a minimal generating set of \( I_A \) of cardinality \( n \)?
Characteristics of Toric ideals

**Definition**

Let $\mu(I_A)$ be the least cardinality of a minimal system of binomial generators of $I_A$.

**Definition**

Let $\nu(I_A)$ be the number of different minimal systems of binomial generators of $I_A$ of least cardinality, where for counting $B$ is the same as $-B$.

**Question**

Are these numbers: $\mu(I_A)$ and $\nu(I_A)$ computable? Are they finite?
A recent problem, arising from Algebraic Statistics asks what conditions are needed for $\nu(I_A) = 1$.

To study this problem Ohsugi and Hibi introduced the notion of indispensable binomials while Aoki, Takemura and Yoshida introduced the notion of indispensable monomials.
Minimal and Indispensable binomials

Definition

A binomial \( B = x^u - x^v \in \mathcal{I}_A \) is indispensable binomial if every system of binomial generators of \( \mathcal{I}_A \) contains \( B \) or \(-B\).

Definition

A monomial \( x^u \) is indispensable monomial if every system of binomial generators of \( \mathcal{I}_A \) contains a binomial \( B \) such that the \( x^u \) is a monomial of \( B \).
Markov binomials

Definition

A binomial $B = x^u - x^v \in I_A$ is a minimal binomial if there exists a minimal system of binomial generators of $I_A$ which contains $B$.

A binomial $B \in I_A$ is Markov if there exists a minimal system of binomial generators of $I_A$ of least cardinality which contains $B$.

The Universal Markov Basis of $I_A$ is the union of all minimal generating sets of $I_A$ of least cardinality.

Is the Markov basis of $I_A$ finite? What is the relation of the Markov basis with the Universal Grobner basis?
Introduction
Generating $I_A$

Example (Pointed case)

Example

Let $A = \{(2, 1, 0), (1, 2, 0), (2, 0, 1), (1, 0, 2), (0, 2, 1), (0, 1, 2)\}$.

- $I_A = \langle x_1x_6 - x_2x_4, x_1x_6 - x_3x_5, x_2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6, \rangle.$
- $I_A = \langle x_1x_6 - x_2x_4, x_2x_4 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6, \rangle.$
- $I_A = \langle x_3x_5 - x_2x_4, x_1x_6 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6, \rangle.$

The last 7 generators are common in all generating sets. Are they indispensable? Which of the monomial terms of the binomials that appear in these generating sets are indispensable?
Example (non-Pointed case)

Let $A = \{1, -1\}$.

- $I_A = (x_1x_2 - 1)$.
- $I_A = (x_1^2x_2^2 - 1, x_1^3x_2^3 - 1)$.

$x_1x_2 - 1$ is Markov. There are no indispensable binomials. $x_1^0x_2^0 = 1$ is the only indispensable monomial. What about the degrees of the elements in minimal generating sets of $I_A$?
Betti $A$-degrees

An $A$-degree $b$ is called $Betti$ $A$-degree if there exists a minimal binomial generator $B$ of $I_A$ with $\text{deg}_A(B) = b$.

The number of times that a Betti $A$-degree $b$ appears as an $A$-degree of a binomial in a given minimal generating set is called $Betti$ $number$.

Theorem

Let $NA$ be pointed. The Betti $A$-degrees of $I_A$ and their corresponding Betti numbers are independent of the choice of a minimal generating set of $I_A$, (graded Nakayama Lemma).

When $NA$ is pointed the notions of minimal and Markov are the same.
Example

Let $A = \{(2, 1, 0), (1, 2, 0), (2, 0, 1), (1, 0, 2), (0, 2, 1), (0, 1, 2)\}$.

$I_A = \langle x_1x_6 - x_2x_4, x_1x_6 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1x_4^2, x_1x_5^2 - x_2^2x_6, x_1x_4^2 - x_3^2x_6, x_4^2x_5 - x_3x_6^2, x_1x_4x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2 \rangle$.

The Betti $A$-degrees are:

$(2, 2, 2), (2, 2, 5), (1, 4, 4), (4, 1, 4), (2, 5, 2), (4, 4, 1), (5, 2, 2), (3, 3, 3)$.

The Betti number for $(2, 2, 2)$ is 2 while the other $A$-graded Betti numbers are 1.
Questions on generating Toric ideals

- How is $I_A$ generated?
- What are the Betti degrees of $I_A$?
- What are the Betti numbers of $I_A$?
- How do we find minimal binomials for $I_A$?
- How do we find the Markov binomials of $I_A$?
- Why are there indispensable binomials of $I_A$?
- What are the values of $\mu(I_A)$ and of $\nu(I_A)$?
Order in $\mathbb{N}A$

**Definition**

*When $\mathbb{N}A$ is pointed we can partially order it with the relation*

$$c \geq d \iff \text{there is } e \in \mathbb{N}A \text{ such that } c = d + e.$$  

**Example**

*When $\mathbb{N}A$ is not pointed the above relation is not an order.*

Consider $A = \{-1, 1\}$. Then $1 = 0 + 1$ and $0 = 1 + (-1)$. So we would get to situations where $1 > 0$ and $0 > 1$. 
Fibers of $\mathbb{N}A$

**Definition**

Let $b \in \mathbb{N}A$. The fiber at $b$ is the following set of monomials:

$$\deg_A^{-1}(b) = \{x^u \mid \deg_A(x^u) = b\}$$

**Example**

Let $A = \{(2,1,0), (1,2,0), (2,0,1), (1,0,2), (0,2,1), (0,1,2)\}$.

$$\deg_A^{-1}(2,2,2) = \{x_1x_6, x_2x_4, x_3x_5\}$$

$$\deg_A^{-1}(1,4,4) = \{x_2^2x_3, x_1^2x_5\}$$

**Example**

Let $A = \{1, -1\}$.

$$\deg_A^{-1}(0) = \{1, x_1x_2, x_1^2x_2^2, \ldots\}$$

Toric ideals
Cardinality of Fibers

Remark

When $NA$ is pointed the fiber $\deg_A^{-1}(b)$ is finite for every $b \in NA$.

When $NA$ is not pointed then there is a $b \in NA$ such that the fiber $\deg_A^{-1}(b)$ is not finite.
For what follows \( \mathbb{N}A \) will be pointed.

**Definition**

For any \( b \in \mathbb{N}A \) we let

\[
I_{A,b} = (x^u - x^v \mid \deg_A(x^u) = \deg_A(x^v) \leq b) \subset I_A.
\]
Let $A = \{(2, 2, 2, 0, 0), (2, -2, -2, 0, 0), (2, 2, -2, 0, 0), (2, -2, 2, 0, 0), (3, 0, 0, 3, 3), (3, 0, 0, -3, -3), (3, 0, 0, 3, -3)(3, 0, 0, -3, 3)\}$. 

$I_A = (x_1 x_2 - x_3 x_4, x_5 x_6 - x_7 x_8, x_1^3 x_2^3 - x_5^2 x_6^2)$

The Betti $A$-degrees are $b_1 = (4, 0, 0, 0, 0)$, $b_2 = (6, 0, 0, 0, 0)$ and $b_3 = (12, 0, 0, 0, 0)$. Note that

$b_1, b_2 < b_3$.

- $I_{A,b_1} = I_{A,b_2} = 0$ (why?): $b_1$ and $b_2$ are minimal binomial $A$-degrees
- $I_{A,b_3} = (x_1 x_2 - x_3 x_4, x_5 x_6 - x_7 x_8)$. 
The graph of a $b \in \mathbb{N}A$

**Definition**

Let $G(b)$ be the graph with vertices the elements of the fiber $\deg^{-1}_A(b)$ and edges all the sets $\{x^u, x^v\}$ whenever $x^u - x^v \in I_{A,b}$.

**Example**

Let $A = \{(2, 2, 2, 0, 0), (2, -2, -2, 0, 0), (2, 2, -2, 0, 0), (2, -2, 2, 0, 0), (3, 0, 0, 3, 3), (3, 0, 0, -3, -3), (3, 0, 0, 3, -3)(3, 0, 0, -3, 3)\}$.

$$I_A = (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8, x_1^3x_2^3 - x_5^2x_6^2)$$

$b_1 = (4, 0, 0, 0, 0)$, $b_2 = (6, 0, 0, 0, 0)$, $b_3 = (12, 0, 0, 0, 0)$, and $b_1, b_2 < b_3$.

- $G(b_1)$ and $G(b_2)$ consist of two points each.
- Connected components of $G(b_1)$: $\{x_1x_2\}$ and $\{x_3x_4\}$.
- Connected components of $G(b_2)$: $\{x_5x_6\}$ and $\{x_7x_8\}$.
Example (continued)

Example

\[ I_A = (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8, x_3^3x_2^3 - x_5^2x_6^2) \]

\[ b_3 = \deg_A (x_1^3x_2^3 - x_5^2x_6^2) = (12, 0, 0, 0, 0) \]

It is clear that \( x_1^3x_2^3, x_5^2x_6^2 \) belong to two different components of \( G(b_3) \).

\[ \deg_A^{-1}(b_3) = \{ x_1^3x_2^3, x_1^2x_2^2x_3x_4, x_1x_2x_3^2x_4^2, x_3^3x_4^3, x_5^2x_6^2, x_5x_6x_7x_8, x_7^2x_8^2 \} \]

\( G(b_3) \) has two connected components: \( \{ x_1^3x_2^3, x_1^2x_2^2x_3x_4, x_1x_2x_3^2x_4^2, x_3^3x_4^3 \} \) and \( \{ x_5^2x_6^2, x_5x_6x_7x_8, x_7^2x_8^2 \} \). For example

\[ x_1^3x_2^3 - x_1^2x_2^2x_3x_4 = x_1^2x_2^2(x_1x_2 - x_3x_4) \in I_{A,(b_3)} \]

and indeed \( x_1^3x_2^3, x_1^2x_2^2x_3x_4 \) are in the same component of \( G(b_3) \).

How many components does \( G(b) \) for all other \( b \in \mathbb{N}A \)?
Remarks on $G(b)$

Suppose that $G(b)$ has $n_b$ connected components $G(b)_i$, i.e.

$$G(b) = \bigcup_{i=1}^{n_b} G(b)_i,$$

and let $t_i(b)$ be the number of vertices of the $i$-component. Every connected component of $G(b)$ is a complete subgraph. If $b$ is not a Betti $A$-degree, then $G(b)$ is connected.

**Theorem**

$b \in \mathbb{NA}$ is a minimal binomial $A$-degree if and only if every connected component of $G(b)$ is a singleton.
The graph on the components of $G(b)$

We consider the complete graph with vertices the connected components $G(b)_i$ of $G(b)$,
Spanning trees and generators

Let $T_b$ be a spanning tree of this graph.
Spanning trees and generators

For every edge of $T_b$ joining two components $G(b)_i$, $G(b)_j$ of $G(b)$ we choose a binomial $x^u - x^v$ with $x^u \in G(b)_i$ and $x^v \in G(b)_j$. 

![Diagram of spanning trees and generators](image-url)
Markov basis for $I_A$

We call $\mathcal{F}_{T_b}$ the collection of these binomials. Note that if $b$ is not a Betti $A$-degree, then $\mathcal{F}_{T_b} = \emptyset$.

**Theorem**

The set $\mathcal{F} = \bigcup_{b \in \mathbb{N}^A} \mathcal{F}_{T_b}$ is a minimal generating set of $I_A$.
Cardinality of minimal generating sets of $I_A$

**Theorem**

*For a toric ideal $I_A$ we have that*

$$
\mu(I_A) = \sum_{b \in \mathbb{N}A} (n_b - 1)
$$

*where $n_b$ is the number of connected components of $G(b)$.*
Number of minimal generating sets of $I_A$

**Theorem**

$$\nu(l_A) = \prod_{b \in NA} t_1(b) \cdots t_{n_b}(b)(t_1(b) + \cdots + t_{n_b}(b))^{n_b-2}$$

where $n_b$ is the number of connected components of $G(b)$ and $t_i(b)$ is the number of vertices of the $G(b)_i$. 

[Diagram showing a network with labeled vertices $t_i(b)$]
Indispensable binomials

**Theorem**

Let \( B = x^u - x^v \in I_A \) with \( A \)-degree \( b \). Then \( B \) is indispensable if and only if the graph \( G(b) \) has only two connected components: \( \{x^u\} \) and \( \{x^v\} \).

**Theorem**

Let \( b_1, \ldots, b_q \) be the degrees of a minimal generating set of \( I_A \). The ideal \( I_A \) is generated by indispensable binomials if and only if for \( i = 1, \ldots q \) the connected components of \( G(b_i) \) are two one element sets.

If \( I_A \) is generated by indispensable binomials then the degrees of a minimal generating set of \( I_A \) are minimal binomial \( A \)-degrees.
Example

Let \( A = \{3, 1, 1\} \).

\[ I_A = (x_1 - x_2^3, x_2 - x_3) \]

The Betti \( A \)-degrees are \( b_1 = 1 \) and \( b_2 = 3 \).

The \( A \)-graded Betti numbers are: \( \beta_{0,1} = 1 \) and \( \beta_{0,3} = 1 \).

\( G(1) \) consists of 2 vertices and has two connected components.

\( G(3) \) has two connected component: the singleton \( \{x_1\} \) and \( \{x_2^3, x_2^2x_3, x_2x_3^2, x_3^3\} \).

\[ \nu(I_A) = 4. \]

Exercise

Let \( A = \{a_0 = k, a_1 = 1, \ldots, a_n = 1\} \subset \mathbb{N} \) be a set of \( n + 1 \) natural numbers with \( k > 1 \). Find \( \nu(I_A) \).
Generic binomial ideals

Connection to Integer Programming.

**Theorem**

*(Peeva, Sturmfels)* If $I_A$ is generated by binomials of full support then $\nu(I_A) = 1$.

**Example**

Let $A = \{20, 24, 25, 31\}$.

$$I_A = (x_3^3 - x_1x_2x_4, x_1^4 - x_2x_3x_4, x_4^3 - x_1^2x_2x_3, x_2^4 - x_1^2x_3x_4, x_3^3 - x_2x_4^2, x_1^2x_2^2 - x_3x_4^2, x_2^3x_4^2 - x_3^2x_2^2)$$

$$\nu(I_A) = 1$$
Ohsugi and Hibi proved that a binomial $B$ is indispensable if and only if either $B$ or $-B$ belongs to the reduced Gröbner base of $I_A$ for all lexicographic term orders.

Aoki, Takemura and Yoshida have shown that a monomial $x^u$ is indispensable if the reduced Gröbner base of $I_A$, with respect to any lexicographic term order, contains a binomial that has $x^u$ as one of its terms.

There are quite few lexicographic term orders for large $m$!
We let $\mathcal{M}_A$ be the monomial ideal generated by all $x^u$ for which there exists a nonzero $x^u - x^v \in I_A$. Let $T_A := \{M_1, \ldots, M_k\}$ be the unique minimal monomial generating set of $\mathcal{M}_A$.

**Theorem**

The indispensable monomials of $I_A$ are precisely the elements of $T_A$.

**Remark**

To compute $T_A$ it is enough to find one generating set of $I_A$.  

---

**Toric ideals**
Indispensable binomials and indispensable monomials

**Theorem**

A binomial $B = x^u - x^v \in I_A$ is indispensable if and only if $\deg_A(B)$ is a minimal binomial $A$-degree and $\{x^u, x^v\}$ is the maximal (with respect to inclusion) subset of $T_A$ whose elements have degree $\deg_A(B)$.

**Algorithm**

- Let $\{B_1 = x^{u_{11}} - x^{u_{12}}, \ldots, B_s = x^{u_{s1}} - x^{u_{s2}}\}$ a system of binomial generators or a Gröbner base of $I_A$ with respect to any term order on $K[x_1, \ldots, x_m]$.
- $\mathcal{M}_A = (x^{u_{11}}, x^{u_{12}}, \ldots, x^{u_{s1}}, x^{u_{s2}})$
- compute the elements of $T_A$ and their $A$-degrees
- Check how many monomials of $T_A$ have the same $A$-degree.
- Check minimality of degrees whenever you find exactly two such monomials.
Let \( A = \{ (2, 1, 0), (1, 2, 0), (2, 0, 1), (1, 0, 2), (0, 2, 1), (0, 1, 2) \} \).

\[
I_A = (x_1 x_6 - x_2 x_4, x_1 x_6 - x_3 x_5, x_2^2 x_3 - x_1^2 x_5, x_2 x_3^2 - x_1^2 x_4, \\
x_1 x_5^2 - x_2^2 x_6, x_1 x_4^2 - x_3^2 x_6, x_4^2 x_5 - x_3 x_6^2, x_1 x_4 x_5 - x_2 x_3 x_6, x_4 x_5^2 - x_2 x_6^2) \cdot
\]

\[
T_A = \{ x_1 x_6, x_2 x_4, x_3 x_5, x_3 x_6, x_4^2 x_5, x_2 x_6^2 x_4 x_5^2, x_3^2 x_6, \\
x_1 x_4^2, x_2^2 x_6, x_1 x_5^2, x_1^2 x_5, x_2^2 x_3, x_1^2 x_4, x_2 x_3^2, x_2 x_3 x_6, x_1 x_4 x_5 \} \cdot
\]

The \( A \)-degrees of the elements of \( T_A \) are \( (2, 2, 2), (2, 2, 5), (1, 4, 4), (4, 1, 4), (2, 5, 2), (4, 4, 1), (5, 2, 2), (3, 3, 3) \). All are minimal.

\( I_A \) has 7 indispensable binomials.

• H. Charalambous, A. Sinefakopoulos, A. Thoma, “Markov Bases of Binomial ideals”, work in progress

• H. Charalambous, A. Thoma, M. Vladoiu “On the Universal Markov basis and the generalized Lawrence Liftings”, work in progress
