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Toric Ideals and Minimal systems of generators

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Circuits, Universal Grobner bases and Primitive polynomials

Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset \mathbb{Z}^n$ and I_A the corresponding toric ideal in $K[x_1, \ldots, x_m]$.

- $\deg_A(x_1^{u_1} \cdots x_m^{u_m}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A$
- $I_A = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \ \deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}}) \rangle$.

Theorem

(Sturmfels) For any toric ideal I_A the following containments hold:

$$Circuits_A \subset UGB_A \subset Graver_A$$

Definition

An irreducible binomial $B \in I_A$ is called a circuit *if there is no binomial $B' \in I_A$ such that $supp(B') \subsetneqq supp(B)$.*

where if

$$B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \text{ then } supp(B) = supp(x^{\mathbf{u}}) \cup supp(x^{\mathbf{v}})$$

and

$$supp(x^{\mathbf{u}}) = \{i \mid x_i \text{ divides } x^{\mathbf{u}}\}$$

Definition

An irreducible binomial $x^{\mathbf{u}^+} - x^{\mathbf{u}^-} \in I_A$ is called primitive *if there exists no other binomial $x^{\mathbf{v}^+} - x^{\mathbf{v}^-} \in I_A$ such that $x^{\mathbf{v}^+}$ divides $x^{\mathbf{u}^+}$ and $x^{\mathbf{v}^-}$ divides $x^{\mathbf{u}^-}$.*

Example

Let $A = \{1, 2, 4\}$.
Then $I_A = \langle x_1^2 - x_2, x_1^4 - x_3, x_2^2 - x_3 \rangle$.

- $x_1^2 - x_2, x_1^4 - x_3, x_2^2 - x_3$ are circuits.
- $x_1^2 x_2 - x_3$ is primitive, but not a circuit.
- $x_1^2 x_2 - x_3$ is primitive, but not in the Universal Grobner basis of I_A.
  Proof: Let $x_1^2 x_2 - x_3$ be in the reduced Grobner basis G.
  Fact: $x_1^2 - x_2 \in I_A$. Thus there is $g \in G$ such that $\mathrm{in}_<(g)$ divides $\mathrm{in}_<(x_1^2 - x_2)$.
  Case 1: $\mathrm{in}_<(x_1^2 - x_2) = x_1^2$. Then $\mathrm{in}_<(g)$ divides $x_1^2 x_2$, a contradiction.
  Case 2: $\mathrm{in}_<(x_1^2 - x_2) = x_2$. Again we derive a contradiction.

What is a minimal generating set of I_A? Is it unique?

Question

When is a primitive binomial not in the Universal Grobner basis? When is $UGB_A = Graver_A$?

Pointed semigroups

Definition

The affine semigroup $\mathbb{N}A := \{l_1\mathbf{a}_1 + \cdots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}\}$ *is pointed if*

$$\{x : \ x \in \mathbb{N}A \ \mathit{and} \ -x \in \mathbb{N}A\} = \{\mathbf{0}\} \ .$$

Example

- $A = \{1, -1\}$. $\mathbb{N}A$ is not pointed.
- $A = \{1, 2, 3\}$. $\mathbb{N}A$ is pointed.
- $A \subset \mathbb{N}^n$ then $\mathbb{N}A$ is pointed.
- The semigroups of toric ideals of graphs are pointed.

Example where $\mathbb{N}A$ is pointed

Example

Let $A = \{(2,1,0),(1,2,0),(2,0,1),(1,0,2),(0,2,1),(0,1,2)\}$.

- $I_A = \langle x_1x_6 - x_2x_4, x_1x_6 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6, x_1x_4^2 - x_3^2x_6, x_4^2x_5 - x_3x_6^2, x_1x_4x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2\rangle$.
- $I_A = \langle x_1x_6 - x_2x_4, x_2x_4 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6, x_1x_4^2 - x_3^2x_6, x_4^2x_5 - x_3x_6^2, x_1x_4x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2\rangle$.
- $I_A = \langle x_3x_5 - x_2x_4, x_1x_6 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6, x_1x_4^2 - x_3^2x_6, x_4^2x_5 - x_3x_6^2, x_1x_4x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2\rangle$.

Are there other generating sets of I_A? What do these minimal generating sets of I_A have in common?
The A-degrees of the binomials are:
$(2,2,2)$, $(2,2,2)$, $(2,2,5)$, $(1,4,4)$, $(4,1,4)$,$(2,5,2)$, $(4,4,1)$, $(5,2,2)$, $(3,3,3)$.

Example where $\mathbb{N}A$ is not pointed

Example

Let $A = \{1, -1\}$. In $k[x_1, x_2]$, $\deg_A(x_1) = 1$, $\deg_A(x_2) = -1$.

- $I_A = (x_1x_2 - 1)$.
- **Claim**: $I_A = (x_1^2x_2^2 - 1, x_1^3x_2^3 - 1)$.
  Proof $x_1x_2 - 1 = x_1x_2(x_1^2x_2^2 - 1) - (x_1^3x_2^3 - 1)$.
- $I_A = (x_1^6x_2^6 - 1, x_1^{10}x_2^{10} - 1, x_1^{15}x_2^{15} - 1)$.
- Is it true that for every n there is a minimal generating set of I_A of cardinality n?

Characteristics of Toric ideals

Definition

Let $\mu(I_A)$ be the least cardinality of a minimal system of binomial generators of I_A.

Definition

Let $\nu(I_A)$ be the number of different minimal systems of binomial generators of I_A of least cardinality, where for counting B is the same as $-B$.

Question

Are these numbers: $\mu(I_A)$ and $\nu(I_A)$ computable? Are they finite?

Motivation

A recent problem, arising from Algebraic Statistics asks what conditions are needed for $\nu(I_A) = 1$.

To study this problem Ohsugi and Hibi introduced the notion of indispensable binomials while Aoki, Takemura and Yoshida introduced the notion of indispensable monomials.

Minimal and Indispensable binomials

Definition

A binomial $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A$ is indispensable binomial if every system of binomial generators of I_A contains B or $-B$.

Definition

A monomial $\mathbf{x}^{\mathbf{u}}$ is indispensable monomial if every system of binomial generators of I_A contains a binomial B such that the $\mathbf{x}^{\mathbf{u}}$ is a monomial of B.

Markov binomials

Definition

A binomial $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A$ is a minimal binomial if there exists a minimal system of binomial generators of I_A which contains B.

A binomial $B \in I_A$ is Markov if there exists a minimal system of binomial generators of I_A of least cardinality which contains B.

The Universal Markov Basis of I_A is the union of all minimal generating sets of I_A of least cardinality.

Is the Markov basis of I_A finite? What is the relation of the Markov basis with the Universal Grobner basis?

Example (Pointed case)

Example

Let $A = \{(2,1,0),(1,2,0),(2,0,1),(1,0,2),(0,2,1),(0,1,2)\}$.

- $I_A = \langle x_1x_6 - x_2x_4, x_1x_6 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6, x_1x_4^2 - x_3^2x_6, x_4^2x_5 - x_3x_6^2, x_1x_4x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2\rangle$.
- $I_A = \langle x_1x_6 - x_2x_4, x_2x_4 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6, x_1x_4^2 - x_3^2x_6, x_4^2x_5 - x_3x_6^2, x_1x_4x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2\rangle$.
- $I_A = \langle x_3x_5 - x_2x_4, x_1x_6 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6, x_1x_4^2 - x_3^2x_6, x_4^2x_5 - x_3x_6^2, x_1x_4x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2\rangle$.

The last 7 generators are common in all generating sets. Are they indispensable? Which of the monomial terms of the binomials that appear in these generating sets are indispensable?

Example (non-Pointed case)

Example

Let $A = \{1, -1\}$.

- $I_A = (x_1x_2 - 1)$.
- $I_A = (x_1^2x_2^2 - 1, x_1^3x_2^3 - 1)$.

$x_1x_2 - 1$ is Markov. There are no indispensable binomials. $x_1^0x_2^0 = 1$ is the only indispensable monomial. What about the degrees of the elements in minimal generating sets of I_A?

Betti A-degrees

An A-degree $\mathbf{b}$ is called *Betti A-degree* if there exists a minimal binomial generator B of I_A with $\deg_A(B) = \mathbf{b}$.

The number of times that a Betti A-degree $\mathbf{b}$ appears as an A-degree of a binomial in a given minimal generating set is called *Betti number*.

Theorem

Let $\mathbb{N}A$ be pointed. The Betti A-degrees of I_A and their corresponding Betti numbers are independent of the choice of a minimal generating set of I_A, (graded Nakayama Lemma).

When $\mathbb{N}A$ is pointed the notions of minimal and Markov are the same.

Example

Example

Let $A = \{(2,1,0),(1,2,0),(2,0,1),(1,0,2),(0,2,1),(0,1,2)\}$.
$I_A = \langle x_1x_6 - x_2x_4, x_1x_6 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4, x_1x_5^2 - x_2^2x_6,$
$x_1x_4^2 - x_3^2x_6, x_4^2x_5 - x_3x_6^2, x_1x_4x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2\rangle$.
The Betti A-degrees are:
$(2,2,2)$, $(2,2,5)$, $(1,4,4)$, $(4,1,4)$,$(2,5,2)$, $(4,4,1)$, $(5,2,2)$, $(3,3,3)$.

The Betti number for $(2,2,2)$ is 2 while the other A-graded Betti numbers are 1.

Questions on generating Toric ideals

- How is I_A generated?
- What are the Betti degrees of I_A?
- What are the Betti numbers of I_A?
- How do we find minimal binomials for I_A?
- How do we find the Markov binomials of I_A?
- Why are there indispensable binomials of I_A?
- What are the values of $\mu(I_A)$ and of $\nu(I_A)$?

Order in $\mathbb{N}A$

Definition

When $\mathbb{N}A$ is pointed we can partially order it with the relation

$$\mathbf{c} \geq \mathbf{d} \iff \textit{there is } \mathbf{e} \in \mathbb{N}A \textit{ such that } \mathbf{c} = \mathbf{d} + \mathbf{e}.$$

Example

When $\mathbb{N}A$ is not pointed the above relation is not an order.

Consider $A = \{-1, 1\}$. Then $1 = 0 + 1$ and $0 = 1 + (-1)$. So we would get to situations where $1 > 0$ and $0 > 1$.

Fibers of $\mathbb{N}A$

Definition

Let $b \in \mathbb{N}A$. The fiber at b is the following set of monomials:

$$\deg_A^{-1}(\mathbf{b}) = \{\mathbf{x}^{\mathbf{u}} \mid \deg_A(\mathbf{x}^{\mathbf{u}}) = \mathbf{b}\}$$

Example

Let $A = \{(2,1,0),(1,2,0),(2,0,1),(1,0,2),(0,2,1),(0,1,2)\}$.

$$\deg_A^{-1}(2,2,2) = \{x_1x_6, x_2x_4, x_3x_5\}$$

$$\deg_A^{-1}(1,4,4) = \{x_2^2x_3, x_1^2x_5\}$$

Example

Let $A = \{1,-1\}$.

$$\deg_A^{-1}(0) = \{1, x_1x_2, x_1^2x_2^2, \ldots\}$$

Cardinality of Fibers

Remark

When $\mathbb{N}A$ is pointed the fiber $\deg_A^{-1}(\mathbf{b})$ is finite for every $b \in \mathbb{N}A$.

When $\mathbb{N}A$ is not pointed then there is a $b \in \mathbb{N}A$ such that the fiber $\deg_A^{-1}(\mathbf{b})$ is not finite.

Subideals of I_A in degrees less than $b \in \mathbb{N}A$

For what follows $\mathbb{N}A$ will be pointed.

Definition

For any $\mathbf{b} \in \mathbb{N}A$ we let

$$I_{A,\mathbf{b}} = (\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}}) \lneq \mathbf{b}) \subset I_A.$$

Example

Example

Let $A = \{(2,2,2,0,0),(2,-2,-2,0,0),(2,2,-2,0,0),(2,-2,2,0,0),$ $(3,0,0,3,3),(3,0,0,-3,-3),(3,0,0,3,-3)(3,0,0,-3,3)\}$.

$$I_A = (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8, x_1^3x_2^3 - x_5^2x_6^2)$$

The Betti A-degrees are $\mathbf{b}_1 = (4,0,0,0,0)$, $\mathbf{b}_2 = (6,0,0,0,0)$ and $\mathbf{b}_3 = (12,0,0,0,0)$. Note that

$$b_1, b_2 < b_3 \ .$$

- $I_{A,\mathbf{b}_1} = I_{A,\mathbf{b}_2} = 0$ (why?):
  $\mathbf{b}_1$ and $\mathbf{b}_2$ are minimal binomial A-degrees
- $I_{A,\mathbf{b}_3} = (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8)$.

The graph of a $b \in \mathbb{N}A$

Definition

Let $G(\mathbf{b})$ be the graph with vertices the elements of the fiber $\deg_A^{-1}(\mathbf{b})$ and edges all the sets $\{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}\}$ whenever $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_{A,\mathbf{b}}$.

Example

Let $A = \{(2,2,2,0,0), (2,-2,-2,0,0), (2,2,-2,0,0), (2,-2,2,0,0), (3,0,0,3,3), (3,0,0,-3,-3), (3,0,0,3,-3)(3,0,0,-3,3)\}$.

$$I_A = (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8, x_1^3x_2^3 - x_5^2x_6^2)$$

$\mathbf{b}_1 = (4,0,0,0,0)$, $\mathbf{b}_2 = (6,0,0,0,0)$, $\mathbf{b}_3 = (12,0,0,0,0)$, and $b_1, b_2 < b_3$.

- $G(\mathbf{b}_1)$ and $G(\mathbf{b}_2)$ consist of two points each.
- Connected components of $G(\mathbf{b}_1)$: $\{x_1x_2\}$ and $\{x_3x_4\}$,
- Connected components of $G(\mathbf{b}_2)$: $\{x_5x_6\}$ and $\{x_7x_8\}$

Example (continued)

Example

$$I_A = (x_1x_2 - x_3x_4, x_5x_6 - x_7x_8, x_1^3x_2^3 - x_5^2x_6^2)$$

$\mathbf{b}_3 = \deg_A(x_1^3x_2^3 - x_5^2x_6^2) = (12, 0, 0, 0, 0)$

It is clear that $x_1^3x_2^3, x_5^2x_6^2$ belong to two different components of $G(\mathbf{b}_3)$.

$$\deg_A^{-1}(\mathbf{b_3}) = \{x_1^3x_2^3, x_1^2x_2^2x_3x_4, x_1x_2x_3^2x_4^2, x_3^3x_4^3, x_5^2x_6^2, x_5x_6x_7x_8, x_7^2x_8^2\}$$

$G(\mathbf{b}_3)$ has two connected components: $\{x_1^3x_2^3, x_1^2x_2^2x_3x_4, x_1x_2x_3^2x_4^2, x_3^3x_4^3\}$ and $\{x_5^2x_6^2, x_5x_6x_7x_8, x_7^2x_8^2\}$. For example

$$x_1^3x_2^3 - x_1^2x_2^2x_3x_4 = x_1^2x_2^2(x_1x_2 - x_3x_4) \in I_{A,(\mathbf{b_3})}$$

and indeed $x_1^3x_2^3, x_1^2x_2^2x_3x_4$ are in the same component of $G(\mathbf{b}_3)$.
How many components does $G(\mathbf{b})$ for all other $b \in \mathbb{N}A$?

Remarks on $G(\mathbf{b})$

Suppose that $G(\mathbf{b})$ has $n_{\mathbf{b}}$ connected components $G(\mathbf{b})_i$, i.e.

$$G(\mathbf{b}) = \bigcup_{i=1}^{n_{\mathbf{b}}} G(\mathbf{b})_i,$$

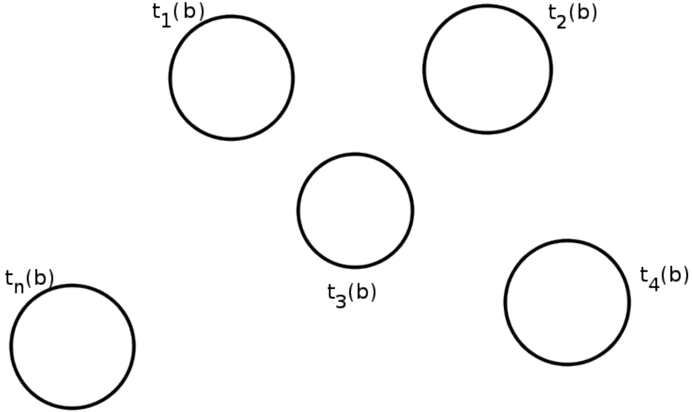
and let $t_i(\mathbf{b})$ be the number of vertices of the i-component.
Every connected component of $G(\mathbf{b})$ is a complete subgraph.
If $\mathbf{b}$ is not a Betti A-degree, then $G(\mathbf{b})$ is connected.

Theorem

$\mathbf{b} \in \mathbb{N}A$ is a minimal binomial A-degree if and only if every connected component of $G(\mathbf{b})$ is a singleton.

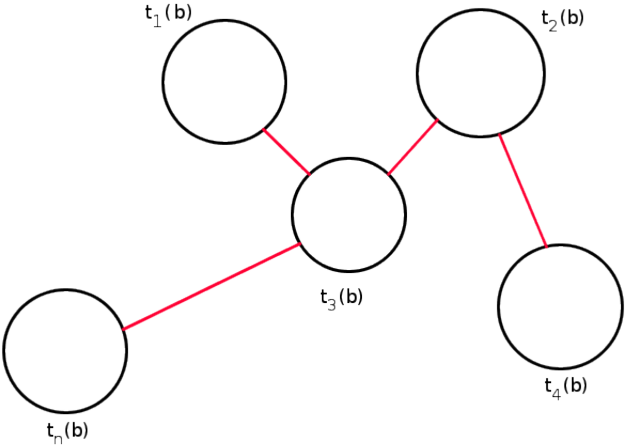
The graph on the components of $G(\mathbf{b})$

We consider the complete graph with vertices the connected components $G(\mathbf{b})_i$ of $G(\mathbf{b})$,



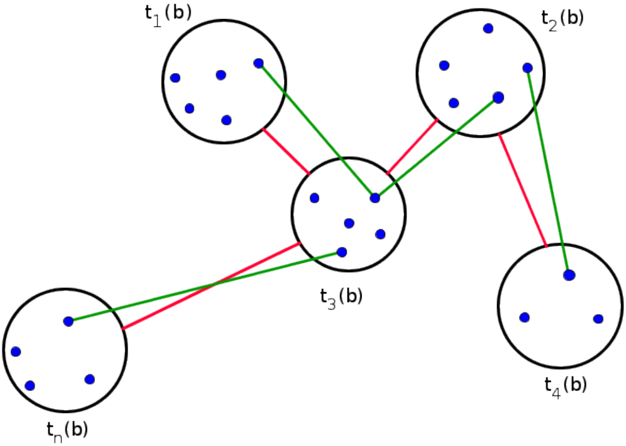
Spanning trees and generators

Let $T_{\mathbf{b}}$ be a spanning tree of this graph.



Spanning trees and generators

For every edge of $T_{\mathbf{b}}$ joining two components $G(\mathbf{b})_i$, $G(\mathbf{b})_j$ of $G(\mathbf{b})$ we choose a binomial $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ with $\mathbf{x}^{\mathbf{u}} \in G(\mathbf{b})_i$ and $\mathbf{x}^{\mathbf{v}} \in G(\mathbf{b})_j$.

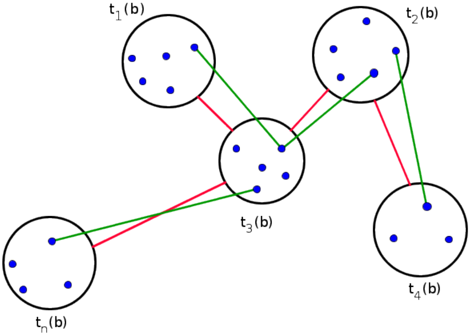


Markov basis for I_A

We call $\mathcal{F}_{T_{\mathbf{b}}}$ the collection of these binomials. Note that if $\mathbf{b}$ is not a Betti A-degree, then $\mathcal{F}_{T_{\mathbf{b}}} = \emptyset$.

Theorem

The set $\mathcal{F} = \cup_{\mathbf{b}\in\mathbb{N}A}\mathcal{F}_{T_{\mathbf{b}}}$ is a minimal generating set of I_A.



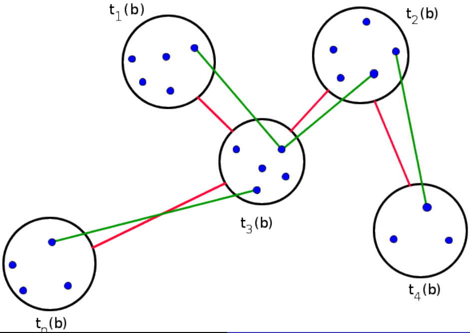
Cardinality of minimal generating sets of I_A

Theorem

For a toric ideal I_A we have that

$$\mu(I_A) = \sum_{\mathbf{b} \in \mathbb{N}A} (n_{\mathbf{b}} - 1)$$

where $n_{\mathbf{b}}$ is the number of connected components of $G(\mathbf{b})$.

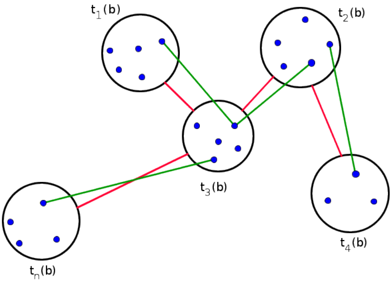


Number of minimal generating sets of I_A

Theorem

$$\nu(I_A) = \prod_{\mathbf{b} \in \mathbb{N}A} t_1(\mathbf{b}) \cdots t_{n_\mathbf{b}}(\mathbf{b})(t_1(\mathbf{b}) + \cdots + t_{n_\mathbf{b}}(\mathbf{b}))^{n_\mathbf{b}-2}$$

where $n_\mathbf{b}$ is the number of connected components of $G(\mathbf{b})$ and $t_i(\mathbf{b})$ is the number of vertices of the $G(\mathbf{b})_i$.



Indispensable binomials

Theorem

Let $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A$ with A-degree $\mathbf{b}$. Then B is indispensable if and only if the graph $G(\mathbf{b})$ has only two connected components: $\{\mathbf{x}^{\mathbf{u}}\}$ and $\{\mathbf{x}^{\mathbf{v}}\}$.

Theorem

Let $\mathbf{b}_1, \dots, \mathbf{b}_q$ be the degrees of a minimal generating set of I_A. The ideal I_A is generated by indispensable binomials if and only if for $i = 1, \dots q$ the connected components of $G(b_i))$ are two one element sets.

If I_A is generated by indispensable binomials then the degrees of a minimal generating set of I_A are minimal binomial A-degrees

Example

Example

Let $A = \{3, 1, 1\}$.

$$I_A = (x_1 - x_2^3, x_2 - x_3)$$

The Betti A-degrees are $\mathbf{b}_1 = 1$ and $\mathbf{b}_2 = 3$.
The A-graded Betti numbers are: $\beta_{0,1} = 1$ and $\beta_{0,3} = 1$.
$G(1)$ consists of 2 vertices and has two connected components.
$G(3)$ has two connected component: the singleton $\{x_1\}$ and $\{x_2^3, x_2^2x_3, x_2x_3^2, x_3^3\}$.

$$\nu(I_A) = 4.$$

Exercise

Let $A = \{a_0 = k, a_1 = 1, \dots, a_n = 1\} \subset \mathbb{N}$ be a set of $n+1$ natural numbers with $k > 1$. Find $\nu(I_A)$.

Generic binomial ideals

Connection to Integer Programming.

Theorem

(Peeva, Sturmfels) If I_A is generated by binomials of full support then $\nu(I_A) = 1$.

Example

Let $A = \{20, 24, 25, 31\}$.

$$I_A = (x_3^3 - x_1x_2x_4, x_1^4 - x_2x_3x_4, x_4^3 - x_1x_2^2x_3, x_2^4 - x_1^2x_3x_4,$$

$$x_1^3x_3^2 - x_2^2x_4^2, x_1^2x_2^3 - x_3^2x_4^2, x_1^3x_4^2 - x_2^3x_3^2)$$

$$\nu(I_A) = 1$$

Recognizing indispensable binomials and monomials

- Ohsugi and Hibi proved that a binomial B is indispensable if and only if either B or $-B$ belongs to the reduced Gröbner base of I_A for all lexicographic term orders.
- Aoki, Takemura and Yoshida have shown that a monomial $\mathbf{x}^{\mathbf{u}}$ is indispensable if the reduced Gröbner base of I_A, with respect to any lexicographic term order, contains a binomial that has $\mathbf{x}^{\mathbf{u}}$ as one of its terms.

There are quite few lexicographic term orders for large m!

Indispensable monomials

We let $\mathcal{M}_A$ be the monomial ideal generated by all $\mathbf{x}^{\mathbf{u}}$ for which there exists a nonzero $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A$. Let $T_A := \{M_1, \ldots, M_k\}$ be the unique minimal monomial generating set of $\mathcal{M}_A$.

Theorem

The indispensable monomials of I_A are precisely the elements of T_A.

Remark

To compute T_A it is enough to find ***one*** *generating set of I_A.*

Indispensable binomials and indispensable monomials

Theorem

A binomial $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A$ is indispensable if and only if $\deg_A(B)$ is a minimal binomial A-degree and $\{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}\}$ is the maximal (with respect to inclusion) subset of T_A whose elements have degree $\deg_A(B)$.

Algorithm

- *Let $\{B_1 = \mathbf{x}^{\mathbf{u}_{11}} - \mathbf{x}^{\mathbf{u}_{12}}, \ldots, B_s = \mathbf{x}^{\mathbf{u}_{s1}} - \mathbf{x}^{\mathbf{u}_{s2}}\}$ a system of binomial generators or a Gröbner base of I_A with respect to any term order on $K[x_1, \ldots, x_m]$.*
- $\mathcal{M}_A = (\mathbf{x}^{\mathbf{u}_{11}}, \mathbf{x}^{\mathbf{u}_{12}}, \ldots, \mathbf{x}^{\mathbf{u}_{s1}}, \mathbf{x}^{\mathbf{u}_{s2}})$
- *compute the elements of T_A and their A-degrees*
- *Check how many monomials of T_A have the same A-degree.*
- *Check minimality of degrees whenever you find exactly two such monomials.*

Example

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Let $A = \{(2,1,0),(1,2,0),(2,0,1),(1,0,2),(0,2,1),(0,1,2)\}$.

$$I_A = (x_1x_6 - x_2x_4, x_1x_6 - x_3x_5, x_2^2x_3 - x_1^2x_5, x_2x_3^2 - x_1^2x_4,$$

$$x_1x_5^2 - x_2^2x_6, x_1x_4^2 - x_3^2x_6, x_4^2x_5 - x_3x_6^2, x_1x_4x_5 - x_2x_3x_6, x_4x_5^2 - x_2x_6^2)\ .$$

$$T_A = \{x_1x_6, x_2x_4, x_3x_5,\ x_3x_6^2, x_4^2x_5, x_2x_6^2x_4x_5^2,\ x_3^2x_6,$$

$$x_1x_4^2, x_2^2x_6, x_1x_5^2,\ x_1^2x_5, x_2^2x_3, x_1^2x_4,\ x_2x_3^2, x_2x_3x_6, x_1x_4x_5\}\ .$$

The A-degrees of the elements of T_A are $(2,2,2),(2,2,5),(1,4,4),(4,1,4),\ (2,5,2),(4,4,1),(5,2,2),(3,3,3)$. All are minimal .

I_A has 7 indispensable binomials.

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