The 20th National School on Algebra:
DISCRETE INVARIANTS IN COMMUTATIVE ALGEBRA AND IN ALGEBRAIC GEOMETRY
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Tutorial

Toric ideals of graphs

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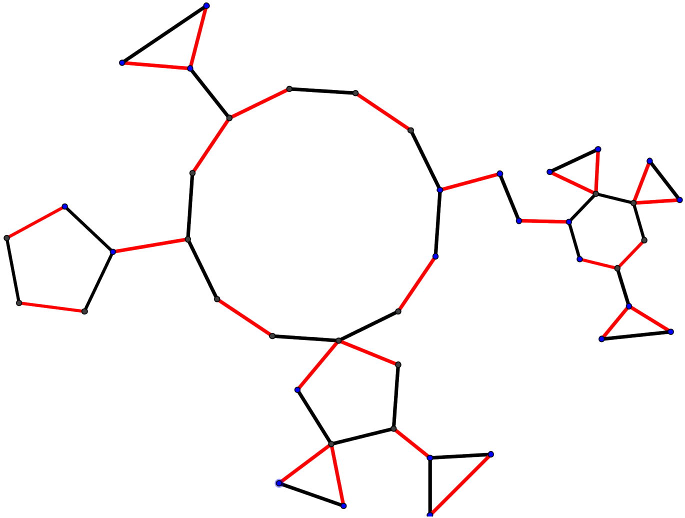
Toric ideals of Graphs



Toric ideals of Graphs

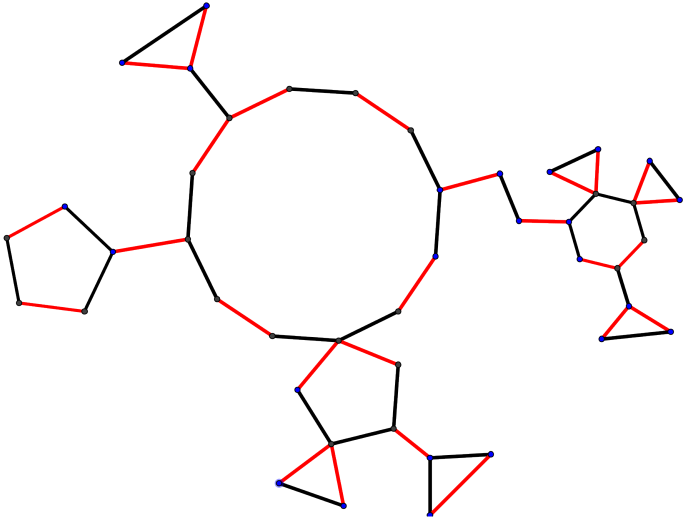
The following theorem describes the underlying graph of a primitive walk.

Theorem

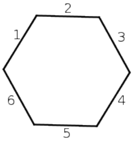
Let G be a graph and let W be a connected subgraph of G. The subgraph W is the graph $\mathbf{w}$ of a primitive walk w if and only if

1. *W is an even cycle or*
2. *W is not biconnected and*
   1. *every block of W is a cycle or a cut edge and*
   2. *every cut vertex of W belongs to exactly two blocks and separates the graph in two parts, the total number of edges of the cyclic blocks in each part is odd.*

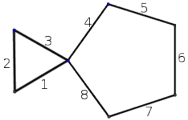
Toric ideals of graphs



Circuits



w is an even cycle



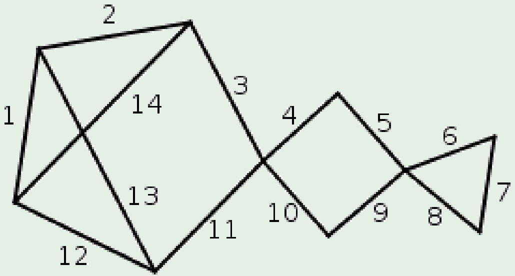
two odd cycles intersecting in exactly one vertex

two vertex disjoint odd cycles joined by a path

Example

Example

let G be the graph with 10 vertices and 14 edges.

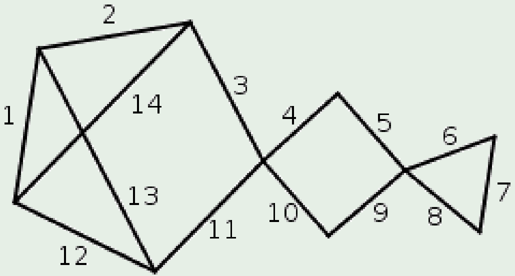


Find the elements in the Graver basis and the circuits.

Example

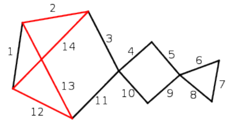
Example

let G be the graph with 10 vertices and 14 edges.

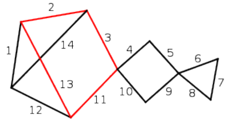


Find the elements in the Graver basis and the circuits.
The Graver basis has twenty eight elements.

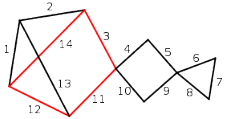
Example of a Graver basis



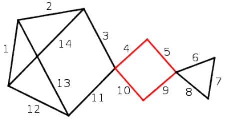
$$B_1 = e_2 e_{12} - e_{13} e_{14}$$



$$B_2 = e_2 e_{11} - e_3 e_{13}$$

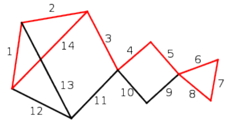


$$B_3 = e_3 e_{12} - e_{11} e_{14}$$

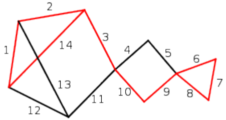


$$B_4 = e_4 e_9 - e_5 e_{10}$$

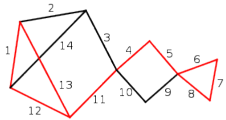
Example of a Graver basis



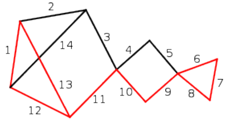
$$B_5 = e_{14}e_2e_4^2e_6e_8 - e_1e_3^2e_5^2e_7$$



$$B_6 = e_{14}e_2e_{10}^2e_6e_8 - e_1e_3^2e_9^2e_7$$

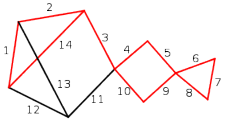


$$B_7 = e_{13}e_{12}e_4^2e_6e_8 - e_1e_{11}^2e_5^2e_7$$

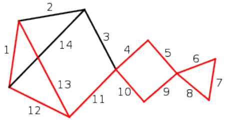


$$B_8 = e_{13}e_{12}e_{10}^2e_6e_8 - e_1e_{11}^2e_9^2e_7$$

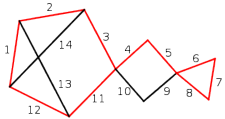
Example of a Graver basis



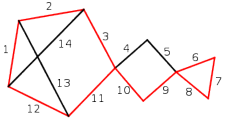
$$B_9 = e_{14}e_2e_4e_6e_8e_{10} - e_1e_3^2e_5e_7e_9$$



$$B_{10} = e_{13}e_{12}e_4e_6e_8e_{10} - e_1e_{11}^2e_5e_7e_9$$

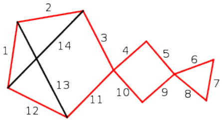


$$B_{11} = e_3e_1e_{11}e_5^2e_7 - e_2e_{12}e_4^2e_6e_8$$

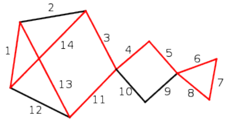


$$B_{12} = e_3e_1e_{11}e_9^2e_7 - e_2e_{12}e_{10}^2e_6e_8$$

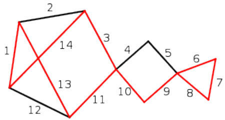
Example of a Graver basis



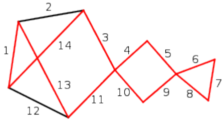
$$B_{13} = e_3e_1e_{11}e_5e_9e_7 - e_2e_{12}e_4e_{10}e_6e_8$$



$$B_{14} = e_3e_1e_{11}e_5^2e_7 - e_{14}e_{13}e_4^2e_6e_8$$

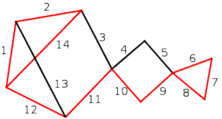


$$B_{15} = e_3e_1e_{11}e_9^2e_7 - e_{14}e_{13}e_{10}^2e_6e_8$$

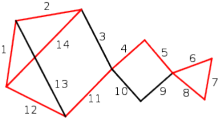


$$B_{16} = e_3e_1e_{11}e_5e_9e_7 - e_{14}e_{13}e_4e_{10}e_6e_8$$

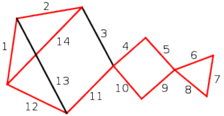
Example of a Graver basis



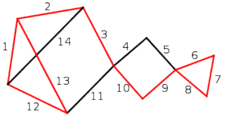
$$B_{17} = e_{14}e_1e_{11}^2e_9^2e_7 - e_2e_{12}^2e_{10}^2e_6e_8$$



$$B_{18} = e_{14}e_1e_{11}^2e_5^2e_7 - e_2e_{12}^2e_4^2e_6e_8$$

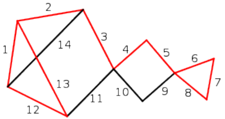


$$B_{19} = e_{14}e_1e_{11}^2e_9e_5e_7 - e_2e_{12}^2e_4e_{10}e_6e_8$$

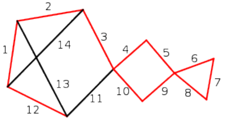


$$B_{20} = e_{13}e_1e_3^2e_9^2e_7 - e_{12}e_2^2e_{10}^2e_6e_8$$

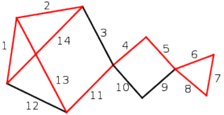
Example of a Graver basis



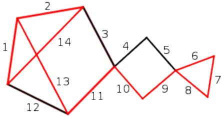
$$B_{21} = e_{13}e_1e_3^2e_5^2e_7 - e_{12}e_2^2e_4^2e_6e_8$$



$$B_{22} = e_{13}e_1e_3^2e_9e_5e_7 - e_{12}e_2^2e_4e_{10}e_6e_8$$

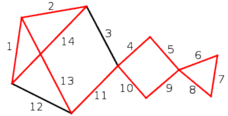


$$B_{23} = e_1e_2e_{11}^2e_5^2e_7 - e_{14}e_{13}^2e_4^2e_6e_8$$

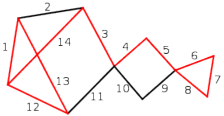


$$B_{24} = e_1e_2e_{11}^2e_9^2e_7 - e_{14}e_{13}^2e_{10}^2e_6e_8$$

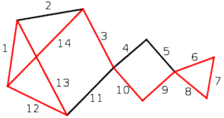
Example of a Graver basis



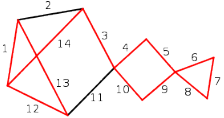
$$B_{25} = e_1 e_2 e_{11}^2 e_9 e_7 e_5 - e_{14} e_{13}^2 e_{10} e_6 e_8 e_4$$



$$B_{26} = e_1 e_{12} e_3^2 e_5^2 e_7 - e_{13} e_{14}^2 e_4^2 e_8 e_6$$



$$B_{27} = e_1 e_{12} e_3^2 e_9^2 e_7 - e_{13} e_{14}^2 e_{10}^2 e_8 e_6$$



$$B_{28} = e_1 e_{12} e_3^2 e_9 e_5 e_7 - e_{13} e_{14}^2 e_{10} e_8 e_6 e_4$$

Example of a Graver basis

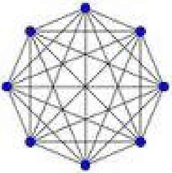
The Graver basis has 28 elements.

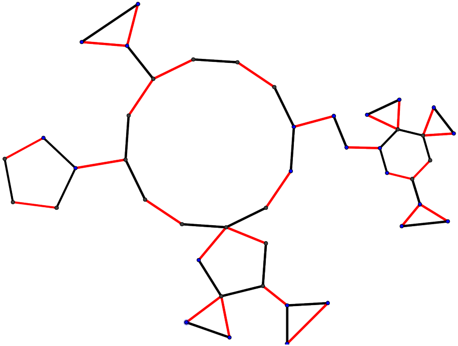
Twenty of them are circuits, all except the binomials $B_9, B_{10}, B_{13}, B_{16}, B_{19}, B_{22}, B_{25}, B_{28}$.

The universal Markov basis has 16 elements, the first sixteen binomials.
There are 8 different Markov bases, each one with 13 elements.
The first 10 binomials are indispensable and each Markov basis contains also one binomial among B_{11}, B_{14}, one among B_{12}, B_{15} and one among B_{13}, B_{16}.

Toric ideals of Graphs

The number of elements in the Graver basis is usually very large.





Graver basis

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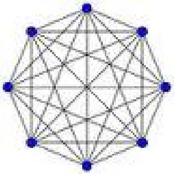

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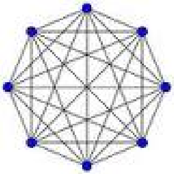
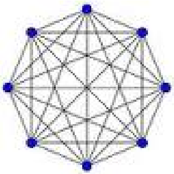


The number of the elements in the Graver basis of I_{K_8} is 45570.

Graver basis

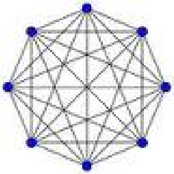
The number of elements in the Graver basis is usually very large.



The number of the elements in the Graver basis of I_{K_8} is 45570. The number of elements in the Universal Gröbner basis is 45570.

Graver basis

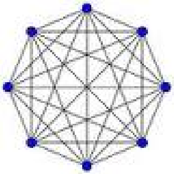
The number of elements in the Graver basis is usually very large.



The number of the elements in the Graver basis of I_{K_8} is 45570. The number of elements in the Universal Gröbner basis is 45570. The number of circuits is 38010.

Graver basis

The number of elements in the Graver basis is usually very large.



The number of the elements in the Graver basis of I_{K_8} is 45570. The number of elements in the Universal Gröbner basis is 45570. The number of circuits is 38010. The number of elements in the Universal Markov basis is 210.
There are 3^{70} different Markov bases, each one having 140 binomials.

Degree Bounds

An estimate for the size of a Graver basis can be a bound for the degrees of the elements in the Graver basis.

Definition

We denote by d_n the largest degree of a binomial in the Graver basis of I_{K_n}.

Theorem

(J. De Loera, B. Sturmfels and R. Thomas)

$$n - 2 \leq d_n \leq \binom{n}{2}.$$

Degree bounds

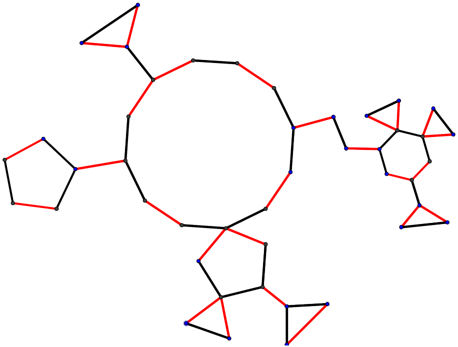
Theorem

The largest degree d_n of any binomial in the Graver basis for I_{K_n} is $d_n = n - 2$, for $n \geq 4$.

Proof. The graph of a primitive walk w consists of blocks which are cut edges and cyclic blocks.

If w is an even cycle in K_n then has length at most n therefore $deg(B_w) = n/2$.

Toric ideals of Graphs



In the case that w is not a cycle then it has at least two cyclic blocks.

Suppose that w has s_0 cyclic blocks and s_1 cut edges. Thus

$$s = s_0 + s_1$$

is the total number of blocks.

Degree bounds

We know that there are exactly $s-1$ cut vertices and each one belongs to exactly two blocks.

Let $B_1, \dots, B_{s_0}$ be the cyclic blocks and t_i denotes the number of edges (vertices) of the cyclic block B_i. Then the total number of vertices of w is

$$t_1 + \cdots + t_{s_0} + 2s_1 - (s-1) \leq n,$$

since the cut vertices are counted twice. Therefore

$$t_1 + \cdots + t_{s_0} + 2s_1 \leq n + s - 1.$$

Two times the degree of B_w is the sum of edges of the cyclic blocks $t_1 + \cdots + t_{s_0}$ plus two times the number of cut edges s_1, since cut edges are double edges of the walk w and edges of cycles are always single. Therefore

$$2\deg(B_w) = t_1 + \cdots + t_{s_0} + 2s_1 \leq n + s - 1.$$

Degree bounds

$$2\deg(B_w) \leq n + s - 1.$$

So the largest degree is attained when the number of blocks of w is the largest possible and equality is achieved only if the walk w passes through all n vertices.

But from $t_1 + \cdots + t_{s_0} + 2s_1 \leq n + s - 1 = n + s_0 + s_1 - 1$ we get

$$s_0 + s_1 + (t_1 - 2) + \cdots + (t_{s_0} - 2) \leq n - 1.$$

Note that $2 \leq (t_1 - 2) + \cdots + (t_{s_0} - 2)$ since cyclic blocks have at least three vertices and the walk has at least two cyclic blocks. Thus

$$s \leq n - 3.$$

$$2\deg(B_w) \leq n + s - 1 \leq 2(n - 2).$$

The largest degree d_n of a binomial in the Graver basis for I_{K_n} is $d_n = n - 2$ and it is attained by a circuit with $n - 3$ blocks, $n - 5$ cut edges and 2 cyclic blocks of three vertices each.

Toric ideals of Graphs

Theorem

Let G be a graph with m vertices, $m \geq 4$. The largest degree d of any binomial in the Graver basis for I_G is $d \leq m-2$.

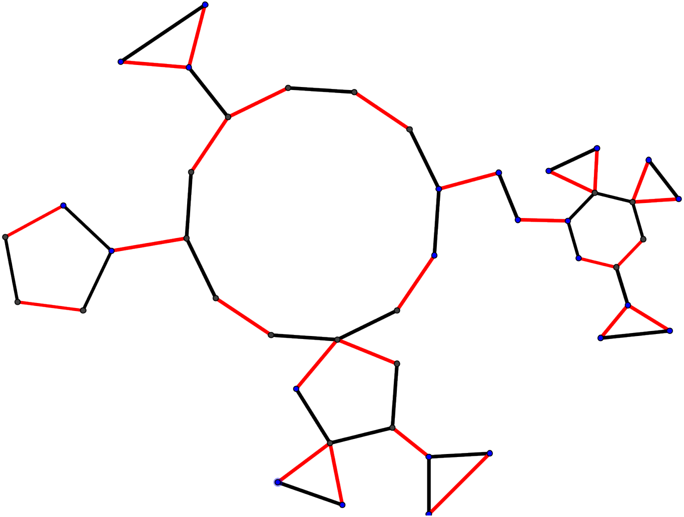
True circuit conjecture

B. Sturmfels in 1995 with the help of S. Hosten and R. Thomas made the following conjecture:

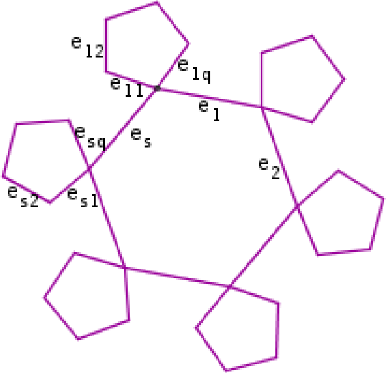
Conjecture

The degree of any element in the Graver basis Gr_A of a toric ideal I_A is bounded above by the maximal true degree of any circuit in $\mathcal{C}_A$.

Toric ideals of Graphs

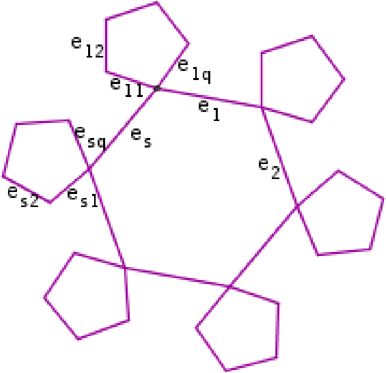


True circuit conjecture



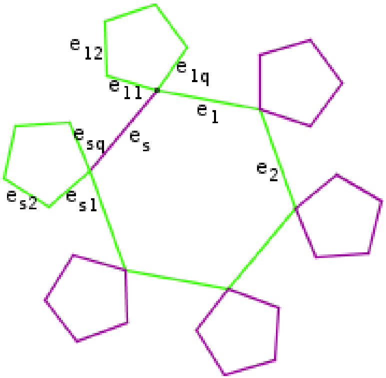
Next we give examples of circuits for which their true degrees are less than the degrees of some elements of the Graver basis.
Let us consider a graph G consisting of a cycle of length s and s odd cycles of length q, each one attached to a vertex of the initial cycle. Let w be the walk that passes through every edge of the graph G.

True circuit conjecture



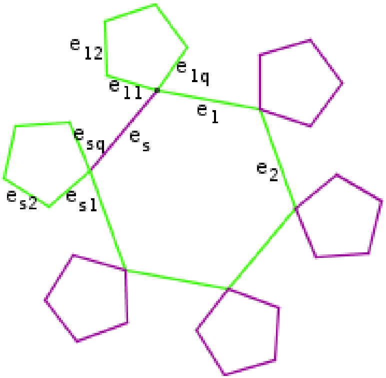
Let w be the walk that passes through every edge of the graph G. The length of the walk w is $qs + s = s(q+1)$, which is even. Thus, B_w is an element of the Graver basis of I_G and has degree $s(q+1)/2$.

True circuit conjecture



In the graph G there are a lot of circuits.
The longest one consists of two odd cycles joined by a path of length $s-1$.

True circuit conjecture



Its degree is
$(2q + 2(s-1))/2 = q + s - 1$.
Note that s, q are greater than two, as lengths of cycles, which implies that
$s(q+1)/2 > q + s - 1$.

True circuit conjecture

So there exists an element B_w in the Graver basis that has larger degree than any of the circuits.

$$deg(B_w) = s(q+1)/2 > q+s-1 \geq deg(C)$$

The difference of the degrees can be made as large as one wishes, by choosing large values for q and s.
Note that an easy, but lengthy, computation of the true degree of these circuits shows that the true degree is equal to the usual degree.

Therefore this family provides infinitely many counterexamples to the True circuit conjecture.