# Stanley Depth and Sequentially Cohen-Macaulay Lexsegment Ideals

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# Basic Definitions and Notations

$S = K[x_1, \dots, x_n]$,
$M$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module,
$w(homogenous) \in M$, $Z \subset \{x_1, \dots, x_n\}$.
$wK[Z]$ denotes the $K$-subspace of $M$ generated by
$\{wv : v(monomial) \in K[Z]\}$.
Then $K$-subspace $wK[Z]$ is called a Stanley space of dimension $|Z|$ if it is a free $K[Z]$-module.
A Stanley decomposition of $M$ is a presentation of the $K$-vector space $M$ as a finite direct sum of Stanley spaces

$$\mathcal{D} : M = \bigoplus_{i=1}^{s} u_i K[Z_i].$$

The Stanley depth of a decomposition $\mathcal{D}$ is
$\operatorname{sdepth} \mathcal{D} = \min\{|Z_i|, i = 1, \dots, s\}$. The Stanley depth of $M$ is

$$\operatorname{sdepth}_S(M) = \max\{\operatorname{sdepth} \mathcal{D} : \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

- In 1982, Stanley conjectured that

$$\mathrm{sdepth}(M) \geq \mathrm{depth}(M)$$

$$I = (x_1x_2^3, x_1^3x_2) \subset K[x_1, x_2]$$

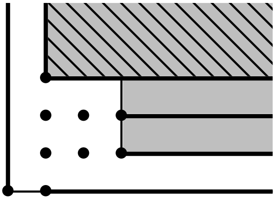

Figure:

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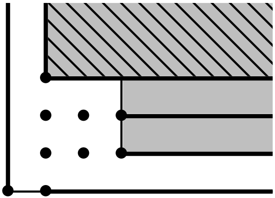

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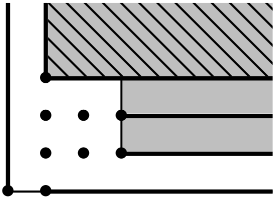

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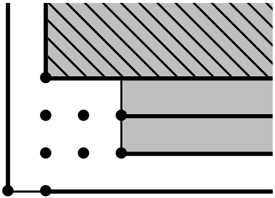

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$$S/I = K[x_2] \oplus x_1K[x_1] \oplus x_1x_2K \oplus x_1x_2^2K \oplus x_1^2x_2K \oplus x_1^2x_2^2K.$$

# Stanley's Conjecture

Let $I \subset S$ be a monomial ideal. Then in the following cases Stanleys conjecture holds for $S/I$.

- If $n \leq 3$ (J. Apel)
- If $n = 4$ (I. Anwar, D. Popescu)
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- Let $I$ be an intersection of four monomial prime ideals of $S$. Then the Stanley's conjecture holds for $I$. (D. Popescu)

(Herzog, Vladoiu and Zheng)
$M = I/J$, $J \subset I$ are monomial ideals of $S$. Let "$\leq$" be the natural partial order on $\mathbb{N}^n$ given by $a \leq b$ if $a(i) \leq b(i)$ for all $i \in [n]$. We denote $x^a = x_1^{a(1)} \ldots x_n^{a(n)}$ for an $a \in \mathbb{N}^n$. Suppose that $I$ is generated by the monomials $x^{a_1}, \ldots, x^{a_r}$ and $J$ by the monomials $x^{b_1}, \ldots, x^{b_s}$, $a_i, b_j \in \mathbb{N}^n$. Choose $g \in \mathbb{N}^n$ such that $a_i \leq g, b_j \leq g$ for all $i, j$. Let $P^g_{I/J}$ be the subposet of $\mathbb{N}^n$ given by all $c \in \mathbb{N}^n$ with $c \leq g$ and such that $a_i \leq c$ for some $i$ and $c \not\geq bj$ for all $j$. We call $P^g_{I/J}$ the characteristic poset of $I/J$ with respect to $g$. Clearly $P^g_{I/J}$ is finite.
Given a finite poset $P$ and $a, b \in P$ we call $[a, b] = \{c \in P : a \leq c \leq b\}$ interval. A partition of $P$ is a disjoint union

$$\mathcal{P} : P = \bigcup_{i=1}^{r} [a_i, b_i]$$

of intervals for $c \in P$ we set $Z_c = \{x_j : c(j) = g(j)\}$ and let $\rho : P \longrightarrow \mathbb{N}$ be the map given by $c \longrightarrow |Z_c|$.

**Theorem(Herzog, Vladoiu, Zheng)**

Let $\mathcal{P} : P^g_{I/J} = \bigcup_{i=1}^r [c_i, d_i]$ be a partition of $P^g_{I/J}$. Then

$$\mathcal{D}(\mathcal{P}) : I/J = \bigoplus_{i=1}^{r} (\bigoplus_{c} x^c [Z_{d_i}])$$

is a Stanley decomposition of $I/J$, where the inner direct sum is taken over all $c \in [c_i, d_i]$ for which $c(j) = c_i(j)$ for all $j$ with $x_j \in Z_{d_i}$. Moreover
$\operatorname{sdepth} \mathcal{D}(\mathcal{P}) = \min\{\rho(d_i) : i \in [r]\} \leq \operatorname{sdepth} I/J$.

Let $I = (x_1x_3, x_1x_4, x_2x_3, x_2x_4) \subset K[x_1, x_2, x_3, x_4]$ and $J = 0$. Set $a_1 = (1, 0, 1, 0)$, $a_2 = (1, 0, 0, 1)$, $a_3 = (0, 1, 1, 0)$ and $a_4 = (0, 1, 0, 1)$. Thus $I$ is generated by $x^{a_1}, x^{a_2}, x^{a_3}, x^{a_4}$ and we may choose $g = (1, 1, 1, 1)$. The poset $P = P^g_{I/J}$ is given by

$$P = \{(1,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,0,1),(1,1,1,0), \\ ,(1,1,0,1),(1,0,1,1),(0,1,1,1),(1,1,1,1)\}$$

A partition $\mathcal{P}$ of $P$ is given by

$$[(1,0,1,0),(1,0,1,1)]\bigcup[(1,0,0,1),(1,1,0,1)]\bigcup[(0,1,1,0), \\ (1,1,1,0)]\bigcup[(0,1,0,1),(0,1,1,1)]\bigcup[(1,1,1,1),(1,1,1,1)].$$

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By above theorem the corresponding Stanley decomposition is

$$I = x_1x_3K[x_1, x_3, x_4] \oplus x_1x_4K[x_1, x_2, x_4] \oplus x_2x_3K[x_1, x_2, x_3] \oplus \\ x_2x_4K[x_2, x_3, x_4] \oplus x_1x_2x_3x_4K[x_1, x_2, x_3, x_4].$$

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**Theorem(Shen)**

Let $I \subset S$ be a complete intersection monomial ideal minimally generated by $m$ elements. Then $\operatorname{sdepth}(I) = n - \lfloor \frac{m}{2} \rfloor$.

## Question(Shen)

Let $I \subset S$ be a squarefree monomial ideal minimally generated by $m$ elements. Is it true that $\operatorname{sdepth}(I) \geq n - \lfloor \frac{m}{2} \rfloor$?

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- Let $I = (x_1, x_2, x_3) \cap (x_4, x_5, x_6) \cap (x_7, x_8, x_9) \subset K[x_1, \ldots, x_9]$, here $m = 27$ and $n = 9$ by above result we have $\operatorname{sdepth}(I) \geq -4$.

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- Let $s$ be the largest integer such that $n + 1 \geq (2s + 1)(s + 1)$. Then the Stanley depth of any squarefree monomial ideal in n variables is greater or equal to $2s + 1$. (G. Floystad, J. Herzog)

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- For the above example by Floystad and Herzog we have $\mathrm{sdepth}(I) \geq 3$.
- Can we give an upper bound for the Stanley depth of a monomial ideal?

# Motivations

- Let $I \subset S$ be a monomial ideal. It is well known that

$$\operatorname{depth} S/I \leq \operatorname{depth} S/\sqrt{I}$$

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- J. Apel showed that the first inequality holds also for sdepth, that is $\operatorname{sdepth} S/I \leq \operatorname{sdepth} S/\sqrt{I}$.
- Is the inequality $\operatorname{sdepth} I \leq \operatorname{sdepth} \sqrt{I}$ holds?

- Let $P$ be an associated prime ideal of $S/I$. We know that

$$\operatorname{depth}_S S/I \leq \operatorname{depth}_S S/P = \dim S/P$$

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- Is the inequality $\operatorname{sdepth}_S I \leq \operatorname{sdepth}_S P$ holds?

# Results

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**Corollary**

Let $I \subset S$ be a monomial ideal. Then $\operatorname{sdepth}(S/I) \leq \operatorname{sdepth}(S/\sqrt{I})$ and $\operatorname{sdepth}(I) \leq \operatorname{sdepth}(\sqrt{I})$.

**Corollary**

Let $I$ and $J$ be two monomial ideals of $S$ such that $I \subset J$. If $\operatorname{sdepth}(J/I) = \dim(J/I)$, then $\operatorname{sdepth}(\sqrt{J}/\sqrt{I}) = \dim(\sqrt{J}/\sqrt{I})$.

**Theorem**

Let $Q$ and $Q'$ be two primary ideals with $\sqrt{Q} = (x_1, \ldots, x_t)$ and $\sqrt{Q'} = (x_{t+1}, \ldots, x_n)$, where $t \geq 2$ *and* $n \geq 4$. Then

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**Corollary**

Let $Q$ and $Q'$ be two irreducible monomial ideals such that $\sqrt{Q} = (x_1, \ldots, x_t)$ and $\sqrt{Q'} = (x_{t+1}, \ldots, x_n)$. Suppose that $n$ is odd. Then $\operatorname{sdepth}(Q \cap Q') = \lceil \frac{n}{2} \rceil$.

**Corollary**

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$$\operatorname{sdepth}(Q \cap Q') = \begin{cases} \frac{n}{2} + 1, & \textit{if} \quad t \textit{ is odd}; \\ \\ \frac{n}{2} \textit{ or } \frac{n}{2} + 1, & \text{if} \quad t \textit{ is even}. \end{cases}$$

**Theorem**

Let $Q$ and $Q'$ be two primary monomial ideals with $\sqrt{Q} = (x_1, \ldots, x_t)$ and $\sqrt{Q'} = (x_{r+1}, \ldots, x_p)$, where $1 < r \leq t < p \leq n$, $n \geq 4$. Then

$$\operatorname{sdepth}(Q \cap Q') \leq \min\{\frac{2n+t-p-r+2}{2},\ n - \lfloor \frac{t}{2} \rfloor,\ n - \lfloor \frac{p-t}{2} \rfloor\}.$$

The inequality becomes equality if $t = r$, $n$ is odd and $Q$, $Q'$ are irreducible.

**Theorem**

Let $I \subset S$ be a monomial ideal and let $P \in \operatorname{Ass}(S/I)$. Then

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**Corollary**

Let $I \subset S$ be a monomial ideal such that $Ass(S/I) = \{P_1, \ldots, P_s\}$. Then

$$\text{sdepth}(I) \leq \min\{n - \lfloor \frac{\text{ht}(P_i)}{2} \rfloor,\ 1 \leq i \leq s\}.$$

**Corollary**

Let $I \subset S$ be a monomial ideal with $|G(I)| = m$. Suppose that $m$ is even, and let there exists a prime ideal $P \in \mathrm{Ass}(S/I)$ such that $\mathrm{ht}(P) = m$. Then

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Let $I \subset S$ be a monomial ideal with $|G(I)| = m$. Suppose that $m$ is odd, and let there exists a prime ideal $P \in \mathrm{Ass}(S/I)$ such that $\mathrm{ht}(P) \geq m - 1$. Then $\mathrm{sdepth}_S(I) = n - \left\lfloor \frac{m}{2} \right\rfloor$.

**Theorem**

Let $I \subset S$ be a monomial ideal such that all associated prime ideals of $S/I$ are generated in disjoint sets of varables. Then Stanley's conjecture holds for $S/I$ and $I$.

## Definition

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A graph $G(V, E)$ with vertex set $V$ and edge set $E$ is called complete $k$-partite if the vertex set $V$ is partitioned into $k$ disjoint subset $V_1, V_2, \ldots, V_k$ and $E = \{\{u, v\} : u \in V_i, v \in V_j, i \neq j\}$.

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## Definition

Let $G$ be a graph. Then the edge ideal $I$ associated to $G$ is the squarefree monomial ideal $I = (x_i x_j : \{v_i, v_j\} \in E)$ of $S$.

After relabeling the elements of $V$, we may assume that

$$V_i = \{v_j : r_1 + r_2 + \cdots + r_{i-1} + 1 \leq j \leq r_1 + \cdots + r_i\}.$$

Now let $G$ be a complete $k$-partite graph with vertex set $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ with $|V_i| = r_i$, where $r_i \in \mathbb{N}$ and $2 \leq r_1 \leq \cdots \leq r_k$. Let $r_1 + \cdots + r_k = n$. Let $I_1 = (x_1, \ldots, x_{r_1})$, $I_2 = (x_{r_1+1}, \ldots, x_{r_1+r_2}), \ldots, I_k = (x_{r_1+\cdots+r_{k-1}+1}, \ldots, x_n)$. Then the edge ideal of $G$ is of the form

$$I = (\sum_{i \neq j} I_i \cap I_j).$$

**Lemma(Ishaq, Qureshi)**

$$\operatorname{sdepth}(I) \leq 2 + \frac{\binom{n}{3} - (\sum_{i=1}^{k} \binom{r_i}{3})}{\sum_{1\leq i<j\leq k} r_i r_j}.$$

**Lemma(Ishaq, Qureshi)**

$$\operatorname{sdepth}(I) \leq 2 + \frac{\binom{n}{3} - (\sum\limits_{i=1}^{k} \binom{r_i}{3})}{\sum\limits_{1\leq i<j\leq k} r_i r_j}.$$

**Proposition(Ishaq, Qureshi)**

Let $I$ be the edge ideal of complete $k$-partite graph then Stanley's conjecture holds for $I$.

Let $\mathbf{H} = (V, E)$ denote a hypergraph with vertex set $V$ and hyperedge set $E$. A hyperedge $e \in E$ is a subset of the vertices. That is, $e \subset V$ for each $e \in E$. A hypergraph is called complete $k$-partite if the vertices are partitioned into $k$ disjoint subsets $V_i$, $i = 1, \ldots, k$ and $E$ consists of all hyperedges containing exactly one vertex from each of the $k$ subsets.

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**Definition**

Let $\mathbf{H} = (V, E)$ be a hypergraph with vertex set $V$ and hyperedge set $E$. Then the edge ideal associated to hypergraph $\mathbf{H}$ is a square free monomial ideal

$$I = (x_{i_1}x_{i_2}\dots x_{i_r} : \{v_{i_1}, v_{i_2}, \dots, v_{i_r}\} \in E).$$

## Question(Nill, Vorwerk)

Let $I$ be the edge ideal of a complete $k$-partite hypergraph $\mathbf{H}_d^k$. Here, $\mathbf{H}_d^k$ has $kd$ vertices divided into $k$ independent sets $V^{(i)}$ (for $i = 1, \ldots, k$) each with $d$ vertices $v_1^{(i)}, \ldots, v_d^{(i)}$, and $\mathbf{H}_d^k$ has $d^k$ hyperedges consisting of exactly $k$ vertices. Then $I$ is squarefree monomial ideal in the polynomial ring $K[v_j^{(i)} : i \in \{1, \ldots, k\}, j \in \{1, \ldots, d\}]$:

$$I = (v_1^{(1)}, \ldots, v_d^{(1)}) \cdots (v_1^{(k)}, \ldots, v_d^{(k)}).$$

What is sdepth$(S/I)$ in this case?

We consider this question even in more general frame. We consider the case where each vertex set $V^{(i)}$ is not necessarily of the same cardinality. Let $I$ be the edge ideal of a complete $k$-partite hypergraph $\mathbf{H}^k$, where $\mathbf{H}^k$ has $n$ vertices divided into $k$ independent sets $V^{(i)}$ (for $i = 1, \ldots, k$) each with $d_i$ vertices $v_1^{(i)}, \ldots, v_{d_i}^{(i)}$, and $\mathbf{H}^k$ has $d_1 d_2 \cdots d_k$ hyperedges consisting of exactly $k$ vertices. To each vertex set $V^{(i)}$ we associate a set of variables $\{x_{i_1}, \ldots, x_{i_{d_i}}\}$ and set $S = K[(x_{i_j})]$. Now let $V^{(i)}$ and $V^{(j)}$ be two vertex sets, $\{x_{i_1}, \ldots, x_{i_{d_i}}\}$ and $\{x_{j_1}, \ldots, x_{j_{d_j}}\}$ be the sets of variables associated to $V^{(i)}$ and $V^{(j)}$ respectively. Since $V^{(i)}$ and $V^{(j)}$ are independent we have $\{x_{i_1}, \ldots, x_{i_{d_i}}\} \cap \{x_{j_1}, \ldots, x_{j_{d_j}}\} = \emptyset$. Then $I$ is the squarefree monomial ideal in the polynomial ring $S$:

$$I = P_1 P_2 \cdots P_k = P_1 \cap P_2 \cap \cdots \cap P_k,$$

where $P_i = (x_{i_1}, \ldots, x_{i_{d_i}})$ and $\sum_{i=1}^{k} P_i = \mathfrak{m} = (x_1, \ldots, x_n)$.

**Lemma**

Let $I$ be a squarefree monomial ideal of $S$ generated by monomials of degree $d$. Let $A$ be the number of monomials of degree $d$ and $B$ be the number of squarefree monomials of degree $d+1$ in $I$. Then

$$d \leq \operatorname{sdepth}(I) \leq d + \lfloor \frac{B}{A} \rfloor,$$

**Corollary**

Let $I$ be a squarefree monomial ideal of $S$ generated by monomials of degree $d$. If $\binom{n}{d+1} < |G(I)|$ then $\operatorname{sdepth}(I) = d$.

**Theorem(Ishaq, Qureshi)**

Let $I = \bigcap_{i=1}^{k} Q_i \subset S$ be a monomial ideal such that each $Q_i$ is irreducible and $G(\sqrt{Q_i}) \cap G(\sqrt{Q_j}) = \emptyset$ for all $i \neq j$, then $\operatorname{sdepth}(I) = \operatorname{sdepth}(\bigcap_{i=1}^{k} \sqrt{Q_i})$.

**Theorem(Ishaq, Qureshi)**

Let $I = \bigcap_{i=1}^{k} Q_i \subset S$ be a monomial ideal such that each $Q_i$ is irreducible and $G(\sqrt{Q_i}) \cap G(\sqrt{Q_j}) = \emptyset$ for all $i \neq j$, then $\mathrm{sdepth}(I) = \mathrm{sdepth}(\bigcap_{i=1}^{k} \sqrt{Q_i})$.

In the setting of the above theorem, if $Q_i$ are not irreducible for all $i$ then the result is false. For example if $n = 4$, $I = (x_1^2, x_1x_2, x_2^2) \cap (x_3^2, x_3x_4, x_4^2)$ and $\mathcal{P}$ is a partition of $\mathcal{P}_I^g$, $g = (2,2,2,2)$ then we must have 9 intervals $[a, b]$ in $\mathcal{P}$ starting with the generators $a$ of $I$ but only 8 monomials $b$ are in $\mathcal{P}_I^g$ with $\rho(b) = 3$, the biggest one $x_1^2x_2^2x_3^2x_4^2$ cannot be taken. Thus $\mathrm{sdepth}\, I < 3$. But clearly $\mathrm{sdepth}(\sqrt{I}) = 3$.

**Theorem(Ishaq, Qureshi)**

Let $I = \bigcap_{i=1}^{k} P_i$ be a monomial ideal in $S$ where each $P_i$ is a monomial prime ideal and $\sum_{i=1}^{k} P_i = \mathfrak{m}$. Suppose that $G(P_i) \cap G(P_j) = \emptyset$ for all $i \neq j$. Then

$$\operatorname{sdepth}(I) \leq \frac{n+k}{2}.$$

# Results

Let $I = \bigcap_{i=1}^{k} P_i$ be a monomial ideal such that each $P_i$ is irreducible and $G(P_i) \cap G(P_j) = \emptyset$ for all $i \neq j$, $\mathrm{ht}(P_i) = d_i$ and $\sum_{i=1}^{k} P_i = \mathfrak{m}$.
We define a set

$$A := \{P_i \ : \ \ \mathrm{ht}(P_i) \text{ is odd } \}.$$

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**Corollary(Ishaq, Qureshi)**

Let $I = \bigcap_{i=1}^{k} P_i$ be a squarefree monomial ideal such that each $P_i$ is monomial prime ideal and $G(P_i) \cap G(P_j) = \emptyset$ for all $i \neq j$, $\operatorname{ht}(P_i) = d_i$ and $\sum_{i=1}^{k} \sqrt{P_i} = \mathfrak{m}$. Then

$$\frac{n + |A|}{2} \leq \operatorname{sdepth}(I) \leq \lfloor \frac{n+k}{2} \rfloor.$$

**Corollary(Ishaq, Qureshi)**

Let $I$ be the edge ideal of a complete $k$-partite hypergraph $\mathbf{H}_d^k$. Then

$$\operatorname{sdepth}(I) = \frac{n+k}{2}, \qquad \textit{if } d \textit{ is odd};$$

$$\frac{n}{2} \leq \operatorname{sdepth}(I) \leq \frac{n+k}{2}, \quad \textit{if } d \textit{ is even}.$$

**Theorem(Ishaq, Qureshi)**

Let $I$ be a monomial ideal and let $\mathrm{Min}(S/I) = \{P_1, \dots, P_s\}$ with $\sum_{i=1}^{s} P_i = \mathfrak{m}$. Let $d_i := |G(P_i) \setminus G(\sum_{i \neq j}^{s} P_j)|$, and $r := |\{d_i : d_i \neq 0\}|$. Suppose that $r \geq 1$. Then

$$\mathrm{sdepth}(I) \leq (2n + r - \sum_{i=1}^{s} d_i)/2$$

**Example**

*Let* $I = (x_1 \dots, x_9) \cap (x_9, \dots, x_{18}) \cap (x_{18}, \dots, x_{27}) \cap (x_{27}, \dots, x_{36}) \subset K[x_1, \dots, x_{36}]$. *We have* $d_1 = 8$, $d_2 = 8$, $d_3 = 8$, $d_4 = 9$ *and* $s = 4$, *then by above theorem we have* $\operatorname{sdepth}(I) \leq 21$. *And by one of our stated result we have* $\operatorname{sdepth}(I) \leq 31$.

**Theorem(Ishaq, Qureshi)**

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring and $Q_1, Q_2, \dots, Q_k$ monomial irreducible ideals of $S$ such that $G(\sqrt{Q_i}) \cap G(\sqrt{Q_j}) = \emptyset$ for all $i \neq j$. Let $r_i := \mathrm{ht}(Q_i)$, $\sum_{i=1}^{k} r_i = n$. If $I = Q_1 \cap Q_2 \cap \dots \cap Q_k$, then

$$\mathrm{sdepth}(S/I) \geq \min\left\{n - r_1, \min_{2 \leq i \leq k}\{\lceil\frac{r_1}{2}\rceil + \dots + \lceil\frac{r_{i-1}}{2}\rceil + r_{i+1} + \dots + r_k\}\right\},$$

**Proposition(Ishaq, Qureshi)**

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring and $Q_1, Q_2, \dots, Q_k$ monomial primary ideals of $S$ such that $G(\sqrt{Q_i}) \cap G(\sqrt{Q_j}) = \emptyset$ for all $i \neq j$. Let $r_i := \mathrm{ht}(Q_i)$. Suppose that $r_1 \geq r_2 \geq \cdots \geq r_k$, $k \geq 3$ and $\sum_{i=1}^{k} r_i = n$. If $I = Q_1 \cap Q_2 \cap \ldots \cap Q_k$, then

$$\mathrm{sdepth}(S/I) \leq \lceil \frac{r_{k-1}}{2} \rceil + r_1 + r_2 + \cdots + r_{k-2}.$$

**Corollary(Ishaq, Qureshi)**

Let $I$ be the edge ideal of a complete $k$-partite hypergraph $\mathbf{H}_d^k$. Then

$$(k-1)\lceil \frac{d}{2} \rceil \leq \text{sdepth}(S/I) \leq (k-2)d + \lceil \frac{d}{2} \rceil.$$

# Results

## Definition

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $K$. We consider the lexicographical order on the monomials of $S$ induced by $x_1 > x_2 > \ldots > x_n$. Let $d \geq 2$ be an integer and $\mathcal{M}_d$ the set of monomials of degree $d$ of $S$. For two monomials $u, v \in \mathcal{M}_d$, with $u \geq_{lex} v$, the set

$$L(u, v) = \{w \in \mathcal{M}_d \mid u \geq_{lex} w \geq_{lex} v\}$$

is called a *lexsegment set*. A *lexsegment ideal* in $S$ is a monomial ideal of $S$ which is generated by a lexsegment set.

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- A lexsegment ideal of the form $(L(x_1^d, v))$, $v \in \mathcal{M}_d$, is called an *initial lexsegment ideal* determined by $v$.
- An ideal generated by a lexsegment set of the form $L(u, x_n^d)$ is called a *final lexsegment ideal* determined by $u \in \mathcal{M}_d$.

**Proposition**

Let $v \in \mathcal{M}_d$ be a monomial and let $I = (L^i(v))$ the initial ideal determined by $v$. Then

$$\mathrm{Ass}(S/I) = \{(x_1, \ldots, x_j) : j \in \mathrm{supp}(v) \cup \{n\}\}.$$

## Results

### Proposition

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### Proposition

Let $u \in \mathcal{M}_d, u \neq x_1^d$, with $x_1|u$ and $I = (L^f(u))$ be the final lexsegment ideal defined by $u$. Then

$$\mathrm{Ass}(S/I) = \{(x_1, \ldots, x_n), (x_2, \ldots, x_n)\}.$$

**Proposition**

Let $I = (L(u, v))$ be a lexsegment ideal which is neither initial nor final, with $x_1 \nmid v$, and such that $\operatorname{depth}(S/I) = 0$ Then

$$\operatorname{Ass}(S/I) = \{(x_1, \ldots, x_j) : j \in \operatorname{supp}(v) \cup \{n\}\} \cup \{(x_2, \ldots, x_n)\}.$$

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For $2 \leq j, t \leq n$ such that $2 \leq j \leq t - 2$, we denote $P_{j,t} = (x_2, \dots, x_j, x_t, \dots, x_n)$.

**Proposition**

# Results

Let $I = (L(u,v))$ be a lexsegment ideal with $x_1 \nmid v$ and such that $\operatorname{depth}(S/I) > 0$.

(i) Let $\operatorname{depth}(S/I) = 1$. Then,

(a) for $a_l < d-1$, we have

$$\operatorname{Ass}(S/I) = \{(x_2,\dots,x_n)\} \cup \{(x_1,\dots,x_j) : j \in \operatorname{supp}(v) \setminus \{n\}\} \cup$$
$$\cup \{P_{j,l} : j \in \operatorname{supp}(v), j \le l-2\} \cup \{P_{j,l+1} : j \in \operatorname{supp}(v), j \le l-1\};$$

(b) for $a_l = d-1$, we have
$\operatorname{Ass}(S/I) = \{(x_2,\dots,x_n)\} \cup \{(x_1,\dots,x_j) : j \in \operatorname{supp}(v) \setminus \{n\}\} \cup$

$$\cup \{P_{j,l} : j \in \operatorname{supp}(v), j \le l-2\}.$$

(ii) Let $\operatorname{depth}(S/I) > 1$. Then

(a) for $a_l < d-1$, we have $\operatorname{Ass}(S/I) =$

$$\{(x_1,\dots,x_j) : j \in \operatorname{supp}(v) \setminus \{n\}\} \cup \{P_{j,l} : j \in \operatorname{supp}(v)\}$$
$$\cup \{P_{j,l+1} : j \in \operatorname{supp}(v)\};$$

(b) for $a_l = d-1$, we have

$$\operatorname{Ass}(S/I) = \{(x_1,\dots,x_j) : j \in \operatorname{supp}(v) \setminus \{n\}\} \cup \{P_{j,l} : j \in \operatorname{supp}(v)\}.$$

**Definition**

Let $M$ be a finitely generated multigraded $S$-module. A multigraded prime filtration of $M$,

$$\mathcal{F}: \quad 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{r-1} \subseteq M_r = M,$$

where $M_i/M_{i-1} \cong S/P_i$, with $P_i$ a monomial prime ideal, is called *pretty clean* if for all $i < j$, $P_i \subseteq P_j$ implies $i = j$. In other words, a proper inclusion $P_i \subseteq P_j$ is possible only if $i > j$. A multigraded $S$-module is *called pretty* clean if it admits a pretty clean filtration.

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**Lemma**

Let $M$ be a finitely generated multigraded $S$-module such that $\mathrm{Ass}(M)$ is totally ordered by inclusion. Then $M$ is pretty clean.

**Definition**

Let $M$ be a finitely generated multigraded $S$-module. We say that $M$ is sequentially Cohen-Macaulay if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$$

of $M$ by graded submodules $M_i$ satisfying the two conditions:

- Each quotient $M_i/M_{i-1}$ is Cohen-Macaulay;
- $\dim(M_1/M_0) < \dim(M_2/M_1) < \ldots < \dim(M_r/M_{r-1})$.

**Theorem**

Let $I \subseteq S$ be a lexsegment ideal. Then $S/I$ is a pretty clean module.

**Theorem**

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**Corollary**

Let $I \subseteq S$ be a lexsegment ideal. Then $S/I$ is sequentially Cohen-Macaulay.

**Theorem**

Let $I \subseteq S$ be a lexsegment ideal. Then $S/I$ is a pretty clean module.

**Corollary**

Let $I \subseteq S$ be a lexsegment ideal. Then $S/I$ is sequentially Cohen-Macaulay.

**Corollary**

Let $I \subseteq S$ be a lexsegment ideal. Then $S/I$ satisfies the Stanley's conjecture.

Let $I \subset S$ be a monomial ideal and $I = \cap_{i=1}^{s} Q_i$ an irredundant primary decomposition of $I$, where the $Q_i$ are monomial ideals. Let $Q_i$ be $P_i$-primary. Then each $P_i$ is a monomial prime ideal and $\mathrm{Ass}(S/I) = \{P_1, \dots, P_s\}$.
According to Lyubeznik the size of $I$, denoted $\mathrm{size}(I)$, is the number $a + (n - b) - 1$, where $a$ is the minimum number $t$ such that there exist $j_1 < \cdots < j_t$ with

$$\sqrt{\sum_{l=1}^{t} Q_{j_l}} = \sqrt{\sum_{j=1}^{s} Q_j},$$

and where $b = \mathrm{ht}(\sum_{j=1}^{s} Q_j)$. It is clear from the definition that $\mathrm{size}(I)$ depends only on the associated prime ideals of $S/I$. In the above definition if we replaced "there exists $j_1 < \cdots < j_t$" by "for all $j_1 < \cdots < j_t$", we obtain the definition of $\mathrm{bigsize}(I)$, introduced by D. Clearly $\mathrm{bigsize}(I) \geq \mathrm{size}(I)$.

Let $I$ be a squarefree monomial ideal with minimal monomial generating set $G(I) = \{u_1, \dots, u_m\}$. Let $u$ be a monomial of $S$ then $\mathrm{supp}(u) := \{i : x_i \text{ divides } u\}$. Then we call a monomial ideal $J$ a *modification* of $I$, if $G(J) = \{v_1, \dots, v_m\}$ and $\mathrm{supp}(v_i) = \mathrm{supp}(u_i)$ for all $i$. Obviously, $\sqrt{J} = I$. Let $\alpha = (a_1, \dots, a_n) \in \mathbb{N}^n$, $a_i \neq 0$ for all $i$ and $\sigma_\alpha$ be the $K$-morphism of $S$ given by $x_i \to x_i^{a_i}$, $i \in [n]$. Let $I^\alpha := \sigma_\alpha(I)S$. Then $I^\alpha$ is called a *trivial modification* of $I$.

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**Theorem(Lyubeznik)**

Let $I \subset S$ ba a monomial ideal then $\mathrm{depth}(I) \geq 1 + \mathrm{size}(I)$.

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Herzog, Popescu and Vladoiu say a monomial ideal $I$ has *minimal depth*, if $\text{depth}(I) = \text{size}(I) + 1$.

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**Theorem(Herzog, Popescu and Vladoiu)**

Let $I \subset S$ be a monomial ideal then $\text{sdepth}(I) \geq 1 + \text{size}(I)$. In particular, Stanley's conjecture holds for the monomial ideals of minimal depth.

# Results

**Theorem**

Stanley's conjecture holds for $I$, if it satisfies one of the following statements:

1. $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant presentation of $I$ as an intersection of primary monomial ideals. Let $P_i := \sqrt{Q_i}$. If $P_i \not\subset \sum_{1=i\neq j}^{s} P_j$ for all $i \in [s]$
2. the bigsize of $I$ is one,
3. $I$ is a lexsegment ideal.

**Theorem**

Let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant presentation of $I$ as an intersection of primary monomial ideals. Let $P_i := \sqrt{Q_i}$. If $P_i \not\subset \sum_{1=i\neq j}^{s} P_j$ for all $i \in [s]$ then $\operatorname{sdepth}(S/I) \geq \operatorname{depth}(S/I)$, that is the Stanley's conjecture holds for $S/I$.

**Theorem**

Let $\alpha \in \mathbb{N}^n$, then $\operatorname{sdepth}(I^\alpha) = \operatorname{sdepth}(I)$.

**Corollary**

Let $I \subset S$ be a squarefree monomial ideal if the Stanley conjecture holds for $I$, then the Stanley conjecture also holds for $I^\alpha$.

# THANK YOU