Lefschetz properties of Artinian algebras and Hilbert series

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Outline

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Background

Lefschetz properties

Application to barycentric subdivisions

Application to Veronese algebras and edgewise subdivisions

Standard graded algebras

k field

$A = \bigoplus_{i \geq 0} A_i$ standard graded k-algebra $\Leftrightarrow$

- $A_0 = k$
- A generated in degree 1
- $A_i A_j \subseteq A_{i+j}$ for all $i, j \geq 0$
- $\dim_k(A_i) < \infty$ for all i

A Artinian $\Leftrightarrow$ $\dim_k(A) < \infty$ $\Leftrightarrow$ $A = \bigoplus_{i=0}^{s} A_i$

Example:

- $A = k[x_1, \dots, x_n]$ standard graded ✓, not Artinian ✗
- $A = k[x_1, x_2, x_3]/(x_1x_2, x_1x_3, x_2x_3, x_1^3, x_2^3, x_3^3)$ standard graded ✓, Artinian ✓

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Hilbert series

k field, $A = \bigoplus_{i \geq 0} A_i$ standard graded k-algebra

$$\mathrm{Hilb}(A,t) = \sum_{i \geq 0} \dim_k(A_i) t^i = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d}$$ Hilbert series of A

Notation:

- $h_A(t)$ — h-polynomial of A
- $h(A) = (h_0, h_1, \ldots, h_s)$ — h-vector of A
- $g(A) = (1, h_1 - h_0, \ldots, h_{\lfloor \frac{s}{2} \rfloor} - h_{\lfloor \frac{s}{2} \rfloor - 1})$ — g-vector of A

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Goal: Study h-polynomials/vectors. $\Rightarrow$ How?

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Our Tool: Lefschetz properties

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A is called

- m-Lefschetz if there exists $\omega \in A_1$ such that the multiplication map
$$\omega^{m-2i} : A_i \to A_{m-i} : p \mapsto \omega^{m-2i} p$$
is injective for $0 \leq i \leq \lfloor \frac{m-1}{2} \rfloor$.

- strong Lefschetz if A is s-Lefschetz.

- almost strong Lefschetz if A is $(s-1)$-Lefschetz.

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- weak Lefschetz if there exists $\omega \in A_1$ and $1 \leq g \leq s$ such that the multiplication map

$$\omega : \; A_i \to A_{i+1} : \; p \mapsto \omega p$$

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- quasi weak Lefschetz if there exists $\omega \in A_1$ and $1 \leq g \leq s$ such that the multiplication map

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ω is called ($m-$ / (almost) strong / (quasi) weak) Lefschetz element.

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Why can Lefschetz elements serve our purposes?

Example 1: A m-Lefschetz with Lefschetz element ω.

$\Rightarrow \omega : A_i \to A_{i+1} : \ p \mapsto \omega p$ injective for $0 \leq i \leq \lfloor \frac{m-1}{2} \rfloor$.

$\Rightarrow (g_0, g_1, \ldots, g_{\lfloor \frac{m+1}{2} \rfloor})$ is the Hilbert function of $A/(\omega A + \mathfrak{m}^{\lfloor \frac{m+1}{2} \rfloor + 1})$.

$\Rightarrow (g_0, g_1, \ldots, g_{\lfloor \frac{m+1}{2} \rfloor})$ is an M-sequence.

Moreover: $h_i \leq h_{m-i}$ for $0 \leq i \leq \lfloor \frac{m-1}{2} \rfloor$.

Example 2: A quasi weak Lefschetz.

$\Rightarrow h_0 \leq h_1 \leq \ldots \leq h_p \quad h_{p+1} \geq \ldots \geq h_s$ (unimodality).

What to remember: Lefschetz properties are a tool to prove properties of the Hilbert series of Artinian algebras.

Question: What about non-Artinian algebras?

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What to do for non-Artinian algebras?

A standard graded k-algebra, not Artinian ✗

Assume: A Cohen-Macaulay, d-dimensional

Then: Exists linear system of parameters $\Theta = \{\theta, \ldots, \theta_d\}$ for A.

Comparing the Hilbert series of A and $A/\Theta A$ one gets

$$\mathrm{Hilb}(A,t) = \frac{\mathrm{Hilb}(A/\Theta A, t)}{(1-t)^d}.$$

This means $h_A(t) = h_{A/\Theta A}(t)$ (independent of Θ).

Consequence: Can reduce to the Artinian case, i.e., try to find Lefschetz elements for $A/\Theta A$.

Remark: If there exists a l.s.o.p. Θ such that $A/\Theta A$ is Lefschetz, then this will be true for generic l.s.o.p. and a generic linear form ω. (Call A Lefschetz in this case.)

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Consequence: Can reduce to the Artinian case, i.e., try to find Lefschetz elements for $A/\Theta A$.

Remark: If there exists a l.s.o.p. Θ such that $A/\Theta A$ is Lefschetz, then this will be true for generic l.s.o.p. and a generic linear form ω. (Call A Lefschetz in this case.)

Lefschetz elements in Combinatorics

A simplicial complex Δ is called Lefschetz if its Stanley-Reisner ring has this property.

- Stanley: necessity part of the g-theorem for simplicial polytopes (uses Hard Lefschetz theorem for toric variety)
- Babson/Nevo: Preservation of Lefschetz properties under taking joins, connected sums, stellar subdivisions, unions of simplicial complexes.
- Murai: strongly edge decomposable complexes are strong Lefschetz.
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Lefschetz elements in Commutative Algebra

Theorem

k field of characteristic 0, $S = k[x_1, \dots, x_n]$, $I = \langle x_1^{a_1}, \dots, x_n^{a_n} \rangle$ Artinian monomial complete intersection, ω general linear form. Then for any positive integers d, i the multiplication

$$\omega^d : \ (S/I)_i \to (S/I)_{i+d} : \ p \mapsto \omega^d \cdot p$$

has maximal rank.

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  - Stanley (1980, algebraic topology),
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Outline

Background

Lefschetz properties

Application to barycentric subdivisions

Application to Veronese algebras and edgewise subdivisions

Stanley-Reisner rings and h-vectors

Δ simplicial complex on vertex set $[n] = \{1, \ldots, n\}$

$I_\Delta = (\prod_{i \in F} x_i \mid F \notin \Delta)$ Stanley-Reisner ideal of Δ

$k[\Delta] = k[x_1, \ldots, x_n]/I_\Delta$ Stanley-Reisner ring of Δ

$h(\Delta) = h(k[\Delta])$ h-vector of Δ

Example:

- Δ $(d-1)$-simplex $\Rightarrow k[\Delta] = k[x_1, \ldots, x_d]$, $h(\Delta) = (1, 0, \ldots, 0)$
- $k[\Delta] = k[x_1, x_2, x_3, x_4, x_5]/(x_1x_4, x_1x_5, x_2x_5, x_4x_5)$ and $h(\Delta) = (1, 2, -1, 0)$

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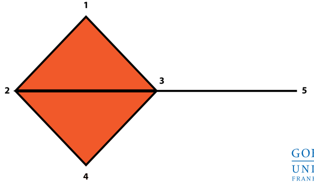

Barycentric subdivisions

Δ simplicial complex

The barycentric subdivision of Δ is the simplicial complex $\mathrm{sd}(\Delta)$ on vertex set $\Delta \setminus \{\emptyset\}$, whose faces are chains

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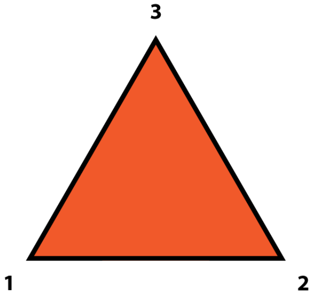

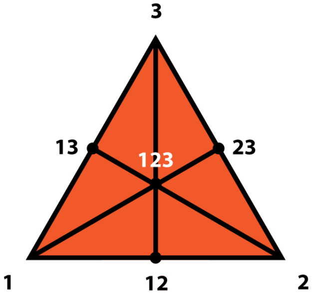

Shellability

Δ pure simplicial complex on vertex set $[n]$.

Δ shellable $\Leftrightarrow \exists$ linear ordering $F_1, \ldots, F_t$ of the facets of Δ such that for $2 \le i \le t$ the set

$$\langle F_1, \ldots, F_i \rangle \setminus \langle F_1, \ldots, F_{i-1} \rangle$$

has exactly one minimal element, the so-called restriction face.

Facts:

- Δ shellable $\Rightarrow k[\Delta]$ Cohen-Macaulay.
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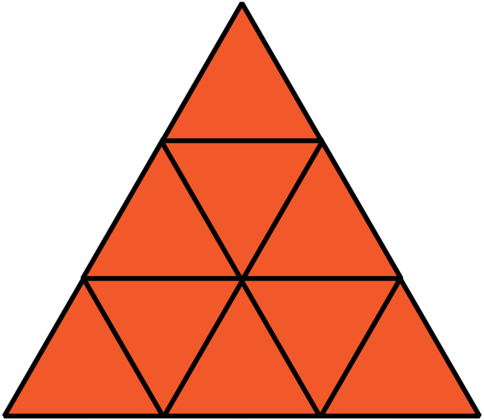

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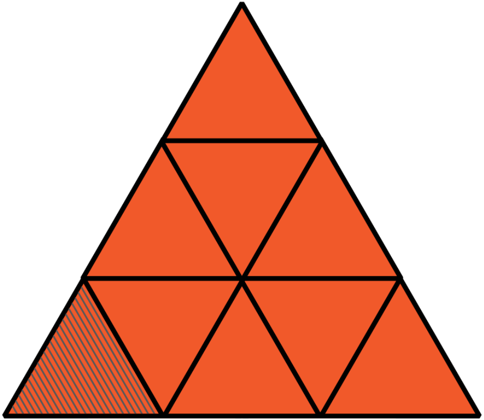

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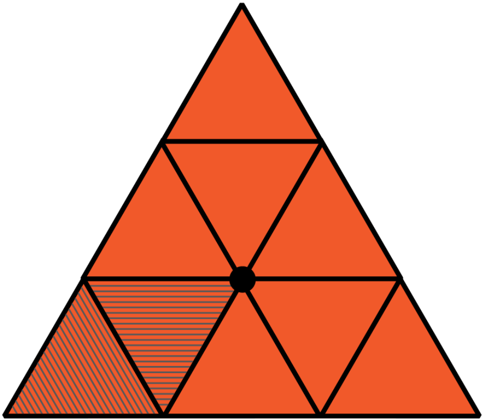

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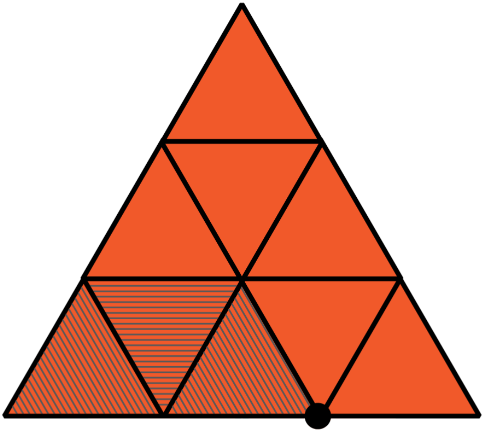

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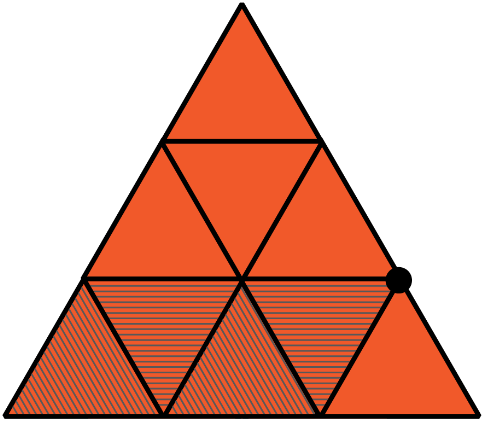

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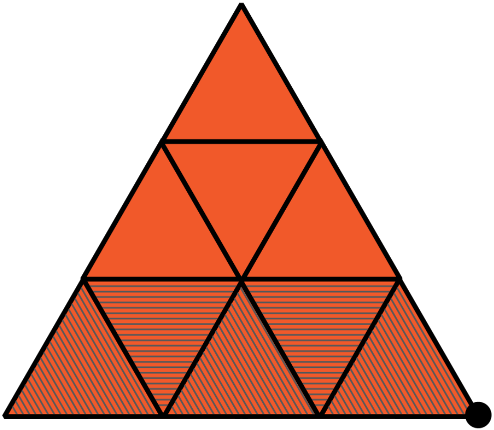

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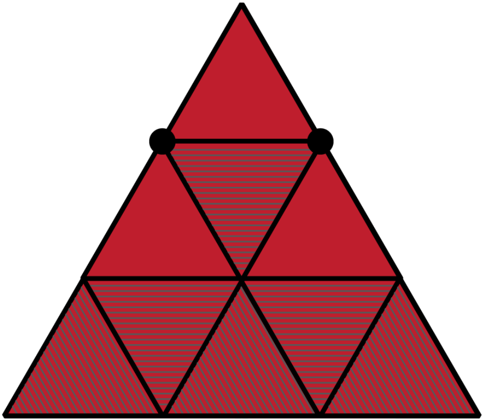

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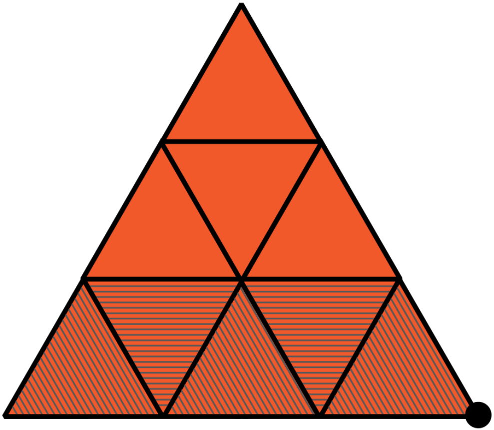

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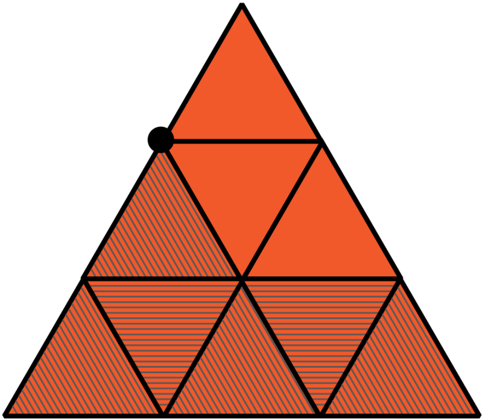

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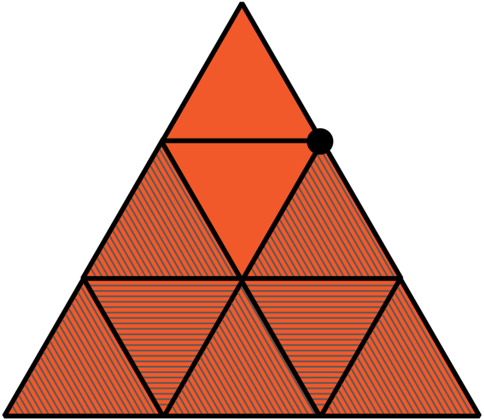

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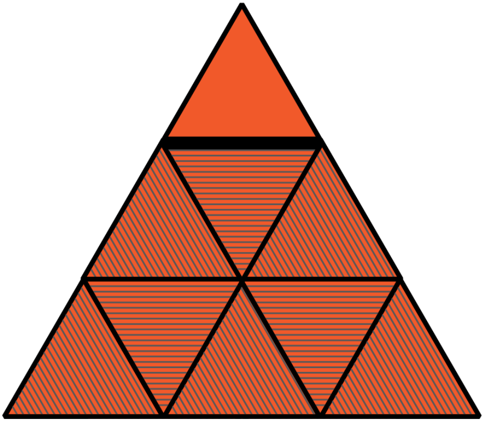

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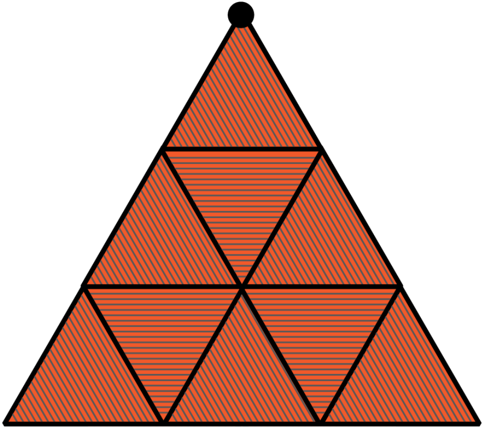

Motivation

Brenti/Welker studied the h-vector transformation of barycentric subdivisions. They showed:

- $h(\mathrm{sd}(\Delta))$ can be obtained from $h(\Delta)$ by a linear transformation with positive integral coefficients.
- Δ Cohen-Macaulay $\Rightarrow$ $h(\mathrm{sd}(\Delta))$ is unimodal.

Conjecture

Δ Cohen-Macaulay $\Rightarrow$ $k[\Delta]$ has some type of Lefschetz property.

$\Rightarrow$ $g(\mathrm{sd}(\Delta))$ is an M-sequence.

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Barycentric subdivisions: The result

Theorem (K., Nevo)

(i) *k infinite field, Δ shellable $(d-1)$-dimensional simplicial complex. Then: $\mathrm{sd}(\Delta)$ is $(d-1)$-Lefschetz over k.*

(ii) *Δ Cohen-Macaulay $\Rightarrow$ $g(\mathrm{sd}(\Delta))$ M-sequence.*

(iii) *Δ $(d-1)$-dimensional Cohen-Macaulay complex $\Rightarrow$ $h_i(\mathrm{sd}(\Delta)) \leq h_{d-1-i}(\mathrm{sd}(\Delta))$ for $0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor$.*

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Sketch of the proof

Double induction over the number of facets and the dimension of Δ.

Base of the induction: ✓

Induction step: Take a shelling $F_1, \ldots, F_t$ of Δ. Set

$$\widetilde{\Delta} = \langle F_1, \ldots, F_{t-1} \rangle \qquad \text{and} \qquad \sigma = \widetilde{\Delta} \cap F_t$$

This yields for the barycentric subdivision:

$$\mathrm{sd}(\Delta) = \mathrm{sd}(\widetilde{\Delta}) \cup \mathrm{sd}(F_t) \qquad \text{and} \qquad \mathrm{sd}(\sigma) = \mathrm{sd}(\widetilde{\Delta}) \cap \mathrm{sd}(F_t)$$

We will use the following exact sequence:

$$0 \to k[\mathrm{sd}(\Delta)] \to k[\mathrm{sd}(\widetilde{\Delta})] \oplus k[\mathrm{sd}(F_t)] \to k[\mathrm{sd}(\sigma)] \to 0.$$

Choose $\Theta = \{\theta_1, \ldots, \theta_d\}$ l.s.o.p. for $k[\mathrm{sd}(\Delta)]$, $k[\mathrm{sd}(\widetilde{\Delta})]$, $k[\mathrm{sd}(F_t)]$ such that $\{\theta_1, \ldots, \theta_{d-1}\}$ is l.s.o.p. for $k[\sigma]$.

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Dividing out by Θ yields the following long exact sequence of Tor-modules:

$$\begin{aligned} \ldots \quad & \to \mathrm{Tor}_1(k[\mathrm{sd}(\Delta)], S/\Theta) \to \mathrm{Tor}_1(k[\mathrm{sd}(\widetilde{\Delta})] \oplus k[\mathrm{sd}(F_t)], S/\Theta) \\ \to \quad & \mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta) \xrightarrow{\delta} \mathrm{Tor}_0(k[\mathrm{sd}(\Delta)], S/\Theta) \\ \to \quad & \mathrm{Tor}_0(k[\mathrm{sd}(\widetilde{\Delta})] \oplus k[\mathrm{sd}(F_t)], S/\Theta) \to \mathrm{Tor}_0(k[\mathrm{sd}(\sigma)], S/\Theta) \to 0, \end{aligned}$$

where δ is the connecting homomorphism.

Use: $\mathrm{Tor}_0(M, R/I) = M \otimes_R R/I = M/IM$ for an R-module M and an ideal $I \subseteq R$.

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The exact Tor-sequence simplifies:

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Induction hypothesis: ω Lefschetz element for $k[\mathrm{sd}(\widetilde{\Delta})]$ and $k[\mathrm{sd}(F_t)]$.

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Remains to show: $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta)_i = 0$ for $0 \leq i \leq \lfloor \frac{d-2}{2} \rfloor$ (uses that θ_d can be used as a Lefschetz element for $k[\mathrm{sd}(\sigma)]$). $\square$

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GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Outline

Veronese algebras

$A = \bigoplus_{i \geq 0} A_i$ standard graded k-algebra

For $r \geq 1$: $A^{\langle r \rangle} = \bigoplus_{i \geq 0} A_{ir}$ r^{th} Veronese algebra of A

$A^{\langle r \rangle}$ is standard graded and $(A^{\langle r \rangle})_i = A_{ir}$

Example:

$A = k[x_1, x_2, x_3]$

$$\begin{aligned} A^{\langle 2 \rangle} = \langle 1 \rangle &\oplus \langle x_1x_2, x_1x_3, x_2x_3, x_1^2, x_2^2, x_3^2 \rangle \\ &\oplus \langle \text{ monomials of degree 4 } \rangle \\ &\oplus \langle \text{ monomials of degree 6 } \rangle \oplus \dots \end{aligned}$$

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- Brenti/Welker (2007):
  - give linear transformation $h(A) \overset{C^{r,d}}{\mapsto} h(A^{\langle r \rangle}) = C^{r,d} h(A)$, where $C^{r,d} \in \mathbb{Z}^{(d+1)\times(d+1)}$.
  - $h_{A^{\langle r \rangle}}(t)$ is real-rooted for r sufficiently large.

  $\Rightarrow$ $h(A^{\langle r \rangle})$ is log-concave and unimodal.

- Similar results concerning the asymptotics of $h_{A^{\langle r \rangle}}(t)$ were subsequently obtained by Beck/Stapledon and Diaconis/Fulman.
- K./Welker (2011): $h_i(A) \geq 0$, $r \geq \max(d, \deg h_A(t)) \Rightarrow g(A^{\langle r \rangle})$ is an M-sequence.

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- K./Welker (2011): $h_i(A) \geq 0$, $r \geq \max(d, \deg h_A(t)) \Rightarrow g(A^{\langle r\rangle})$ is an M-sequence.

Question: Is there a nicer proof?

Motivation (cont'd)

Conjecture

A Cohen-Macaulay standard graded k-algebra.

Then: For r sufficiently large $A^{\langle r \rangle}$ has a certain type of Lefschetz property.

$\Rightarrow g(A^{\langle r \rangle})$ *is an M-sequence.*

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Veronese algebras: Almost SLP

Theorem (K., Murai)

k field of characteristic 0,

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$\Theta = \theta_1, \ldots, \theta_d$ l.s.o.p. for A.

$\Theta^{\langle r \rangle} := \{\theta_1^r, \ldots, \theta_d^r\}$

Let $r \geq 1$ be an integer and $s = \lfloor \frac{(r-1)d}{r} \rfloor$.
Then $A^{\langle r \rangle}/\Theta^{\langle r \rangle} A^{\langle r \rangle}$ has the s-Lefschetz property.

Moreover, if $r \geq \deg h_A(t)$, then $A^{\langle r \rangle}/\Theta^{\langle r \rangle} A^{\langle r \rangle}$ is almost strong Lefschetz.

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(i) *If $r \geq \deg h_A(t)$, then $A^{\langle r \rangle}/\Theta^{\langle r \rangle} A^{\langle r \rangle}$ is quasi weak Lefschetz.*

(ii) *If d is even and $r \geq \max\{d, 2\deg h_A(t) - d\}$, then $A^{\langle r \rangle}/\Theta^{\langle r \rangle} A^{\langle r \rangle}$ has the weak Lefschetz property.*

(iii) *If d is odd, $r \geq \frac{d}{2}$ and $\deg h_A(t) \leq \frac{d}{2}$, then $A^{\langle r \rangle}/\Theta^{\langle r \rangle} A^{\langle r \rangle}$ has the weak Lefschetz property.*

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Sketch of the proof

$$\Theta = \{\theta_1, \ldots, \theta_d\}, \qquad \Theta^{\langle r \rangle} = \{\theta_1^r, \ldots, \theta_d^r\}$$

Step 1: Decomposition of A

A Cohen-Macaulay $\Rightarrow$ Θ regular sequence for A, i.e., A finitely generated $k[\theta_1, \ldots, \theta_d]$-module. Hence:

$$A = \bigoplus_{j=1}^{m} u_j \cdot k[\theta_1, \ldots, \theta_d]$$

for homogeneous elements $u_1, \ldots, u_m$ of A.

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$$A^{\langle r\rangle} = \bigoplus_{j=1}^{m} u_j \cdot \left(\bigoplus_{i\geq 0} k[\theta_1,\ldots,\theta_d]_{ir-\deg u_j} \right).$$

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Sketch of proof (cont'd)

Lemma (Stanley, Watanabe)

k field of characteristic 0, $r \geq 1$. For $0 \leq i < j$, the multiplication map

$$\begin{array}{ccc} (K[x_1,\ldots,x_d]/(x_1^r,\ldots,x_d^r))_i & \stackrel{\times(x_1+\cdots+x_d)^{j-i}}{\rightarrow} & (K[x_1,\ldots,x_d]/(x_1^r,\ldots,x_d^r))_j \\ p & \mapsto & (x_1+\cdots+x_d)^{j-i}\cdot p \end{array}$$

is injective if $i+j \leq (r-1)d$ and is surjective if $i+j \geq (r-1)d$.

Sketch of the proof (cont'd)

Step 4: Use this lemma to show that $\theta_1 + \cdots + \theta_d$ is a Lefschetz element for $A^{\langle r \rangle}/\Theta^{\langle r \rangle} A^{\langle r \rangle}$. □

Remark

- *The proof for the result on the (quasi) weak Lefschetz property follows a similar strategy.*
- *The condition $\deg h_A(t) \le \frac{d}{2}$ cannot be dropped in (iii).*

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r^{th} edgewise subdivisions

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- Basic Idea: Edges are sudivided into r pieces.
- gives a regular triangulation of a simplicial complex Δ
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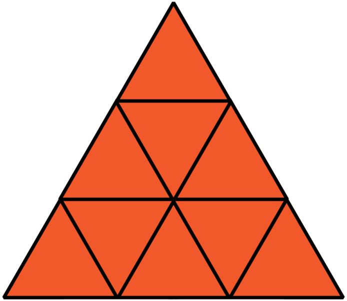
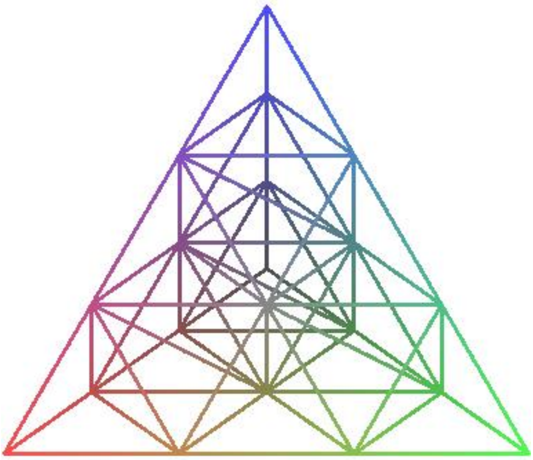


Example: The 3^{rd} edgewise subdivision of the 3-simplex



Edgewise subdivisions and Veronese algebras

Key fact:

Proposition (Brun, Römer)

Δ simplicial complex on ground set $[n]$, $r \geq 1$.

Set $S(r) := k[y_{i_1,\ldots,i_n} \mid (i_1,\ldots,i_n) \in \mathbb{N}^n$ and $i_1 + \cdots + i_n = r]$ and let $I(r)$ be such that $k[\Delta]^{\langle r \rangle} = S(r)/I(r)$.

Then there is a term order $\preceq$ for which $I_{\Delta(r)}$ is the initial ideal of $I(r)$.

Consequence:

$$\mathrm{Hilb}(k[\Delta(r)], t) = \mathrm{Hilb}(k[\Delta]^{\langle r \rangle}, t).$$

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Results for edgewise subdivisions

Being the Hilbert series equal, the numerical results for the Hilbert series of Veronese algebras carry over to edgewise subdivisions.

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Thank you for your attention!

Any questions or remarks?