Hilbert depth and numerical semigroups

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[Join work with Jan Uliczka]

(1) On Hilbert series

Let $\mathbb{F}$ be an arbitrary field. We consider the ring $R = \mathbb{F}[X_1, \ldots, X_n]$, equipped with a positive $\mathbb{Z}$-grading, i. e., $\deg(X_i) = e_i \geq 1$ for $i = 1, \ldots, n$.

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Furthermore let $M \neq 0$ be a f.g. $\mathbb{Z}$-graded R-module.

Every homogenous component of M is a finite-dimensional $\mathbb{F}$-vector space, and since R is positively graded, $M_k = 0$ for $k \ll 0$. Hence the

Hilbert series

$$H_M(t) = \sum_{k \in \mathbb{Z}} (\dim_{\mathbb{F}} M_k) t^k$$

is a well-defined element of $\mathbb{Z}[\![t]\!][t^{-1}]$.

No negative coefficients: nonnegative series.

By a classical result of Hilbert, in the case $e_i = 1$ for all i (*standard graded case*) this series may be written as a rational function of the form

$$H_M(t) = \frac{Q_M(t)}{(1-t)^d}$$

where $Q_M \in Z[t, t^{-1}]$ is a Laurent polynomial with $Q_M(1) \neq 0$, and d equals the Krull dimension of M.

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In the general case (*positively graded case*) the Hilbert-Serre theorem says

$$H_M(t) = \frac{Q_M(t)}{\prod(1-t^{e_i})}.$$

(2) On numerical semigroups

- Let S be a submonoid of $(\mathbb{N}_0, +)$.
- Let A be a nonempty subset of S.
- The *subsemigroup generated by A* is

$$\langle A \rangle := \{\lambda_1 a_1 + \ldots + \lambda_n a_n \mid n, \lambda_i \in \mathbb{N}, a_i \in A\}.$$

- If $S = \langle A \rangle$, then we say: A is a system of generators of S.
- A finite $\Longrightarrow$ S finitely generated.
- If you can't delete generators of $A \Longrightarrow S$ minimally generated.

Definition

A submonoid Γ of $\mathbb{N}_0$ such that $\mathbb{N}_0 \setminus \Gamma$ is a finite set is called a *numerical semigroup*.

The smallest element $c \in \Gamma$ such that $n \in \Gamma$ for all $n \in \mathbb{N}$ with $n \geq c$ is called the *conductor* of Γ.

Lemma

Let A be a nonempty subset of $\mathbb{N}_0$. Then $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

Example: The following semigroup is numerical:

$$\begin{aligned} \langle 5, 7 \rangle &= \{0, 5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24 \longrightarrow\} \\ &= \mathbb{N}_0 \setminus \{1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23\}. \end{aligned}$$

Let $\mathbb{F}$ be an arbitrary field. We consider the ring $R = \mathbb{F}[X_1, \ldots, X_n]$, equipped with a positive $\mathbb{Z}$-grading, i. e., $\deg(X_i) = e_i \geq 1$ for $i = 1, \ldots, n$.

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Furthermore let $M \neq 0$ be a f.g. $\mathbb{Z}$-graded R-module.

Every homogenous component M_k of M is a finite-dimensional $\mathbb{F}$-vector space, and take the Hilbert series

$$H_M(t) = \sum_{k \in \mathbb{Z}} (\dim_{\mathbb{F}} M_k) t^k \in \mathbb{Z}[\![t]\!][t^{-1}].$$

Obviously it has no negative coefficients: Remember that we will call such a series nonnegative.

The depth of a graded module cannot be read off its Hilbert series in general; there may be modules with the same Hilbert series, but different depths. So it makes sense to introduce the

Hilbert depth of M

$$\mathrm{Hdep}(M) := \max\left\{ r \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{There is a f. g. gr. } R\text{–module } N \\ \text{with } H_N = H_M \text{ and } \mathrm{depth}(N) = r. \end{array} \right\}.$$

A priori the Hilbert depth is an opaque quantity, since a vast amount of modules has to be taken into account.

In the standard graded case, i. e. $\deg(X_i) = 1$ for all i , the Hilbert depth turns out to be equal to a simple arithmetic invariant of H_M, the

Positivity of M

$$p(M) = \max\{r \in \mathbb{N} : (1-t)^r H_M(t) \text{ is nonnegative}\}.$$

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The inequality $Hdep(M) \le p(M)$ is easy: We may assume $\mathbb{F}$ to be infinite. Then a maximal M-regular sequence $\underline{a} = a_1, \dots, a_r$ can be composed of elements of degree 1. We have

$$H_{M/\underline{a}M}(t) = (1-t)^r H_M(t),$$

and, being a Hilbert series, this has to be nonnegative.

The reverse inequality $Hdep(M) \geq p(M)$ can be deduced from the existence of a decomposition (Uliczka's theorem)

$$H_M(t) = \sum_{j=0}^{\dim(M)} \frac{Q_j(t)}{(1-t)^j}$$

with nonnegative $Q_j \in \mathbb{Z}[t, t^{-1}]$.

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It seems natural to ask whether one could obtain a description of the Hilbert depth by an arithmetic condition also in the general case.

This talk will provide at least some partial answers.

Before that: Let us consider an example still in the standard grading.

Example

Let $R = \mathbb{F}[X, Y, Z]$ with $\deg(X) = \deg(Y) = \deg(Z) = 1$.
Let $M = R/(XZ, YZ, Z^2)$. Then

$$\mathrm{depth}(M) = 0 < \mathrm{Hdepth}(M) = 1 = p(M) < 2 = \dim(M).$$

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Let $M = R/(XZ, YZ, Z^2)$. Then

$$\mathrm{depth}(M) = 0 < \mathrm{Hdepth}(M) = 1 = p(M) < 2 = \dim(M).$$

Proof: Obviously we have $\mathrm{depth}(M) = 0$, and the computation

$$H_M = t + H_{\mathbb{F}[X,Y]}(t) = t + \frac{1}{(1-t)^2} = \frac{t^3 - 2t^2 + t + 1}{(1-t)^2},$$

shows $p(M) \leq 1 < \dim(M)$, since the numerator of H_M is not positive.

On the other hand $\mathrm{Hdep}(M) \geq 1$, because an R-module with $H_N = H_M$ and $\mathrm{depth}(N) = 1$ is easily constructed;

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we have just to choose

$$N := \Big((R/(Z))(-3)\Big) \oplus \Big((R/(Y,Z))(-1)\Big)^2 \oplus R/(Y,Z),$$

thus $\mathrm{Hdepth}(M) = p(M) = 1$.

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Therefore the notion of positivity has to be modified:

Let $e := \mathrm{lcm}(e_1, \ldots, e_n)$. We define the

e-positivity

$$p_e(M) = \max\{r \in \mathbb{N} : (1 - t^e)^r H_M(t) \text{ is nonnegative}\}$$

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Since an M-regular sequence can always be composed of elements of degree e, the inequality $\mathrm{Hdep(M)} \leq p_e(M)$ follows exactly as in the standard graded case.

The decomposition theorem used in the standard graded case can also be generalized:

Theorem (M.–Uliczka)

The Hilbert series of M admits a decomposition of the form

$$\sum_{I \subseteq \{1,\dots,n\}} \frac{Q_I(t)}{\prod_{j \in I}(1 - t^{e_j})}$$

with nonnegative $Q_I \in \mathbb{Z}$.

For instance: If M is a $\mathbb{F}[X_1, X_2]$–module, we reach

$$H_M(t) = Q_0(t) + \frac{Q_{X_1}(t)}{1 - t^{e_1}} + \frac{Q_{X_2}(t)}{1 - t^{e_2}} + \frac{Q_2(t)}{(1 - t^{e_1})(1 - t^{e_2})}.$$

Any decomposition of H_M in that form yields another R-module with the same Hilbert series: Let

$$H_M(t) = \sum_{I \subseteq \{1,\dots,n\}} \frac{\sum_{k=p_I}^{q_I} h_{k,I} t^k}{\prod_{j \in I}(1 - t^{e_j})}$$

and write J_I for the ideal generated by the X_i with $i \notin I$, then the R-module

$$N := \bigoplus_{I \subseteq \{1,\dots,n\}} \left(\bigoplus_{k=p_I}^{q_I} \Big((R/J_I)(-k) \Big)^{h_{I,k}} \right)$$

has Hilbert series H_M.

Since the module constructed above has

$$\operatorname{depth}(N) = \min\{|I| : Q_I \neq 0\},$$

the Hilbert depth of M is bounded below by

$$\nu(H) := \max\left\{ r \in \mathbb{N} \;\middle|\; \begin{array}{l} H \text{ admits a decomp. of the given form} \\ \text{with } Q_I = 0 \text{ for all } I \text{ with } |I| < r. \end{array} \right\}.$$

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$$\nu(M) \leq \operatorname{Hdep}(M) \leq p_e(M).$$

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It is not hard to show $p(M) \leq \nu(M)$ in the standard graded case. But the argument used there does not work for p_e, since the factors $1 - t^{e_i}$ appearing in the decomposition are different from the factor $1 - t^e$ that is used in the definition of $p(M)$.

For $R = \mathbb{F}[X, Y]$ we know

$$\nu(M) = \mathrm{Hdep}(M) = p_e(M)$$

Aim: Arithmetical characterization of $\nu(M) > 0$ for $\mathbb{F}[X, Y]$ with

$$\alpha := \deg(X),\ \beta := \deg(Y)$$

and

$$\gcd(\alpha, \beta) = 1.$$

This will provide a criterion for $\mathrm{Hdep}(M)$.

The condition

$$p_e(M) = p_{\alpha\beta}(M) > 0$$

is necessary but not sufficient.

If $H_M(t) = \sum_n h_n t^n$, this condition means

$$h_{n+0} \leq h_{n+\alpha\beta}$$

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What to do? $\longrightarrow$ Examples!

Take $\alpha = 3$ and $\beta = 5$

Necessary conditions:

$$h_{n+0} \leq h_{n+15}$$

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and also (after some computations):

$$\begin{aligned}
h_{n+0} + h_{n+1} &\leq h_{n+6} + h_{n+10}, \\
h_{n+0} + h_{n+2} &\leq h_{n+12} + h_{n+5}, \\
h_{n+0} + h_{n+4} &\leq h_{n+9} + h_{n+10}, \\
h_{n+0} + h_{n+7} &\leq h_{n+12} + h_{n+10}
\end{aligned}$$

But $1, 2, 4, 7$ are exactly the gaps of $\langle 3, 5 \rangle$!!!

	0	1	6	10
(3)	0	1	0	1
(5)	0	1	1	0

	0	4	9	10
(3)	0	1	0	1
(5)	0	4	4	0

	0	7	12	10
(3)	0	1	0	1
(5)	0	2	2	0

More computations convinced us that

$$\begin{aligned} h_{n+0} + h_{n+1} + h_{n+2} &\leq h_{n+4} + h_{n+5} + h_{n+6}, \\ h_{n+0} + h_{n+2} + h_{n+4} &\leq h_{n+5} + h_{n+7} + h_{n+9} \end{aligned}$$

are also sufficient!

	0	2	4	5	7	9
(3)	0	2	1	2	1	0
(5)	0	2	4	0	2	4

Look also that:

$$2 < 5 \quad \text{and} \quad 2 < 7$$

$$4 < 7 \quad \text{and} \quad 4 < 9$$

Let L be the set of gaps of $\langle \alpha, \beta \rangle$.

An (α, β)–*fundamental couple* $[I, J]$ consists of two integer sequences $I = (i_k)_{k=0}^m$ and $J = (j_k)_{k=0}^m$, such that

(0) $i_0 = 0$.

(1) $i_1, \dots, i_m, j_1, \dots, j_{m-1} \in L$ and $j_0, j_m \leq \alpha\beta$.

(2)

$$\begin{array}{llll} i_k \equiv j_k & \bmod \alpha \quad \text{and} & i_k < j_k & \text{for } k = 0, \dots, m; \\ j_k \equiv i_{k+1} & \bmod \beta \quad \text{and} & j_k > i_{k+1} & \text{for } k = 0, \dots, m-1; \\ j_m \equiv i_0 & \bmod \beta \quad \text{and} & j_m \geq i_0. & \end{array}$$

(3) $|i_k - i_\ell| \in L$ for $1 \leq k < \ell \leq m$.

The set of (α, β)–fundamental couples will be denoted by $\mathcal{F}_{\alpha,\beta}$.

The number of (α, β)–fundamental couples grows surprisingly with increasing α and β.

We give some examples:

$S = \langle \alpha, \beta \rangle$	$\lvert \mathcal{F}_{\alpha,\beta} \rvert$	genus
$\langle 4, 5 \rangle$	14	6
$\langle 4, 7 \rangle$	30	9
$\langle 6, 11 \rangle$	728	25
$\langle 11, 13 \rangle$	104 006	60

Main Theorem (M.–Uliczka)

Let $R = \mathbb{F}[X, Y]$ be the polynomial ring in two variables s.th.

$$\deg(X) = \alpha, \quad \deg(Y) = \beta, \quad \text{with} \quad \gcd(\alpha, \beta) = 1.$$

Let M be a finitely generated graded R–module. Then

$\mathrm{Hdep}(M) > 0$ if and only if $H_M(t) = \sum_n h_n t^n$ satisfies the condition

$$(\star) \qquad \sum_{i\in I} h_{i+n} \leq \sum_{j\in J} h_{j+n} \quad \text{for all} \quad n \in \mathbb{Z},\ [I, J] \in \mathcal{F}_{\alpha,\beta}.$$

In the special case $\langle 3, 5\rangle$ the criterion is given by the inequalities

$$\begin{aligned}
h_n &\leq h_{n+15}, \\
h_n + h_{n+1} &\leq h_{n+6} + h_{n+10}, \\
h_n + h_{n+2} &\leq h_{n+12} + h_{n+5}, \\
h_n + h_{n+4} &\leq h_{n+9} + h_{n+10}, \\
h_n + h_{n+7} &\leq h_{n+12} + h_{n+10}, \\
h_n + h_{n+1} + h_{n+2} &\leq h_{n+5} + h_{n+6} + h_{n+7}, \\
h_n + h_{n+2} + h_{n+4} &\leq h_{n+5} + h_{n+7} + h_{n+9}
\end{aligned}$$

About the proof:

- Necessity follows from simple arguments even not involving numerical semigroups.

  For any R-module with $\mathrm{Hdep}(M) > 0$ the Hilbert series can be written in the form

  $$H_M(t) = \frac{Q_2(t)}{(1-t^{\alpha})(1-t^{\beta})} + \frac{Q_X(t)}{1-t^{\alpha}} + \frac{Q_Y(t)}{1-t^{\beta}}$$

  with nonnegative $Q_2, Q_X, Q_Y \in \mathbb{Z}[t, t^{-1}]$.

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  with nonnegative $Q_2, Q_X, Q_Y \in \mathbb{Z}[t, t^{-1}]$.

  It is just to show that the constituting elements here satisfy the condition $(\star)$.

- Sufficiency needs a deep understanding of the numerical structure of the (α, β)-fundamental couples.

One shows that condition $(\star)$ is enough to ensure $\mathrm{Hdep}(M) > 0$.

Let M be R-module with its Hilbert series satisfying $(\star)$. By the decomposition theorem there are nonnegative $Q_i \in \mathbb{Z}[t, t^{-1}]$ such that

$$H_M(t) = Q_0(t) + \frac{Q_X(t)}{1 - t^\alpha} + \frac{Q_Y(t)}{1 - t^\beta} + \frac{Q_2(t)}{(1 - t^\alpha)(1 - t^\beta)}.$$

We have to show that it is possible to get rid of Q_0. This is done in two steps:

(i) First we reduce the problem to the one-dimensional case, i. e. we eliminate the Q_2 term.

We may thus assume $Q_2 = 0$. In this case the coefficients in $H_M = \sum_n h_n t^n$ get periodic for, say, $n \geq N$. That is, $h_n = h_{n+\alpha\beta}$ for all $n \geq N$.

Then the sum $h_n + h_{n+1} + \ldots + h_{n+\alpha\beta-1}$ has the same value, say $\sigma(H)$, for every $n \geq N$.

(ii) We apply then induction on $\sigma(H)$.