Combinatorial formulas for the Hilbert coefficients of Schubert varieties in Grassmannians

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Grassmannians

Let V be an n-dimensional vector space over an algebraically closed field K. Let $\{e_1, \ldots, e_n\}$ be the standard basis for V over K, where $e_i = (0, \ldots, 1, \ldots, 0)$, with 1 at the i^{th} place and remaining are zeros.
Fix the flag $\mathcal{F} : V_0 = (0) \subset V_1 \subset \cdots \subset V_n = V$, where $V_i =$ the subspace of V spanned by $e_1, \ldots, e_i$.

Let $G(r, n) =$ the set of all r-dimensional K-subspaces of V. Then $G(r, n)$ is a projective variety via Plücker embedding into $\mathbb{P}(\bigwedge^r V)$, called the r^{th} Grassmannian of V.

Define $I(r, n) := \{(a_1, \ldots, a_r) : 1 \leq a_1 < \cdots < a_r \leq n\}$. For $\theta = (\theta_1, \ldots, \theta_r) \in I(r, n)$, define
$X(\theta) := \{W \in G(r, n) : \dim_K W \cap V_{\theta_j} \geq j, \quad \textit{for } 1 \leq j \leq r\}$, then $X(\theta)$ is a projective variety called Schubert variety defined by θ.

Schubert variety in $G(r, n)$

Let $A(\theta)$ = the homogeneous coordinate ring of $X(\theta)$ and $\mathfrak{m}$ = the homogeneous maximal ideal of $A(\theta)$.

Let $d = \dim A(\theta)$. Note that $d = \theta_1 + \cdots + \theta_r - \binom{r+1}{2} + 1$, where $\theta = (\theta_1, \ldots, \theta_r)$.

- $A(\theta)$ is Cohen-Macaulay
- $A(\theta)$ is a graded ASL with respect to a poset $\Pi(\theta)$ over K

Definitions and Notation:

Let $H_{A(\theta)}(i) := \dim_K A(\theta)_i$, for all $i \in \mathbb{Z}_+$, is the Hilbert function of $A(\theta)$.
It is well known that for i sufficiently large, $H_{A(\theta)}(i)$ is a polynomial of degree equal to $d-1$. This polynomial is called the Hilbert polynomial of $A(\theta)$ and denoted by $P_{A(\theta)}(z)$.

Write $P_{A(\theta)}(z) =$
$e_0(A(\theta))\binom{z+d-1}{d-1} - e_1(A(\theta))\binom{z+d-2}{d-2} + \cdots + (-1)^{d-1}e_{d-1}(A(\theta))$,
where $e_i(A(\theta)) \in \mathbb{Z}$ is called the i^{th} Hilbert coefficient of $A(\theta)$.
The zeroth Hilbert coefficient is called the multiplicity and the first Hilbert coefficient is called the Chern coefficient.

The Hilbert-Poincare series of $A(\theta)$ is denoted and defined as $HS_{A(\theta)}(z) := \sum_{i\in\mathbb{Z}_+} H_{A(\theta)}(i)z^i$, the generating function of $H_{A(\theta)}(i)$.

It is well known that $HS_{A(\theta)}(z)$ is a rational function and one can write $HS_{A(\theta)}(z) = h_{A(\theta)}(z)/(1-z)^d$, for a unique polynomial $h_{A(\theta)}(z) \in \mathbb{Z}[z]$ with $h_{A(\theta)}(1) \neq 0$, called the h-polynomial of $A(\theta)$. The postulation number of $A(\theta)$ is defined as

$$\min\{i \in \mathbb{Z}_+ : H_{A(\theta)}(n) = P_{A(\theta)}(n) \text{ for all } n \geq i\}.$$

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Many authors give various combinatorial formulas for the multiplicity of $A(\theta)$, cf. [Herzog-Trung, 1994], [Conca, 1994] and [Raghavan-Simis, 1995], etc...

Questions

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Question 2: Can we have combinatorial formulas for the h-polynomial of $A(\theta)$?

Question 3: Can we have a combinatorial formula for the Cohen-Macaulay type of $A(\theta)$?

We answer all the above questions.

we explicitly give closed form formulas for

1. $e_i(A(\theta))$, the i^{th} Hilbert coefficient,
2. the h-polynomial $h_{A(\theta)}(z)$ and
3. the Cohen-Macaulay type, $\tau(A(\theta))$, of $A(\theta)$

in terms of the poset $\Pi(\theta)$ associated to $A(\theta)$.

Combinatorial formulas for the Hilbert coefficients of $A(\theta)$

Let $v_j := \lambda(\frac{\mathfrak{m}_{\mathfrak{m}}^{j+1}}{J_{\mathfrak{m}}\mathfrak{m}_{\mathfrak{m}}^{j}})$, where J is a minimal reduction of $\mathfrak{m}$ in $A(\theta)$ generated by homogeneous elements of degree 1. Note that $\lambda(\frac{\mathfrak{m}^{j+1}}{J\mathfrak{m}^{j}}) = v_j$ for all $j \geq 0$.

Let $\mathcal{R} := A(\theta)[\mathfrak{m}t] = \oplus_{i\geq 0}\mathfrak{m}^i t^i \subseteq A(\theta)[t]$ be the Rees algebra of $\mathfrak{m}$ and $G(A(\theta)_{\mathfrak{m}}) := \oplus_{i\geq 0}\mathfrak{m}^i A(\theta)_{\mathfrak{m}}/\mathfrak{m}^{i+1}A(\theta)_{\mathfrak{m}} t^i$ be the associated graded ring of $\mathfrak{m}A(\theta)_{\mathfrak{m}}$ in the local ring $A(\theta)_{\mathfrak{m}}$. For simplicity denote $G(A(\theta)_{\mathfrak{m}})$ by G.
Note that $\dim \mathcal{R} = d + 1 = \dim \mathcal{R}[A(\theta)_{\mathfrak{m}}]$ and $\dim G = d$.

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Note that $\dim \mathcal{R} = d+1 = \dim \mathcal{R}[A(\theta)_{\mathfrak{m}}]$ and $\dim G = d$.

Theorem (Hodge-Pedoe, Theorem III, page 387)

Let $\theta = (\theta_1, \ldots, \theta_r) \in I(r,n)$ and $X(\theta)$ be the Schubert variety corresponding to θ. Then $\dim_K [A(\theta)]_j = \omega_{\theta_1,\ldots,\theta_r}(j)$ for all $j \in \mathbb{Z}_+$, where $\omega_{\theta_1,\ldots,\theta_r}(j)$ is the postulation of $X(\theta)$ given by the determinant,

Combinatorial formulas for the Hilbert coefficients of $A(\theta)$

$$\omega_{\theta_1,\ldots,\theta_r}(j) = \begin{vmatrix} \binom{\theta_r+j-1}{j} & \binom{\theta_r+j-2}{j-1} & \cdots & \binom{\theta_r+j-r}{j-r+1} \\ \binom{\theta_{r-1}+j}{j+1} & \cdot & \cdots & \binom{\theta_{r-1}+j-r+1}{j-r+2} \\ \cdot & \cdot & \cdots & \cdot \\ \binom{\theta_1+j+r-2}{j+r-1} & \cdot & \cdots & \binom{\theta_1+j-1}{j} \end{vmatrix}.$$

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Theorem

The following are true for $A(\theta)$.

(i) *$a_d(A(\theta)) < 0$, where $a_d(A(\theta))$ is the a-invariant of the d^{th} local cohomology, $H^d_{\mathfrak{m}}(A(\theta))$, of $A(\theta)$ with respect to $\mathfrak{m}$.*

(ii) *$\mathcal{R}$ and $\mathcal{R}[\mathfrak{m}A(\theta)_{\mathfrak{m}}]$ are Cohen-Macaulay.*

Recall a theorem of Bruns and Herzog which gives a combinatorial formula for the *a*-invariant of a graded ASL:

Combinatorial formulas for the Hilbert coefficients of $A(\theta)$

Theorem (Bruns-Herzog, 1992)

Let R be a monotonically graded ASL over a field K on an upper semimodular lattice Π with principal chain $\mathcal{P}(\Pi) = \xi_1, \ldots, \xi_m$. Then $a(R) = -\sum_{j=1}^{m} \deg \xi_j$.

Since $A(\theta)$ satisfies the graded ASL property with some lattice say $\Pi(\theta)$, we have:

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Since $A(\theta)$ satisfies the graded ASL property with some lattice say $\Pi(\theta)$, we have:

Corollary

$\operatorname{reg} A(\theta) = a_d(A(\theta)) + d = d - \sum_{j=1}^{m} \deg \xi_j$, *where $\xi_1, \ldots, \xi_m$ is a principal chain in $\Pi(\theta)$ and $\operatorname{reg} A(\theta)$ is the Castelnuovo-Mumford regularity of $A(\theta)$.*

Combinatorial formulas for the Hilbert coefficients of $A(\theta)$

Remark

For an one dimensional Cohen-Macaulay graded ring R, the following holds:

1. $v_j = e_0(R) - H_R(j)$ *for all* $j \geq 0$.
2. $h_i = H_R(i) - H_R(i-1)$ *where* $(h_0, \dots, h_s)$, *is the* h*-polynomial of* R.
3. $h_i = v_{i-1} - v_i$, *for all* $i \geq 1$.

Combinatorial formulas for the Hilbert coefficients of $A(\theta)$

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Theorem

(i) *For all* $j \geq 0$,

$$v_j = e_0(A(\theta)) - \sum_{i=0}^{\min\{j,d-1\}} (-1)^i \omega_{\theta_1,\ldots,\theta_r}(j-i)\binom{d-1}{i};$$

(ii) *the* i^{th} *Hilbert coefficient of* $A(\theta)$ *is given by*

Combinatorial formulas for the Hilbert coefficients of $A(\theta)$

$$\begin{aligned} e_i(A(\theta)) &= e_0(A(\theta)) \sum_{i \le j \le d - \sum_{l=1}^{m} \deg \xi_l} \binom{j-1}{i-1} \\ &- \sum_{i \le j \le d - \sum_{l=1}^{m} \deg \xi_l} \sum_{s=0}^{j-1} (-1)^s \\ &\omega_{\theta_1,\ldots,\theta_r}(j-1-s) \binom{d-1}{s} \binom{j-1}{i-1} \end{aligned}$$

for $i \ge 1$.

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for $i \ge 1$.

Corollary

$$e_0(A(\theta)) = \sum_{i=0}^{d - \sum_{l=1}^{m} \deg \xi_l} (-1)^i \omega_{\theta_1,\ldots,\theta_r}(d - \sum_{l=1}^{m} \deg \xi_l - i) \binom{d-1}{i}.$$

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Corollary

$e_i(A(\theta)) > 0$ for all $i \le d - \sum_{l=1}^{m} \deg \xi_l$ and $e_i(A(\theta)) = 0$ for all $d - \sum_{l=1}^{m} \deg \xi_l + 1 \le i \le d-1$.

Combinatorial formula for the h-polynomial of $A(\theta)$

Let $h_{A(\theta)}(z) = h_0 + h_1 z + \cdots + h_s z^s$ be the h-polynomial of $A(\theta)$, where $h_i \in \mathbb{Z}$ and since $A(\theta)$ is Cohen-Macaulay, we have $s = r(\mathfrak{m}) = d - \sum_{l=1}^{m} \deg \xi_l$. The following proposition explicitly gives the h-polynomial of $A(\theta)$:

Proposition

Let $h_{A(\theta)}(z) = h_0 + h_1 z + \cdots + h_s z^s$ be the h-polynomial of $A(\theta)$, where $h_i \in \mathbb{Z}_+$. Then

$$h_i = \sum_{l=0}^{i-1} (-1)^l \binom{d-1}{l} \left[\omega_{\theta_1,\ldots,\theta_r}(i-l) - \omega_{\theta_1,\ldots,\theta_r}(i-1-l)\right] + (-1)^i \binom{d-1}{i}$$

for all $0 \le i \le s$.

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Corollary

For $d \geq 2$, the postulation number of $A(\theta)$ is equal to

$$\begin{cases} 1 & \text{if } d^{'} = d-1 \\ 0 & \text{otherwise.} \end{cases}$$

A criterion for $X(\theta)$ to be a c.i

Now we give necessary and sufficient conditions for $X(\theta)$ to be a complete intersection.

Theorem

A Schubert variety $X(\theta)$ is a complete intersection $\iff$ there exist $t \in \mathbb{Z}_+$ and $g_1, \ldots, g_t$ positive integers such that $g_1 + \cdots + g_t = d - \sum_{l=1}^{m} \deg \xi_l$ satisfying

$$\sum_{l=0}^{i-1} (-1)^l \binom{d-1}{l} \left[\omega_{\theta_1,\ldots,\theta_r}(i-l) - \omega_{\theta_1,\ldots,\theta_r}(i-1-l)\right] + (-1)^i \binom{d-1}{i} = \binom{t+i-1}{t-1} - \sum_{j=1}^{t} \binom{i+t-g_j-2}{t-1}$$

for all $1 \leq i \leq d - \sum_{l=1}^{m} \deg \xi_l$.

Combinatorial formulas for the Hilbert coefficients of $A(\theta)$ in terms of $\Pi(\theta)$

Let $\mathcal{M} = \{C_1, \cdots, C_t\}$ be the set of all maximal chains in $\Pi(\theta)$ and $|C_i| = |C_j|$ for all $i \neq j$. For any $s \leq t$,

$$M_s(i) = \{(i_1, \ldots, i_s) : \ 1 \leq i_1 < \cdots < i_s \leq t, |C_{i_1} \cap \cdots \cap C_{i_s}| = |C_1| - i\}.$$

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Theorem

Let $A(\theta)$ be the homogeneous coordinate ring of the Schubert variety $X(\theta)$ in the r^{th} Grassmannian $G(r,n)$ defined by $\theta \in I(r,n)$. Then the Hilbert coefficients of $A(\theta)$ are given by the formulas,

$$e_i(A(\theta)) = \sum_{j=2}^{t} (-1)^j |M_j(i)|,$$

for all i, $0 \leq i \leq d-1$, where

$$M_j(i) = \left\{(i_1, \ldots, i_j) : \ 1 \leq i_1 < \cdots < i_j \leq t, |C_{i_1} \cap \cdots \cap C_{i_j}| = |C_1| - i\right\}.$$

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Corollary

$e_0(A(\theta)) =$ *the number of maximal chains in* $\Pi(\theta)$*.*

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Let $d' = d - \sum_{l=1}^{m} \deg \xi_l$. Then $v_{d'-1} = e_{d'}$ and $v_{i-1} = e_i - \sum_{j=i+1}^{d'} \binom{j-1}{i-1} v_{j-1}$ for all i, $1 \leq i \leq d' - 1$.

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Mitsuhiro Miyazaki proved that Schubert varieties in the Grassmannians are level. Using this result we derive a formula for the Cohen-Macaulay type of $A(\theta)$ and a simple combinatorial criterion for the Gorensteinness of $X(\theta)$:

A criterion for $X(\theta)$ to be arithmetically Gorenstein

Corollary

Let $d^{'} = d - \sum_{l=1}^{m} \deg \xi_l$. *Then*

1. $\tau(A(\theta)) = \sum_{i=2}^{t} (-1)^i |M_i(d^{'})|$, *where* $t =$ *the number of maximal chains in* $\Pi(\theta)$;
2. $A(\theta)$ *is Gorenstein* $\iff \sum_{i=2}^{t} (-1)^i |M_i(d^{'})| = 1$.

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Corollary

Let $d^{'} = d - \sum_{l=1}^{m} \deg \xi_l$. *Then*

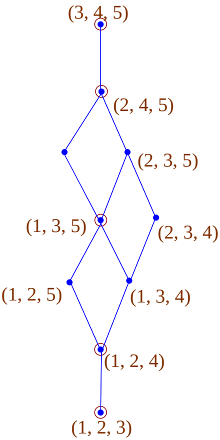
1. $\tau(A(\theta)) = \sum_{i=2}^{t}(-1)^i |M_i(d^{'})|$, *where* $t =$ *the number of maximal chains in* $\Pi(\theta)$*;*
2. $A(\theta)$ *is* *Gorenstein* $\Longleftrightarrow \sum_{i=2}^{t}(-1)^i |M_i(d^{'})| = 1$.

Corollary

$A(\theta)$ *is* *almost Gorenstein* $\Longleftrightarrow \sum_{i=2}^{t}(-1)^i [|M_i(d^{'})| - |M_i(1)|] \geq 0$, *where* $t =$ *the number of maximal chains in* $\Pi(\theta)$.

Example

Let $X(\theta) = G(3,5)$, where $\theta = (3,4,5) \in I(3,5)$. $d = 7$.



The principal chain in $I(3,5)$ is:
$(1,2,3) \leq (1,2,4) \leq (1,3,5) \leq (2,4,5) \leq (3,4,5)$.
Therefore $d^{'} = 7 - 5 = 2$.

Example

$I(3,5)$ has 4 maximal chains:
$C_1 : (1,2,3) \leq (1,2,4) \leq (1,2,5) \leq (1,3,5) \leq (1,4,5) \leq (2,4,5) \leq (3,4,5)$

$C_2 : (1,2,3) \leq (1,2,4) \leq (1,3,4) \leq (1,3,5) \leq (1,4,5) \leq (2,4,5) \leq (3,4,5)$

$C_3 : (1,2,3) \leq (1,2,4) \leq (1,3,4) \leq (1,3,5) \leq (2,3,5) \leq (2,4,5) \leq (3,4,5)$

$C_4 : (1,2,3) \leq (1,2,4) \leq (1,3,4) \leq (2,3,4) \leq (2,3,5) \leq (2,4,5) \leq (3,4,5)$.

Therefore $|M_2(1)| = 2, |M_2(2)| = 3, |M_2(3)| = 1, |M_2(4)| = |M_2(5)| = \cdots = 0$.

Example

$|M_3(1)| = 0, |M_3(2)| = 1, |M_3(3)| = 2, |M_3(4)| = |M_3(5)| = \cdots = 0.$

Now $e_0 =$ the number of maximal chains $= 4$

$e_1 = |M_2(1)| - |M_3(1)| + |M_4(1)| = 2$

$e_2 = |M_2(2)| - |M_3(2)| + |M_4(2)| = 2$

$e_3 = e_4 = \cdots = 0.$

The Cohen-Macaulay type of $A(\theta) = |M_2(2)| - |M_3(2)| + |M_4(2)| = 2$. Therefore $G(3,5)$ is not Gorenstein.

Now consider $\sum_{i=2}^{4}(-1)^i[|M_i(2)| - |M_i(1)|] = 0$. Therefore by one of above Corollary, we have $G(3,5)$ is almost Gorenstein.

Combinatorial formulas for the Hilbert coefficients of points on $X(\theta)$

Let $\mathcal{A} := \{p \in X(\theta) : \text{depth } G(\mathcal{O}_{X(\theta),p}) \geq \dim G(\mathcal{O}_{X(\theta),p}) - 1\}$. For simplicity let $B = \mathcal{O}_{X(\theta),p}$ and $\mathfrak{n}_p =$ the unique maximal ideal of $\mathcal{O}_{X(\theta),p}$. Note that B is Cohen-Macaulay.

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For simplicity let $B = \mathcal{O}_{X(\theta),p}$ and $\mathfrak{n}_p =$ the unique maximal ideal of $\mathcal{O}_{X(\theta),p}$. Note that B is Cohen-Macaulay.
Let $w_j := \lambda\left(\frac{\mathfrak{n}_p^{j+1}}{J^{'}\mathfrak{n}_p^{j}}\right)$, for all $j \geq 0$, where $J^{'}$ is a minimal reduction of $\mathfrak{n}_p$ in B generated by homogeneous elements of degree 1. Let $r(\mathfrak{n}_p)$ denote the reduction number of $\mathfrak{n}_p$.

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Let $w_j := \lambda\left(\frac{\mathfrak{n}_p^{j+1}}{J'\mathfrak{n}_p^j}\right)$, for all $j \geq 0$, where J' is a minimal reduction of $\mathfrak{n}_p$ in B generated by homogeneous elements of degree 1. Let $r(\mathfrak{n}_p)$ denote the reduction number of $\mathfrak{n}_p$.

Theorem (Raghavan-Kodiyalam)

Let $\theta, v \in I(r,n)$ and $X(\theta)$ be the Schubert variety corresponding to θ and e^v be the T-fixed point in $X(\theta)$ corresponding to v. Then $\dim_K \left[G(\mathcal{O}_{X(\theta),e^v})\right]_j = S_\theta^v(j)$ for all $j \in \mathbb{Z}_+$.

Combinatorial formulas for the Hilbert coefficients of points on $X(\theta)$

Theorem

For a T-fixed point $p = e^{v} \in \mathcal{A}$, where $v \in I(r,n)$, the following hold.

(i) *For all $j \geq 0$,* $w_j = e_0(B) - \sum_{i=0}^{\min\{j,d-2\}} (-1)^i S_\theta^v(j-i)\binom{d-2}{i}$;

(ii) *the i^{th} Hilbert coefficient of B is given by*

$$\begin{aligned} e_i(B) &= e_0(B) \sum_{i \leq j \leq r(\mathfrak{n}_p)} \binom{j-1}{i-1} \\ &\quad - \sum_{i \leq j \leq r(\mathfrak{n}_p)} \sum_{s=0}^{\min\{j-1,d-2\}} (-1)^s \\ &\quad S_\theta^v(j-1-s)\binom{d-2}{s}\binom{j-1}{i-1} \end{aligned}$$

Combinatorial formula for the h-polynomial of points on $X(\theta)$

Let $H_{\mathcal{O}_{X(\theta),p}}(z) = H_0 + H_1 z + \cdots + H_s z^s$ be the h-polynomial of $\mathcal{O}_{X(\theta),e^v}$, where $H_i \in \mathbb{Z}$. Then the following proposition gives combinatorial expression for H_i:

Proposition

Let $p = e^v \in \mathcal{A}$ be a T-fixed point corresponding to $v \in I(r,n)$ and $H_{\mathcal{O}_{X(\theta),p}}(z) = H_0 + H_1 z + \cdots + H_s z^s$ be the h-polynomial of $\mathcal{O}_{X(\theta),e^v}$, where $H_i \in \mathbb{Z}$. Then for $i \leq d-2$,

$$H_i = \sum_{l=0}^{i-1} (-1)^l \binom{d-2}{l} \left[S_\theta^v(i-l) - S_\theta^v(i-1-l)\right] + (-1)^i \binom{d-2}{i}$$

and for $i \geq d-1$,

$$H_i = \sum_{l=0}^{d-2} (-1)^l \binom{d-2}{l} \left[S_\theta^v(i-l) - S_\theta^v(i-1-l)\right].$$

Combinatorial Criterion for the tangent cone to $X(\theta)$ at p to be a c.i

Corollary

For $p = e^v \in \mathcal{A}$ be a T-fixed point corresponding to $v \in I(r,n)$. Then the tangent cone of $X(\theta)$ at p is a complete intersection if and only if there exist $t \in \mathbb{Z}_+$ and $g_1, \ldots, g_t$ positive integers such that $g_1 + \cdots + g_t = r(\mathfrak{n}_p)$ satisfying

$$\sum_{l=0}^{i-1}(-1)^l \binom{d-2}{l}\left[S_\theta^v(i-l) - S_\theta^v(i-1-l)\right] + (-1)^i \binom{d-2}{i} = \binom{t+i-1}{t-1} - \sum_{j=1}^{t}\binom{i+t-g_j-2}{t-1}$$

for all $1 \leq i \leq d-2$ and

Combinatorial Criterion for the tangent cone to $X(\theta)$ at p to be a c.i

$$\sum_{l=0}^{d-2}(-1)^l\binom{d-2}{l}\left[S_\theta^v(i-l)-S_\theta^v(i-1-l)\right]$$
$$=\binom{t+i-1}{t-1}-\sum_{j=1}^{t}\binom{i+t-g_j-2}{t-1}$$

for all $i \geq d-1$.

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THANK YOU