Construction of minimal free resolutions of some monomial ideals by algebraic discrete Morse theory

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Outline

1. Introduction
2. Free resolutions of $\mathbb{Z}^n$-graded modules
   - Positively $\mathbf{t}$-determined modules
   - Not necessarily minimal free resolution for $M \in \operatorname{mod}_{\mathbf{t}} S$
3. Algebraic discrete Morse theory
   - Minimal free resolution of monomial ideals with linear quotients
4. Advertising

Introduction

Throughout this talk,

- let $S := \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring over a field $\Bbbk$
- All the $\mathbb{Z}^n$-graded S-modules are assumed to be finitely generated.
- A minimal $\mathbb{Z}^n$-graded S-free resolutions is referred to just as a minimal free resolution, and
- a not necessarily minimal $\mathbb{Z}^n$-graded S-free resolutions just as a free resolution.

Introduction

Short history about graded free resolutions of $\mathbb{Z}^n$-graded S-modules

- (D. Taylor 1960) Free resolutions of monomial ideals.
- (S. Eliahou and M. Kervaire 1990) Minimal free resolutions of stable monomial ideals.
- (J. Herzog and Y. Takayama 2002) Minimal graded free resolutions of monomial ideals with linear quotients and regular decomposition function.
- (A. B. Tchernev 2007) Free resolutions of $\mathbb{Z}^n$-graded modules.

Introduction

Purpose of today's talk

- Introduction of (possibly) "new" free resolutions of $\mathbb{Z}^n$-graded modules. (The connection with Tchernev's is not known).
- Making use of the resolution above, construct a minimal free resolution of any monomial ideal with linear quotients (that does not necessarily have a regular decomposition function).
- The key tool is Algebraic Discrete Morse Theory due to M. Jöllenbeck and V. Welker, and independently E. Sköldberg.
- This is essentially a survey on E. Sköldberg's result on minimal free reslutions of modules with initially linear syzygies (generalization of monomial ideals with linear quotients).

Free resolutions of $\mathbb{Z}^n$-graded modules

Positively $\mathbf{t}$-determined modules

In the sequel,

- bold alphabet $\mathbf{a}, \mathbf{b}, \dots$ denotes elements of $\mathbb{Z}^n$ and $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{b} = (b_1, \dots, b_n)$, and so on.
- $\mathbf{1} := (1, \dots, 1)$, $\mathbf{0} := (0, \dots, 0)$, and let $\mathbf{e}_i$ be the i-th unit vector.
- for $\mathbf{a} \in \mathbb{Z}^n$, set $x^{\mathbf{a}} := \prod_{i=1}^n x_i^{a_i}$.
- $\mathbb{Z}^n$ is regarded as the poset with the order $<$ given as

$$\mathbf{a} \leq \mathbf{b} \iff a_i \leq b_i \quad \forall i,$$

- For $\mathbf{a} \in \mathbb{Z}^n$, set $\mathrm{supp}(x^{\mathbf{a}}) := \mathrm{supp}(\mathbf{a}) := \{i \in [n] = \{1, \dots, n\} \mid a_i > 0\}$.
- Fix a vector $\mathbf{t} \in \mathbb{Z}^n$ with $\mathbf{t} \geq \mathbf{1}$.

Free resolutions of $\mathbb{Z}^n$-graded modules

Positively $\mathbf{t}$-determined modules

Definition (E. Miller)

An $\mathbb{Z}^n$-graded module M is said to be positively $\mathbf{t}$-determined if $M_{\mathbf{a}} = 0$ for $\mathbf{a} \not\geq \mathbf{0}$, and if

$$M_{\mathbf{a}} \xrightarrow{x_i} M_{\mathbf{a}+\mathbf{e}_i}$$

is isomorphic for all i with $a_i = t_i$.

- $\operatorname{mod}_{\mathbf{t}} S :=$ the category consisting of pos. $\mathbf{t}$-det. S-modules and degree-preserving S-homomorphisms.
- Typical examples are monomial ideals which are generated by monomials $x^{\mathbf{a}}$ with $\mathbf{0} \leq \mathbf{a} \leq \mathbf{t}$.

Free resolutions of $\mathbb{Z}^n$-graded modules

Positively $\mathbf{t}$-determined modules

Lemma (Basic facts)

The following are well-known.

- The category $\mathrm{mod}_{\mathbf{t}}\, S$ is abelian. In particular, it is closed under taking kernel and cokernel.
- A $\mathbb{Z}^n$-graded module M is in $\mathrm{mod}_{\mathbf{t}}\, S$ if and only if it has a presentation

$$\bigoplus_{\mathbf{0}\leq\mathbf{a}\leq\mathbf{t}} S(-\mathbf{a})^{\beta_{1,\mathbf{a}}} \longrightarrow \bigoplus_{\mathbf{0}\leq\mathbf{a}\leq\mathbf{t}} S(-\mathbf{a})^{\beta_{0,\mathbf{a}}} \longrightarrow M \longrightarrow 0.$$

- Hence for any $\mathbb{Z}^n$-graded module M, $\exists \mathbf{t} \geq \mathbf{1}$ and $\mathbf{a} \gg \mathbf{0}$ such that $M(-\mathbf{a}) \in \mathrm{mod}_{\mathbf{t}}\, S$.

Free resolutions of $\mathbb{Z}^n$-graded modules

Not necessarily minimal free resolution for $M \in \operatorname{mod}_{\mathbf{t}} S$

We fix $M \in \operatorname{mod}_{\mathbf{t}} S$. For $F \subseteq [n]$ and $\mathbf{a} \in \mathbb{Z}^n$, set

$$\mathbf{e}_F := \sum_{i \in F} \mathbf{e}_i, \quad M_{\leq \mathbf{a}} := \bigoplus_{\mathbf{b} \leq \mathbf{a}} M_{\mathbf{b}}.$$

For i with $i \geq 0$,

$$K_i^M := \bigoplus_{F \subseteq [n],\ \#F = i} S \otimes_{\mathbb{k}} \mathbb{k} \cdot x_F \otimes_{\mathbb{k}} M_{\mathbf{t} - \mathbf{e}_F}, \text{ where } x_F := x^{\mathbf{e}_F},$$

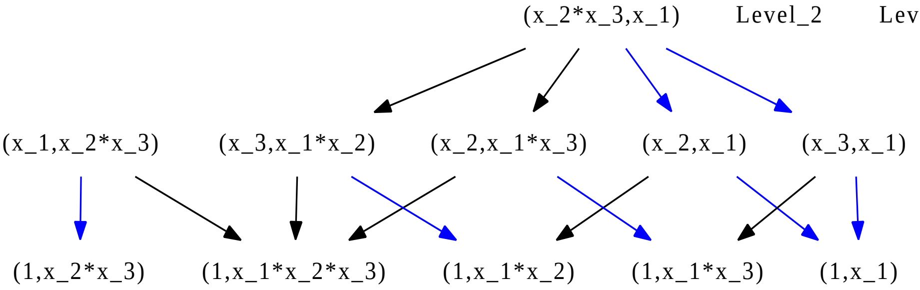
$$\begin{aligned} d_i^M : K_i^M \ni x \otimes x_F \otimes y \mapsto & \sum_{i \in F} (-1)^{\varepsilon(i;F)} (x x_i) \otimes x_{F \setminus \{i\}} \otimes y \\ & - \sum_{i \in F} (-1)^{\varepsilon(i;F)} x \otimes x_{F \setminus \{i\}} \otimes (x_i y) \in K_{i-1}^M, \end{aligned}$$

where $\varepsilon(i;F) := \#\{j \in F \mid j < i\}$.

Free resolutions of $\mathbb{Z}^n$-graded modules

Not necessarily minimal free resolution for $M \in \mathrm{mod}_{\mathbf{t}}\, S$

$I = (x_1, x_2x_3)$, $\mathbf{t} = \{1, 1, 1\}$.



Free resolutions of $\mathbb{Z}^n$-graded modules

Not necessarily minimal free resolution for $M \in \operatorname{mod}_{\mathbf{t}} S$

Theorem

The sequence $(K_\bullet^M, d_\bullet^M)$ is a complex and moreover a free resolution of M.

Sketch of Proof

Set $M_{\geq i} := S \cdot \bigoplus_{\mathbf{0} \leq \mathbf{a} \leq \mathbf{t},\ |\mathbf{a}| \geq i} M_{\mathbf{a}} \in \operatorname{mod}_{\mathbf{t}} S$, where $|\mathbf{a}| = \sum_{i=1}^n a_i$, and consider

$$0 \subseteq M_{\geq |\mathbf{t}|} \subseteq M_{\geq |\mathbf{t}|-1} \subseteq \cdots \subseteq M_0 = M.$$

The exact sequence $0 \to M_{\geq i} \to M_{\geq i-1} \to M_{\geq i-1}/M_{\geq i} \to 0$ induces

$$0 \longrightarrow K_\bullet^{M_{\geq i}} \longrightarrow K_\bullet^{M_{\geq i-1}} \longrightarrow K_\bullet^{M_{\geq i-1}/M_{\geq i}} \longrightarrow 0.$$

Free resolutions of $\mathbb{Z}^n$-graded modules

Not necessarily minimal free resolution for $M \in \operatorname{mod}_{\mathbf{t}} S$

Each $M_{\geq i}/M_{\geq i-1}$ and $M_{|\mathbf{t}|}$ are a direct sum of finitely many copies of $X := S/(x_{i_1}, \dots, x_{i_k})(-\mathbf{a})$ for some $\mathbf{a}$ and $i_1, \dots, i_k$.

For such X, the complex K^X is just a Koszul complex, shifted by $\mathbf{a}$, of $x_{i_1}, \dots, x_{i_k}$, and hence it is a minimal free resolution.

Thus we can use induction.

$\square$

- $K_\bullet^M$ is not minimal in most cases.
- To get a minimal free resolution, we must remove extra free summands from $K_\bullet^M$.
- Algebraic discrete Morse theory is an effective one for such reduction.

Algebraic discrete Morse theory

Henceforth,

- we fix $M =$ a monomial ideal I.
- Each free basis of $K_\bullet := K^I_\bullet$ is labeled by $(x_F, x^{\mathbf{a}})$ with $F \subseteq [n]$, $\mathbf{a} \leq \mathbf{t} - \mathbf{e}_F$, and $I_{\mathbf{a}} \neq 0$. (Thus $K^I_i = \bigoplus S \cdot e(x_F, x^{\mathbf{a}})$).
- $X :=$ all such labels, and $X_i := \{(x_F, x^{\mathbf{a}}) \in X \mid \#F = i\}$
- For $\alpha \in X_i$ and $\beta \in X_{i-1}$, set

$$d_{\beta,\alpha} : S \cdot e(\alpha) \hookrightarrow K_i \longrightarrow K_{i-1} \twoheadrightarrow S \cdot e(\beta).$$

- Let $G = (V, E)$ be the directed graph such that
  - vertices $= X$
  - edges $= \{\alpha \longrightarrow \beta \mid d_{\beta,\alpha} \neq 0\}$.

Algebraic discrete Morse theory

For a subset $A \subseteq E$, let G^A be the directed graph such that

- vertices $= X$.
- edges $= (E \setminus A) \cup \{\beta \longrightarrow \alpha \mid \alpha \longrightarrow \beta \in A\}$

Definition

A subset $A \subseteq E$ is said to be acyclic matching if

- $d_{\beta,\alpha}$ is isomorphic for all $\alpha \longrightarrow \beta \in A$,
- (matching) $\nexists$

$$\longrightarrow \bullet \longrightarrow, \quad \longrightarrow \bullet \longleftarrow, \quad \longleftarrow \bullet \longrightarrow$$

  in A, and
- (acyclic) G^A has no directed cycle.

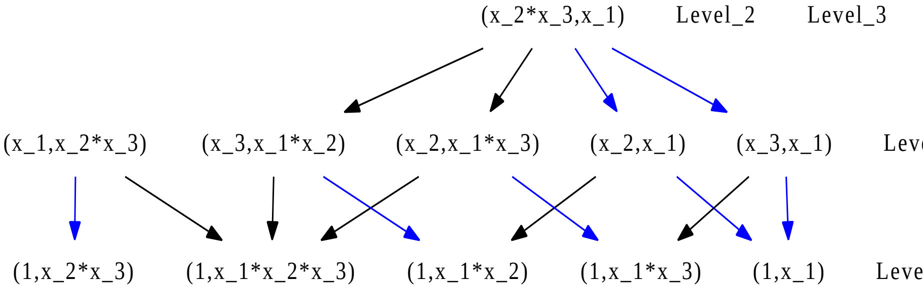
Algebraic discrete Morse theory

For an acyclic matching A, we set

- $X_i^A := \{\alpha \in X_i \mid \alpha \text{ does not appear in any adge in } A\}$, and $X_A = \bigcup_i X_i^A$. The elements of X_A are said to be critical.
- $K_i^A := \bigoplus_{\alpha \in X_i^A} S \cdot e(\alpha)$.
- $$d_{\beta,\alpha}^A := \begin{cases} -d_{\beta,\alpha}^{-1} & \text{if } \alpha \longrightarrow \beta \in A \\ d_{\beta,\alpha} & \text{otherwise,} \end{cases}$$
- For a path $p = \alpha_0 \to \alpha_1 \to \cdots \to \alpha_s$ in G^A,
$$d_p^A := d_{\alpha_s,\alpha_{s-1}}^A \circ d_{\alpha_{s-1},\alpha_{s-2}}^A \circ \cdots \circ d_{\alpha_1,\alpha_0}^A.$$
- $d_i^A := \sum_{\substack{p \in \mathrm{Path}(\alpha,\beta) \\ \alpha \in X_i^A,\ \beta \in X_{i-1}^A}} d_p^A$.

Algebraic discrete Morse theory

$I = (x_1, x_2x_3)$.



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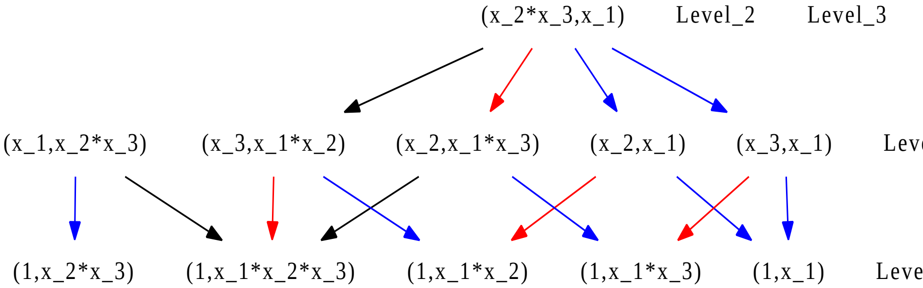

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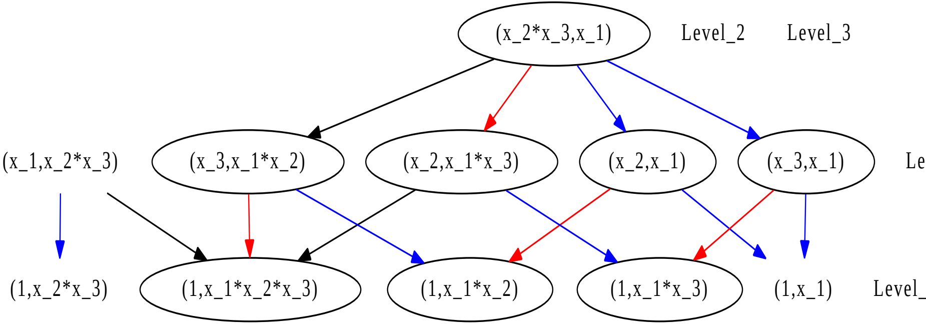

Algebraic discrete Morse theory

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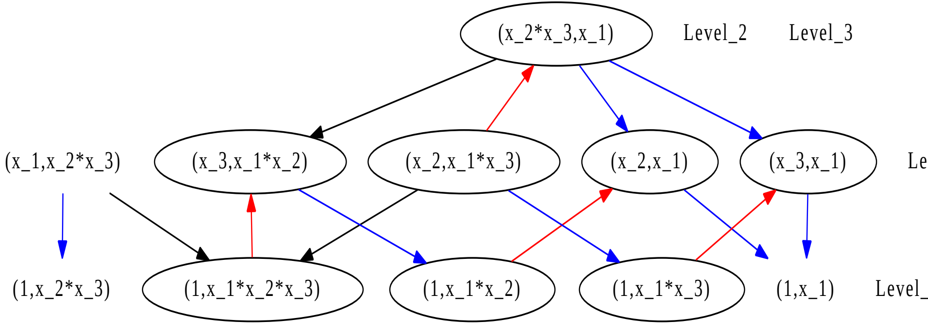


Algebraic discrete Morse theory

Theorem (Jöllenbeck-Welker-Sköldberg)

The sequence (K^A, d^A) is a complex and homotopy equivalent with the original one (K, d), and hence it is also a free resolution of I.

Remark

- Clearly (K^A, d^A) is a free resolution smaller than (K, d).
- If $d_{\beta,\alpha}$ is not zero and isomorphic, then $A := \{\alpha \to \beta\}$ is always an acyclic matching.
- Thus applying the theorem above repeatedly, we get a minimal free resolution.

Algebraic discrete Morse theory

Minimal free resolution of monomial ideals with linear quotients

$G(I) :=$ a minimal system of monomial generators. $M(I) :=$ the set of the monomials in I.

Definition (Herzog-Takayama and Batzies-Welker)

I is said to have linear quotients if $\exists$ a linear ordering $<$ on $G(I)$ such that letting $G(I) = \{u_1, \dots, u_s\}$ with $u_1 < \cdots < u_s$, each colon ideal $(u_1, \dots, u_i) : u_{i+1}$ is generated by some variables of S.

Example

For example,

- $I = (x_1x_2, x_2x_3, x_2x_4)$ has linear quotients, and
- $I = (x_1^2, x_2x_3, x_2x_4)$ does not.

Algebraic discrete Morse theory

Minimal free resolution of monomial ideals with linear quotients

Henceforth, we assume I has linear quotients with the linear ordering $<$ on $G(I)$.

Definition (Herzog-Takayama)

Define $g : M(I) \to G(I)$ by $g(u) := \min_< \{u_i \in G(I) \mid u_i \text{ divides } u\}$. The function g is called the decomposition function (with respect to $<$ and I).

Exercises

Show that $g(x_i u) \leq g(u)$ for all i and $u \in M(I)$.

Algebraic discrete Morse theory

Minimal free resolution of monomial ideals with linear quotients

For $u \in M(I)$, set $c(u) := u/g(u)$ and $\mathrm{stb}(u) := \{i \mid g(u) = g(x_i u)\}$. Thus

$$u = c(u) \cdot g(u)$$

Exercises

Show the following.

(1) $\mathrm{stb}(u) \supseteq \mathrm{supp}(c(u))$ for all $u \in M(I)$.

(2) For $u \in M(I)$ with $g(u) = u_k$,

$$\mathrm{stb}(u) = \mathrm{stb}(g(u)) = [n] \setminus \{i \mid x_i \in (u_1, \ldots, u_{k-1}) : u_k\}.$$

Algebraic discrete Morse theory

Minimal free resolution of monomial ideals with linear quotients

Recall that $G = (V, E)$ is the directed graph associated with $K_\bullet := K_\bullet^I$, and hence

- $V = X = \{(x_F, x^{\mathbf{a}}) \mid F \subseteq [n],\ \mathbf{a} \leq \mathbf{t} - \mathbf{e}_F,\ x^{\mathbf{a}} \in I\}$, and
- $E = \{\alpha \longrightarrow \beta \mid \alpha \in X_i,\ \beta \in X_{i-1},\ d_{\beta,\alpha} \neq 0\}$.

Let A be the subset of E consisting of the edges

$$(x_F, x^{\mathbf{a}}) \longrightarrow (x_{F\setminus\{i\}}, x_i x^{\mathbf{a}})$$

such that

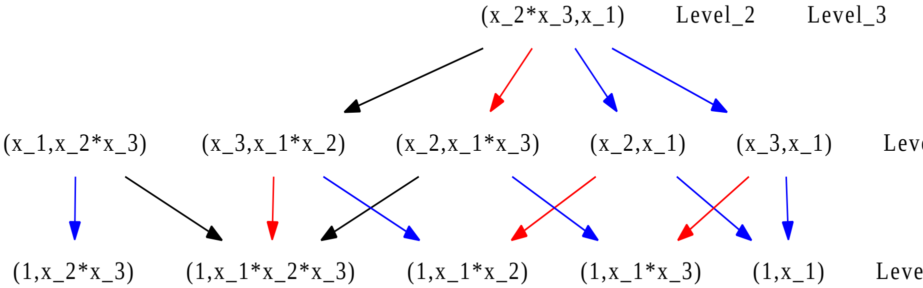
(1) $i \in F$ and

(2) $i = \max(\mathrm{supp}(x_F c(x^{\mathbf{a}})) \cap \mathrm{stb}(x^{\mathbf{a}}))$ (hence $g(x^{\mathbf{a}}) = g(x_i x^{\mathbf{a}})$).

Algebraic discrete Morse theory

Minimal free resolution of monomial ideals with linear quotients

$I = (x_1, x_2 x_3)$.



Algebraic discrete Morse theory

Minimal free resolution of monomial ideals with linear quotients

Theorem (Sköldberg)

The set A is an acyclic matching, and the induced complex $K_\bullet^A$ is a minimal free resolution of I.

For $(x_F, x^{\mathbf{a}}) \to (x_G, x^{\mathbf{b}}) \in A$, it follows that $x_F c(x^{\mathbf{a}}) = x_G c(x^{\mathbf{b}})$.

Exercises

(1) the set A is indeed an acyclic matching..

(2) $X^A = \{(X_F, u) \mid u \in G(I),\ F \cap \operatorname{stb}(u) = \varnothing\}$.

(3) Deduce that any path from $(x_F, x^{\mathbf{a}}) \in X_i^A$ to $(x_G, x^{\mathbf{b}}) \in X_{i-1}^A$ satisfies $x_F x^{\mathbf{a}}/(x_G x^{\mathbf{b}}) \in \mathfrak{m}$, and hence $K_\bullet^A$ is minimal.

Algebraic discrete Morse theory

Minimal free resolution of monomial ideals with linear quotients

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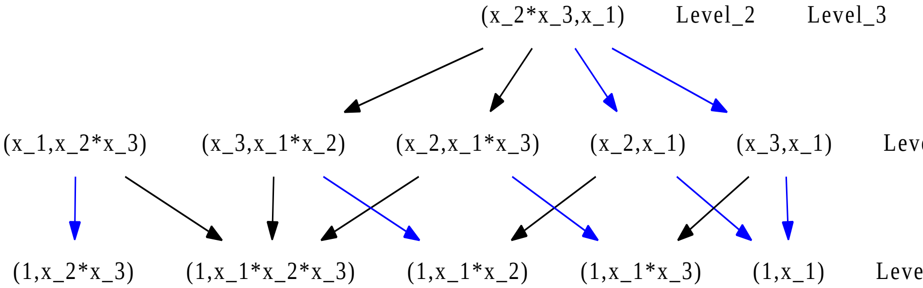

Algebraic discrete Morse theory

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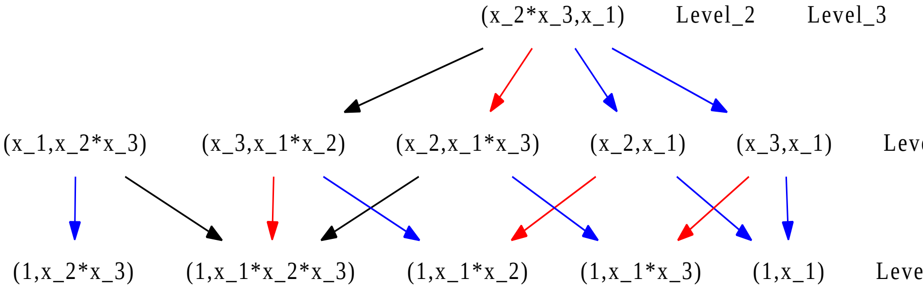

Algebraic discrete Morse theory

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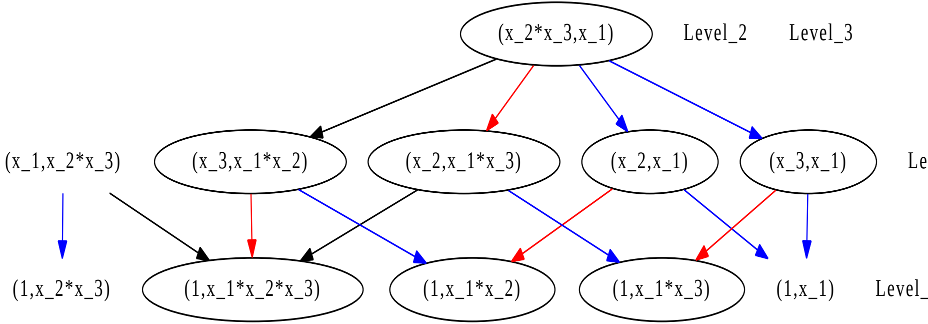

Algebraic discrete Morse theory

Minimal free resolution of monomial ideals with linear quotients

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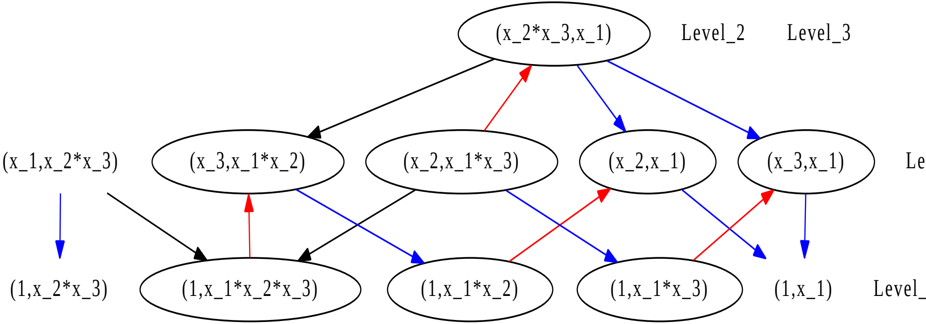

Advertising

Minimal free resolution of monomial ideals with linear quotients

- We can also reduce the Taylor resolution to get a minimal free resolution of a monomial ideal with linear quotients. Indeed, Batzies and Welker gave an acyclic matching for the Taylor resolution, but the construction of the matching is very technical.
- However their method has the advantage that it equips the minimal free resolution with a CW complex. (Such resolutions are called cellular resolutions).
- I and K. Yanagawa concretely described the corresponding minimal free resolution of a strongly stable monomial ideal I without using paths.
- Moreover we showed that the resolution is equipped with a regular CW complex homeomorphic to a ball when I is Cohen-Macaulay.

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References II

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