The type computation of some classes of base rings

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Conjecture: Let $n \geq 4$, $A_i \subset [n]$ for any $1 \leq i \leq n$ and $K[\mathcal{A}]$ be the base ring associated to the transversal polymatroid presented by $\mathcal{A} = \{A_1, \ldots, A_n\}$. If the Hilbert series is:

$$H_{K[\mathcal{A}]}(t) \ = \ \frac{1 + h_1 \ t + \ldots + h_{n-r} \ t^{n-r}}{(1-t)^n},$$

then we have the following:

- 1) If $r = 1$, then $type(K[\mathcal{A}]) = 1 + h_{n-2} - h_1$.
- 2) If $2 \leq r \leq n$, then $type(K[\mathcal{A}]) = h_{n-r}$.

We fix the notation and recall some basic results. For details we refer the reader to [B], [BH], [BG], [MS] and [V].
The subsets of elements ≥ 0 in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ will be referred to by $\mathbb{Z}_+, \mathbb{Q}_+, \mathbb{R}_+$ and the subsets of elements > 0 by $\mathbb{Z}_>, \mathbb{Q}_>, \mathbb{R}_>$.
Fix an integer $n > 0$. If $0 \neq a \in \mathbb{Q}^n$, then H_a will denote the rational hyperplane of $\mathbb{R}^n$ through the origin with normal vector a, that is,

$$H_a = \{x \in \mathbb{R}^n \mid \langle x, a \rangle = 0\},$$

where $\langle\ ,\ \rangle$ is the scalar product in $\mathbb{R}^n$. The two closed rational linear halfspaces bounded by H_a are:

$$H_a^+ = \{x \in \mathbb{R}^n \mid \langle x, a \rangle \geq 0\} \text{ and } H_a^- = H_{-a}^+ = \{x \in \mathbb{R}^n \mid \langle x, a \rangle \leq 0\}.$$

The two open rational linear halfspaces bounded by H_a are:

$$H_a^> = \{x \in \mathbb{R}^n \mid \langle x, a \rangle > 0\} \text{ and } H_a^< = H_{-a}^> = \{x \in \mathbb{R}^n \mid \langle x, a \rangle < 0\}.$$

If $S \subset \mathbb{Q}^n$, then the set

$$\mathbb{R}_+ S = \{\sum_{i=1}^{r} a_i v_i \ : \ a_i \in \mathbb{R}_+, \ v_i \in S, \ r \in \mathbb{N}\}$$

is called the *rational cone* generated by S.
The *dimension* of a cone is the dimension of the smallest vector subspace of $\mathbb{R}^n$ which contains it.

By the theorem of Minkowski-Weyl finitely generated rational cones can also be described as intersection of finitely many rational closed subspaces (of the form H_a^+). We further restrict this presentation to the class of finitely generated rational cones, which will be simply called cones.

If a cone C is presented as

$$C = H_{a_1}^{+} \cap \ldots \cap H_{a_r}^{+}$$

such that no $H_{a_i}^{+}$ can be omitted, then we say that this is an *irredundant representation* of C.
If $\dim(C) = n$, then the halfspaces $H_{a_1}^{+}, \ldots, H_{a_r}^{+}$ in an irredundant representation of C are uniquely determined and we set

$$\operatorname{relint}(C) = H_{a_1}^{>} \cap \ldots \cap H_{a_r}^{>}$$

the *relative interior* of C. If $a_i = (a_{i1}, \ldots, a_{in})$, then we call

$$H_{a_i}(x) := a_{i1}x_1 + \ldots + a_{in}x_n = 0,$$

the *equations of the cone C*.

Discrete polymatroids. Fix an integer $n > 0$ and set $[n] := \{1, 2, \ldots, n\}$. The canonical basis vectors of $\mathbb{R}^n$ will be denoted by $e_1, \ldots, e_n$. For a vector $a \in \mathbb{R}^n$, $a = (a_1, \ldots, a_n)$, we set $\mid a \mid := a_1 + \ldots + a_n$.
A nonempty finite set $B \subset \mathbb{Z}_+^n$ is the *set of bases a discrete polymatroid* $\mathcal{P}$ if:

(a) for every $u,\ v \in B$ one has $\mid u \mid = \mid v \mid$;

(b) (the exchange property) if $u,\ v \in B$, then for all i such that $u_i > v_i$ there exists j such that $u_j < v_j$ and $u + e_j - e_i \in B$.

An element of B is called a *base of the discrete polymatroid* $\mathcal{P}$.

Let K be an infinite field. For $a \in \mathbb{Z}_+^n$, $a = (a_1, \ldots, a_n)$ we denote by $x^a \in K[x_1, \ldots, x_n]$ the monomial $x^a := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and we set $\log(x^a) = a$.
Associated with the set of bases B of a discrete polymatroid $\mathcal{P}$ one has a $K-$algebra $K[B]$, called the *base ring* of $\mathcal{P}$, defined to be the $K-$subalgebra of the polynomial ring in n indeterminates $K[x_1, x_2, \ldots, x_n]$ generated by the monomials x^u with $u \in B$.

$$K[B] = K[x^u \mid u \in B]$$

From [HH], [Vi] the monoid algebra $K[B]$ is known to be normal.

We recall that by a well known result of Danilov and Stanley the canonical module $\omega_{K[B]}$ of $K[B]$, with respect to standard grading, can be expressed as an ideal of $K[B]$ generated by monomials, that is

$$\omega_{K[B]} = (\{x^a \mid a \in \mathbb{Z}_+ B \cap \operatorname{relint}(\mathbb{R}_+ B)\}).$$

Transversal polymatroids.

Consider another integer m such that $1 \leq m \leq n$.
If A_i are some nonempty subsets of $[n]$ for $1 \leq i \leq m$ and $\mathcal{A} = \{A_1, \ldots, A_m\}$, then the set of the vectors $\sum_{k=1}^{m} e_{i_k}$ with $i_k \in A_k$ is the set of bases of a polymatroid, called the *transversal polymatroid presented by $\mathcal{A}$*.
The base ring of the transversal polymatroid presented by $\mathcal{A}$ is the ring

$$K[\mathcal{A}] := K[x_{i_1} \cdots x_{i_m} \mid i_j \in A_j, 1 \leq j \leq m].$$

We denote by

$$A := \{\log(x_{j_1} \cdots x_{j_n}) \mid j_k \in A_k, \text{ for all } 1 \leq k \leq n\} \subset \mathbb{N}^n$$

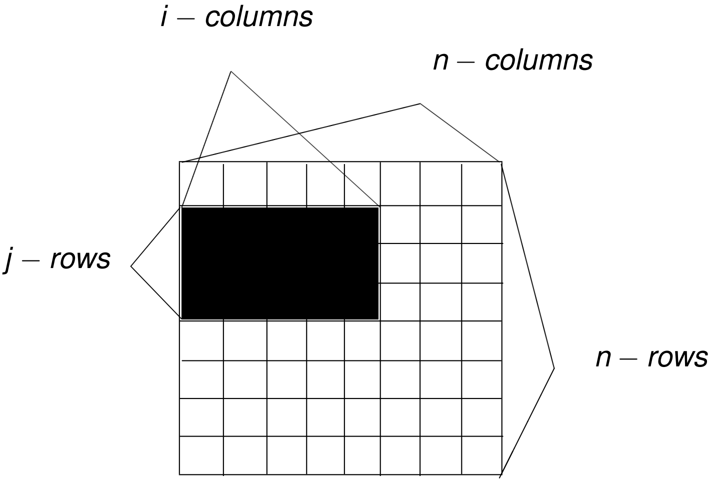
the exponents set of the generators of the base ring $K[\mathcal{A}]$.

Further, for the transversal polymatroid presented by $\mathcal{A}$ we associate a $(n \times n)$ square tiled by unit subsquares, called *boxes*, colored with white and black as follows: the box of coordinate (i, j) is white if $j \in A_i$, otherwise the box is black. We will call this square the *polymatroidal diagram* associated to the presentation $\mathcal{A} = \{A_1, \ldots, A_n\}$([SA1],[SA2]).

In the following we shall restrict our study to a special family of transversal polymatroids.
Fix $n \in \mathbb{Z}_+,\ n \geq 3,\ 1 \leq i \leq n-2$ and $1 \leq j \leq n-1$ and consider the transversal polymatroid presented by

$$\mathcal{A} = \{A_2 = [n] \setminus [i], \ldots, A_{j+1} = [n] \setminus [i],$$

$$A_1 = [n], A_{j+2} = [n], \ldots, A_n = [n]\}.$$



Polymatroidal diagram associated to the presentation $\mathcal{A}$

We recall at this point some previous results contained in [SA]. The cone generated by A has the irredundant representation

$$\mathbb{R}_+ A = \bigcap_{a \in N} H_a^+,$$

where $N = \{\nu_i^j\} \bigcup \{e_k \mid 1 \leq k \leq n\}$ and

$$\nu_i^j := \sum_{k=1}^{i} -j e_k + \sum_{k=i+1}^{n} (n-j) e_k.$$

The extreme rays of the cone $\mathbb{R}_+ A$ are given by

$$E := \{n e_k \mid i+1 \leq k \leq n\} \bigcup$$

$$\{(n-j) e_r + j\, e_s \mid 1 \leq r \leq i \text{ and } i+1 \leq s \leq n\}.$$

The polynomial

$$P_d(k) = \binom{d+k-1}{d-1}$$

counts the number of monomials in degree k over the standard graded polynomial ring $K[x_1, \dots, x_d]$, i.e. $P_d(k)$ is the Hilbert function of $K[x_1, \dots, x_d]$.

Then

$$P_d(k-d) = \binom{k-1}{d-1} = Q_d(k)$$

counts the number of monomials in degree k for which all the variables have nonzero powers, i.e. $Q_d(k)$ is the Hilbert function of the canonical module $\omega_{K[x_1,\dots,x_d]} = K[x_1, \dots, x_d](-d)$

The main result of [SA] is the following theorem.

Theorem

With the above assumptions, the following holds:
If $i + j \leq n - 1$, then the type of $K[\mathcal{A}]$ is

$$\mathrm{type}(K[\mathcal{A}]) = 1 + \sum_{t=1}^{n-i-j-1} Q_i(n + i - j + t) Q_{n-i}(n - i + j - t),$$

Theorem

If $i + j \geq n$, then the type of $K[\mathcal{A}]$ is

$$\mathrm{type}(K[\mathcal{A}]) = \sum_{t=1}^{r(n-j)-i} Q_i(r(n-j)-t)Q_{n-i}(rj+t),$$

where $r = \left\lceil \frac{i+1}{n-j} \right\rceil$ ($\lceil x \rceil$ is the least integer $\geq x$).

Corollary

$K[\mathcal{A}]$ is Gorenstein ring if and only if $i + j = n - 1$.

Let $r \geq 2$, $1 \leq i_1, \ldots, i_r \leq n-2$, $0 = t_1 \leq t_2, \ldots, t_r \leq n-1$ and consider r presentations of transversal polymatroids:

$$\mathcal{A}_s = \{A_{s,k} \mid A_{s,\sigma^{t_s}(k)} = [n], \text{ if } k \in [i_s] \cup \{n\}, \\ A_{s,\sigma^{t_s}(k)} = [n] \setminus \sigma^{t_s}[i_s], \text{ if } k \in [n-1] \setminus [i_s]\}$$

for any $1 \leq s \leq r$.
($\sigma \in S_n$, $\sigma = (1, 2, \ldots, n)$ the cycle of length n,
$\sigma^k[i] := \{\sigma^k(1), \ldots, \sigma^k(i)\}$)

The base rings $K[\mathcal{A}_s]$ are Gorenstein rings.
The exponents set of the generators of the base ring $K[\mathcal{A}_s]$ is:

$$A_s = \{\log(x_{j_1} \cdots x_{j_n}) \mid j_k \in A_{s,k}, 1 \leq k \leq n\} \subset \mathbb{N}^n$$

for any $1 \leq s \leq r$.
We denote by $K[A_1 \cap \ldots \cap A_r]$, the $K-\mathrm{algebra}$ generated by x^α with $\alpha \in A_1 \cap \ldots \cap A_r$.

Lemma

The $K-$ algebra $K[A_1 \cap \ldots \cap A_r]$ is a Gorenstein ring.

Let $n \geq 2$ and consider two transversal polymatroids presented by $\mathcal{A} = \{A_1, \ldots, A_n\}$, respectively $\mathcal{B} = \{B_1, \ldots, B_n\}$.
Let A and B be the set of exponent vectors of monomials defining the base rings $K[\mathcal{A}]$, respectively $K[\mathcal{B}]$, and $K[A \cap B]$ the $K - \mathrm{algebra}$ generated by x^α with $\alpha \in A \cap B$.
Question: There exists a transversal polymatroid such that its base ring is the $K - \mathrm{algebra}$ $K[A \cap B]$?

Example

Let $n=4$, $\mathcal{A}=\{A_1,A_2,A_3,A_4\}$, $\mathcal{B}=\{B_1,B_2,B_3,B_4\}$, where $A_1=A_4=B_2=B_3=\{1,2,3,4\}$, $A_2=A_3=\{2,3,4\}$, $B_1=B_4=\{1,3,4\}$ and $K[\mathcal{A}]$, $K[\mathcal{B}]$ the base rings associated to transversal polymatroids presented by $\mathcal{A}$, respectively $\mathcal{B}$. It is easy to see that the generators set of $K[\mathcal{A}]$, respectively $K[\mathcal{B}]$, is given by
$A=\{y\in\mathbb{N}^4 \mid \mid y\mid=4,\ 0\leq y_1\leq 2,\ y_k\geq 0,\ 1\leq k\leq 4\}$, respectively
$B=\{y\in\mathbb{N}^4 \mid \mid y\mid=4,\ 0\leq y_2\leq 2,\ y_k\geq 0,\ 1\leq k\leq 4\}$. We show that the $K-$ algebra $K[A\cap B]$ is the base ring of the transversal polymatroid presented by $\mathcal{C}=\{C_1,C_2,C_3,C_4\}$, where $C_1=C_4=\{1,3,4\},\ C_2=C_3=\{2,3,4\}$.
Since the base ring associated to the transversal polymatroid presented by $\mathcal{C}$ has the exponent set $C=\{y\in\mathbb{N}^4 \mid \mid y\mid= 4,\ 0\leq y_1\leq 2,\ 0\leq y_2\leq 2,\ y_k\geq 0,\ 1\leq k\leq 4\}$, it follows that $K[A\cap B]=K[\mathcal{C}]$. Thus, in this example $K[A\cap B]$ is the base ring of a transversal polymatroid.

Example

Let $n = 4$, $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$, $\mathcal{B} = \{B_1, B_2, B_3, B_4\}$ where $A_1 = A_2 = A_4 = B_1 = B_2 = B_3 = \{1, 2, 3, 4\}$, $A_3 = \{3, 4\}$, $B_4 = \{1, 4\}$ and $K[\mathcal{A}]$, $K[\mathcal{B}]$ the base rings associated to the transversal polymatroids presented by $\mathcal{A}$, respectively $\mathcal{B}$. It is easy to see that the generators set of $K[\mathcal{A}]$, respectively $K[\mathcal{B}]$, is $A = \{y \in \mathbb{N}^4 \mid \ | y |= 4,\ 0 \leq y_1 + y_2 \leq 3,\ y_k \geq 0,\ 1 \leq k \leq 4\}$, respectively
$B = \{y \in \mathbb{N}^4 \mid \ | y |= 4,\ 0 \leq y_2 + y_3 \leq 3,\ y_k \geq 0,\ 1 \leq k \leq 4\}$.
We claim that there exists no transversal polymatroid $\mathcal{P}$ such that the $K-$ algebra $K[A \cap B]$ is its base ring. Suppose, on the contrary, that $\mathcal{P}$ is presented by $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$ with each $C_k \subset [4]$. Since $(3, 0, 1, 0), (3, 0, 0, 1) \in \mathcal{P}$ and $(3, 1, 0, 0) \notin \mathcal{P}$, we may assume by changing the numerotation of $\{C_i\}_{i=\overline{1,4}}$ that $1 \in C_1, 1 \in C_2, 1 \in C_4$ and $C_3 = \{3, 4\}$. Since $(0, 3, 0, 1) \in \mathcal{P}$, we may assume that $2 \in C_1, 2 \in C_2, 2 \in C_4$. Hence $(0, 3, 1, 0) \in \mathcal{P}$, a contradiction.

Next we give necesary and sufficient conditions such that the $K-\mathrm{algebra}\ K[A \cap B]$ is the base ring associated to some transversal polymatroid.

Theorem

Let $1 \leq i_1, i_2 \leq n-2$, $0 \leq t_2 \leq n-1$. *We consider two presentations of transversal poymatroids presented by:*
$\mathcal{A} = \{A_k \mid A_k = [n],\ \textit{if}\ k \in [i_1] \cup \{n\},\ A_k = [n] \setminus [i_1],\ \textit{if}\ k \in [n-1] \setminus [i_1]\}$ *and*
$\mathcal{B} = \{B_k \mid B_{\sigma^{t_2}(k)} = [n],\ \textit{if}\ k \in [i_2] \cup \{n\},\ B_{\sigma^{t_2}(k)} = [n] \setminus \sigma^{t_2}[i_2],\ \textit{if}\ k \in [n-1] \setminus [i_2]\}$
such that A, *respectively* B, *is the set of exponent vectors of the monomials defining the base ring associated to the transversal polymatroid presented by* $\mathcal{A}$, *respectively* $\mathcal{B}$.

Theorem

Then, the $K-$ *algebra* $K[A\cap B]$ *is the base ring associated to a transversal polymatroid if and only if one of the following conditions holds:*

a) $i_1 = 1$;

b) $i_1 \geq 2$ *and* $t_2 = 0$;

c) $i_1 \geq 2$ *and* $t_2 = i_1$;

d) $i_1 \geq 2$, $1 \leq t_2 \leq i_1 - 1$ *and*
$i_2 \in \{1,\ldots,i_1-t_2\}\cup\{n-t_2,\ldots,n-2\}$;

e) $i_1 \geq 2$, $i_1+1 \leq t_2 \leq n-1$ *and*
$i_2 \in \{1,\ldots,n-t_2\}\cup\{n-t_2+i_1,\ldots,n-2\}$.

Example

Let $\mathcal{A} = \{A_1, \ldots, A_7\}$, where $A_1 = A_2 = A_3 = A_6 = A_7 = [7]$, $A_4 = A_5 = [7] \setminus [3]$. The cone genereated by A, the exponent set of generators of $K-$algebra $K[\mathcal{A}]$, has the irreducible representation

$$\mathbb{R}_+ A = H^+_{\nu_3^2} \cap H^+_{e_1} \cap \ldots \cap H^+_{e_7}.$$

The type of $K[\mathcal{A}]$ is

$$type(K[\mathcal{A}]) = 1 + \binom{8}{2}\binom{4}{3} = 113.$$

The Hilbert series of $K[\mathcal{A}]$ is

$$H_{K[\mathcal{A}]}(t) \;=\; \frac{1 + 1561t + \ldots + 1673t^5 + t^6}{(1-t)^7}.$$

Note that $type(K[\mathcal{A}]) = 1 + h_5 - h_1 = 113$.

Example

Let $\mathcal{A} = \{A_1, \ldots, A_7\}$, where $A_3 = A_4 = [7]$, $A_1 = A_2 = A_5 = A_6 = A_7 = [7] \setminus [4]$. The cone genereated by A, the exponent set of generators of $K-$algebra $K[\mathcal{A}]$, has the irreducible representation

$$\mathbb{R}_+ A = H^+_{\nu^5_4} \cap H^+_{e_1} \cap \ldots \cap H^+_{e_7}.$$

The type of $K[\mathcal{A}]$ is

$$type(K[\mathcal{A}]) = \binom{4}{3}\binom{15}{2} + \binom{3}{3}\binom{16}{2} = 540.$$

The Hilbert series of $K[\mathcal{A}]$ is

$$H_{K[\mathcal{A}]}(t) \;=\; \frac{1 + 351t + 2835t^2 + 3297t^3 + 540t^4}{(1-t)^7}.$$

Note that $type(K[\mathcal{A}]) = h_4 = 540$.

The product of transversal polymatroids.

Fix $n_1, n_2 \in \mathbb{Z}_+,\ \ n_1, n_2 \geq 3$, $n = n_1 + n_2$, $i_1 \in [n_1 - 2]$, $i_2 \in [n_2 - 2]$, $j_1 \in [n_1 - 1]$ and $j_2 \in [n_2 - 1]$.

For the vectors $\alpha \in \mathbb{Z}_+^{n_1}$ and $\beta \in \mathbb{Z}_+^{n_2}$ we denote by $\widetilde{\alpha}, \bar{\beta} \in \mathbb{Z}_+^{n_1+n_2}$ the vectors

$$\widetilde{\alpha} = (\alpha, \underbrace{0, \ldots, 0}_{n_2 \ \ times}) \in \mathbb{Z}_+^{n_1+n_2}\ ,\ \ \bar{\beta} = (\underbrace{0, \ldots, 0}_{n_1 \ \ times}, \beta) \in \mathbb{Z}_+^{n_1+n_2}.$$

Next, we consider the $K-$algebras $K[\mathcal{A}]$ and $K[\mathcal{B}]$ which are the base rings of the transversal polymatroids presented by $\mathcal{A}$, respectively $\mathcal{B}$, where:

$$\mathcal{A} = \{A_2 = [n_1] \setminus [i_1], \ldots, A_{j_1+1} = [n_1] \setminus [i_1], \\ A_1 = [n_1], A_{j_1+2} = [n_1], \ldots, A_{n_1} = [n_1]\}$$

and

$$\mathcal{B} = \{A_{n_1+2} = [n] \setminus [n_1 + i_2], \ldots, A_{n_1+j_2+1} = [n] \setminus [n_1 + i_2], \\ A_{n_1+1} = [n] \setminus [n_1], A_{n_1+j_2+2} = [n] \setminus [n_1], \ldots, A_{n_1+n_2} = [n] \setminus [n_1]\}.$$

Let

$$A = \{\log(x_{t_1} \cdots x_{t_{n_1}}) \mid j_k \in A_k, \text{ for all } 1 \leq k \leq n_1\} \subset \mathbb{Z}_+^{n_1}$$

be the exponent set of generators of $K-$algebra $K[\mathcal{A}]$ and

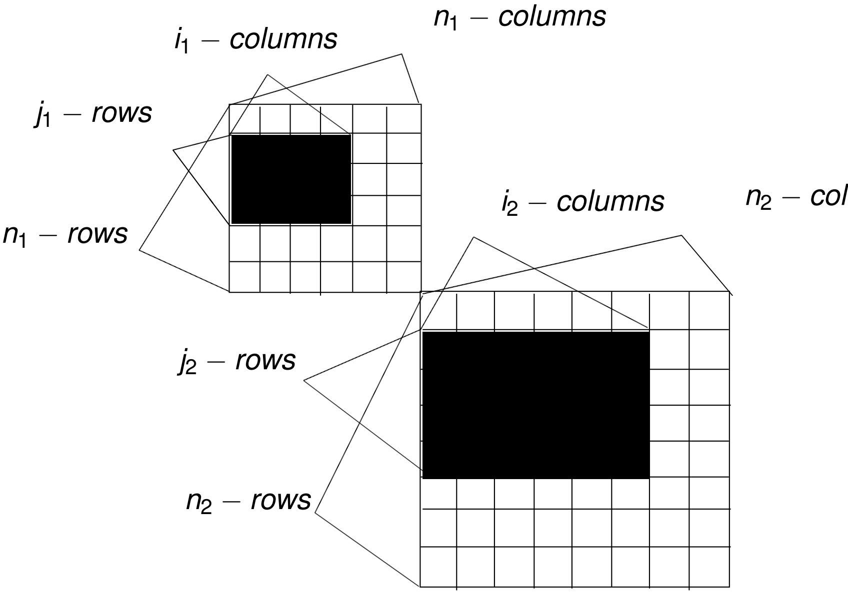
$$B = \{\log(x_{t_1} \cdots x_{t_{n_1}}) \mid j_k \in A_k, \text{for all } n_1 + 1 \leq k \leq n_1 + n_2\} \subset \mathbb{Z}_+^{n_2}$$

be the exponent set of generators of $K-$algebra $K[\mathcal{B}]$.

We denote by $K[\mathcal{A} \diamond \mathcal{B}]$ the $K-$algebra $K[x^{\widetilde{\alpha}+\bar{\beta}} \mid \alpha \in A, \ \beta \in B]$ and by $A \diamond B$ the exponent set of generators of $K[\mathcal{A} \diamond \mathcal{B}]$.

It is easy to see that $K-$algebra $K[\mathcal{A} \diamond \mathcal{B}]$ is the base ring associated to the transversal polymatroid presented by

$$\begin{aligned}\mathcal{A} \diamond \mathcal{B} = \{A_2 &= [n_1] \setminus [i_1], \ldots, A_{j_1+1} = [n_1] \setminus [i_1],\\ A_1 &= [n_1], A_{j_1+2} = [n_1], \ldots, A_{n_1} = [n_1],\\ A_{n_1+2} &= [n] \setminus [n_1 + i_2], \ldots, A_{n_1+j_2+1} = [n] \setminus [n_1 + i_2],\\ A_{n_1+1} &= [n] \setminus [n_1], A_{n_1+j_2+2} = [n] \setminus [n_1], \ldots,\\ A_{n_1+n_2} &= [n] \setminus [n_1]\}.\end{aligned}$$



Polymatroidal diagram associated to the presentation

With the notations from above, the cone generated by $A \diamond B$ has the irreducible representation

$$\mathbb{R}_+(A \diamond B) = \Pi \cap \bigcap_{a \in N} H_a^+,$$

where Π is the hyperplane described by the equation

$$-n_2 x_1 - \cdots - n_2 x_{n_1} + n_1 x_{n_1+1} + \cdots + n_1 x_{n_1+n_2} = 0$$

and $N = \{\widetilde{\nu}_{i_1}^{j_1}, \bar{\nu}_{i_2}^{j_2}\} \bigcup \{ e_k \mid 1 \leq k \leq n\}$.

Theorem

Let $K[\mathcal{A}]$ and $K[\mathcal{B}]$ the base rings of the transversal polymatroids presented by $\mathcal{A}$ and $\mathcal{B}$ from above.
Then:
a) If $i_1 + j_1 \leq n_1 - 1$ and $i_2 + j_2 \leq n_2 - 1$, then the type of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$\begin{aligned} \operatorname{type}(K[\mathcal{A} \diamond \mathcal{B}]) = 1 &+ (\operatorname{type}(K[\mathcal{A}] - 1)Q_2 + (\operatorname{type}(K[\mathcal{B}] - 1)Q_1 \\ &- (\operatorname{type}(K[\mathcal{A}] - 1)(\operatorname{type}(K[\mathcal{B}] - 1), \end{aligned}$$

where

$$Q_r = \sum_{t=i_r}^{2(n_r-j_r)-1} Q_{i_r}(t) Q_{n_r-i_r}(2n_r - t), \text{ for } r \in [2].$$

Theorem

b) If $i_1 + j_1 \geq n_1$ *and* $i_2 + j_2 \geq n_2$ *such that* $r_1 \leq r_2$ *where* $r_1 = \left\lceil \frac{i_1+1}{n_1-j_1} \right\rceil$, $r_2 = \left\lceil \frac{i_2+1}{n_2-j_2} \right\rceil$ *then the type of* $K[\mathcal{A} \diamond \mathcal{B}]$ *is*

$$\operatorname{type}(K[\mathcal{A} \diamond \mathcal{B}]) = [\sum_{t=i_1}^{r_2(n_1-j_1)-1} Q_{i_1}(t) Q_{n_1-i_1}(r_2 n_1 - t)] \operatorname{type}(K[\mathcal{B}]).$$

Theorem

c) If $i_1 + j_1 \leq n_1 - 1$, $i_2 + j_2 \geq n_2$ and $r_2 = \left\lceil \frac{i_2+1}{n_2-j_2} \right\rceil$, then the type of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$\mathrm{type}(K[\mathcal{A} \diamond \mathcal{B}]) = [G + E]\,\mathrm{type}(K[\mathcal{B}]),$$

where

$$G = \sum_{t=0}^{(r_2-1)(n_1-j_1)} P_{i_1}(t) P_{n_1-i_1}((r_2-1)n_1 - t),$$

$$E = \sum_{t=1}^{n_1-i_1-j_1-1} Q_{i_1}(i_1 + (r_2-1)(n_1-j_1) + t) *$$

$$* \, Q_{n_1-i_1}(n_1 - i_1 + (r_2-1)j_1 - t).$$

Corollary

Let $K[\mathcal{A}]$ and $K[\mathcal{B}]$ the base rings of the transversal polymatroids presented by $\mathcal{A}$ and $\mathcal{B}$ and $K[\mathcal{A} \diamond \mathcal{B}]$ the base ring of the transversal polymatroid presented by $\mathcal{A} \diamond \mathcal{B}$, then: $K[\mathcal{A} \diamond \mathcal{B}]$ is Gorenstein ring if and only if $K[\mathcal{A}]$ and $K[\mathcal{B}]$ are Gorenstein rings.

Next we will give some examples.

Example

Let $\mathcal{A} = \{A_1, \dots, A_5\}$, $\mathcal{B} = \{A_6, \dots, A_{12}\}$ and $\mathcal{A} \diamond \mathcal{B} = \{A_1, \dots, A_{12}\}$, where $A_1 = A_3 = A_4 = A_5 = [5]$, $A_2 = [5] \setminus [2]$, $A_6 = A_9 = A_{10} = A_{11} = A_{12} = [12] \setminus [5]$, $A_7 = A_8 = [12] \setminus [8]$.
The type of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$\begin{aligned} \operatorname{type}(K[\mathcal{A} \diamond \mathcal{B}]) &= 1 + (7-1)1680 + (113-1)126 \\ &\quad - (7-1)(113-1) = 23521, \end{aligned}$$

where

$$\operatorname{type}(K[\mathcal{A}]) = 7,\ \operatorname{type}(K[\mathcal{B}]) = 113,\ Q_1 = 126, Q_2 = 1680.$$

Example

The Hilbert series of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$H_{K[\mathcal{A} \diamond \mathcal{B}]}(t) = \frac{1 + 188149t + \ldots + 211669t^9 + t^{10}}{(1-t)^{11}}.$$

Note that $\operatorname{type}(K[\mathcal{A} \diamond \mathcal{B}]) = 1 + h_9 - h_1 = 23521$.

Example

Let $\mathcal{A} = \{A_1, \ldots, A_7\}$, $\mathcal{B} = \{A_8, \ldots, A_{15}\}$ and $\mathcal{A} \diamond \mathcal{B} = \{A_1, \ldots, A_{15}\}$, where $A_1 = A_6 = A_7 = [7]$, $A_2 = A_3 = A_4 = A_5 = [7] \setminus [5]$, $A_8 = A_{15} = [15] \setminus [7]$, $A_9 = A_{10} = A_{11} = A_{12} = A_{13} = A_{14} = [15] \setminus [13]$.
The type of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$\text{type}(K[\mathcal{A} \diamond \mathcal{B}]) = (\sum_{t=5}^{11} \binom{t-1}{4} \binom{27-t}{1})169 = 1327326,$$

where

$$\text{type}(K[\mathcal{B}]) = 169.$$

Example

The Hilbert series of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$H_{K[\mathcal{A}\diamond\mathcal{B}]}(t) \ = \ \frac{1 + 62818t + \ldots + 91435344t^9 + 1327326t^{10}}{(1-t)^{14}}.$$

Note that $\mathrm{type}(K[\mathcal{A} \diamond \mathcal{B}]) = h_{10} = 1327326$.

Example

Let $\mathcal{A} = \{A_1, \ldots, A_8\}$, $\mathcal{B} = \{A_9, \ldots, A_{16}\}$ and $\mathcal{A} \diamond \mathcal{B} = \{A_1, \ldots, A_{16}\}$, where
$A_1 = A_4 = A_5 = A_6 = A_7 = A_8 = [8]$, $A_2 = A_3 = [8] \setminus [3]$,
$A_9 = A_{16} = [16] \setminus [8]$,
$A_{10} = A_{11} = A_{12} = A_{13} = A_{14} = A_{15} = [16] \setminus [14]$.
The type of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$\mathrm{type}(K[\mathcal{A} \diamond \mathcal{B}]) = (2572125 + 42630)169 = 441893595,$$

where

$\mathrm{type}(K[\mathcal{A}]) = 226$, $type(K[\mathcal{B}]) = 169$, $G = 2572125$, $E = 42630$.

Example

The Hilbert series of $K[\mathcal{A} \diamond \mathcal{B}]$ is

$$H_{K[\mathcal{A} \diamond \mathcal{B}]}(t) = \frac{1 + 1266825t + \ldots + 441893595t^{11}}{(1-t)^{15}}.$$

Note that $\operatorname{type}(K[\mathcal{A} \diamond \mathcal{B}]) = h_{11} = 441893595$.

I am grateful to B. Ichim for some extensive computational experiments which was needed in order to deduce the formulas!

The Pappus matroid is the matroid on

$$E = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

whose bases are all triples except

$$\{123, 456, 789, 148, 247, 159, 357, 269, 368\}$$

This is not transversal!

The Hilbert series of the base ring associated to the Pappus matroid is

$$H_{K[P]}(t) = \frac{1 + 66t + 744t^2 + 1915t^3 + 1230t^4 + 147t^5 + t^6}{(1-t)^9}$$

$$1 + h_6 - h_1 = 1 + 147 - 66 = 82,$$

but the type the base ring associated to the Pappus matroid is :

$$type(K[P]) = 181.$$

THANK YOU !

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